Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. II (2003), pp. 379-394

Hörmander Systems and Harmonic Morphisms

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Abstract. Given a Hörmander system $X = \{X_1, \cdots, X_m\}$ on a domain $\Omega \subseteq \mathbf{R}^n$ we show that any *subelliptic harmonic morphism* ϕ from Ω into a ν-dimensional Riemannian manifold N is a (smooth) subelliptic harmonic map (in the sense of J. Jost & C-J. Xu, [9]). Also ϕ is a submersion provided that $\nu \leq m$ and X has rank m. If $\Omega = \mathbf{H}_n$ (the Heisenberg group) and $X = \left\{\frac{1}{2}(L_\alpha + L_{\overline{\alpha}}), \frac{1}{2i}(L_\alpha - L_{\overline{\alpha}})\right\}$, where $L_{\overline{\alpha}} = \partial/\partial \overline{z}^\alpha - i z^\alpha \partial/\partial t$ is the Lewy operator, then a smooth map $\phi : \Omega \to N$ is a subelliptic harmonic morphism if and only if $\phi \circ \pi : (C(\mathbf{H}_n), F_{\theta_0}) \to N$ is a harmonic morphism, where $S^1 \to C(\mathbf{H}_n) \xrightarrow{\pi} \mathbf{H}_n$ is the canonical circle bundle and F_{θ_0} is the Fefferman metric of $(\mathbf{H}_n, \theta_0)$. For any S^1-invariant weak solution to the harmonic map equation on $(C(\mathbf{H}_n), F_{\theta_0})$ the corresponding base map is shown to be a weak subelliptic harmonic map. We obtain a regularity result for *weak* harmonic morphisms from $(C(\{x_1 > 0\}), F_{\theta(k)})$ into a Riemannian manifold, where $F_{\theta(k)}$ is the Fefferman metric associated to the system of vector fields $X_1 = \partial/\partial x_1$, $X_2 = \partial/\partial x_2 + x_1^k \, \partial/\partial x_3$ $(k \geq 1)$ on $\Omega = \mathbf{R}^3 \setminus \{x_1 = 0\}$.

Mathematics Subject Classification (2000): 58E20, 53C43 (primary), 32V20, 35H20 (secondary).

1. – Introduction

J. Jost & C-J. Xu studied (cf. [9]) the existence and regularity of weak solutions $\phi : \Omega \to N$ to the nonlinear subelliptic system

$$(1) \qquad H\phi^i - \sum_{a=1}^{m} \left(\left| \begin{matrix} i \\ jk \end{matrix} \right| \circ \phi \right) X_a(\phi^j) X_a(\phi^k) = 0, \; 1 \leq i \leq \nu,$$

where $H = \sum_{a=1}^m X_a^* X_a$ is the Hörmander operator associated to a system $X = \{X_1, \cdots, X_m\}$ of smooth vector fields on a open set $\Omega \subseteq \mathbf{R}^n$, verifying the Hörmander condition on Ω, and N is a Riemannian manifold. If $\omega \subset \Omega$ is a smooth domain such that $\partial\omega$ is noncharacteristic for X, the result of J. Jost & C-J. Xu (cf. op. cit., Theorem 1 and 2, pp. 4641-4644) is that

Pervenuto alla Redazione il 21 Ottobre 2002 ed in forma definitiva il 27 Marzo 2003.

the Dirichlet problem for (1), with boundary data having values in regular balls of N, may be solved and the solution is continuous on ω, up to the boundary. Any such map is then smooth by a result of C-J. Xu & C. Zuily (cf. [14]), who studied higher regularity of continuous solutions to a quasilinear subelliptic system including (2). Solutions (smooth *a posteriori*) to (1) are *subelliptic harmonic maps* and (1) is the *subelliptic harmonic map system*. Cf. also Z-R. Zhou, [15]. Clearly, if $X_a = \partial/\partial x^a$, $1 \leq a \leq n$, then a subelliptic harmonic map is an ordinary harmonic map (Ω is thought of as a Riemannian manifold, with the Euclidean metric). An important class of harmonic maps are *harmonic morphisms*, i.e. smooth maps of Riemannian manifolds pulling back local harmonic functions to harmonic functions. That these are indeed harmonic maps is a classical result by T. Ishihara (cf. [7]), actually holding in general for harmonic morphisms between semi-Riemannian manifolds (cf. B. Fuglede, [5]). In the present paper we extend the notion of a harmonic morphism to the context of systems of vector fields and generalize the Fuglede-Ishihara theorem.

A localizable (in the sense of [8], p. 434, i.e. for any $x_0 \in \Omega$ there is an open neighborhood $U \subset \Omega$ of x_0 and a coordinate neighborhood (V, y^i) on N such that $\phi(U) \subset V$) map $\phi : \Omega \to N$ is a (weak) *subelliptic harmonic morphism* if for any $v : V \to \mathbf{R}$, with $V \subseteq N$ open and $\Delta_N v = 0$ in V, one has i) $v \circ \phi \in L^1_{loc}(U)$, for any open set $U \subset \Omega$ such that $\phi(U) \subset V$, and ii) $H(v \circ \phi) = 0$, in distributional sense. Our main result is

THEOREM 1. *Let $X = \{X_1, \cdots, X_m\}$ be a Hörmander system on a domain $\Omega \subseteq \mathbf{R}^n$ and N a ν-dimensional Riemannian manifold. If $\nu > m$ there are no subelliptic harmonic morphisms of Ω into N, except for the constant maps. If $\nu \leq m$ then any subelliptic harmonic morphism $\phi : \Omega \to N$ is an actually smooth subelliptic harmonic map and there is a smooth function $\lambda : \Omega \to [0, +\infty)$ such that*

$$\sum_{a=1}^{m} (X_a \phi^i)(x)(X_a \phi^j)(x) = \lambda(x)\delta^{ij}\,,\ 1 \leq i, j \leq \nu\,, \tag{2}$$

for any $x \in \Omega$ and any normal coordinate system (V, y^i) at $\phi(x) \in N$, where $\phi^i = y^i \circ \phi$. In particular if $x \in U = \phi^{-1}(V)$ is such that $\lambda(x) \neq 0$ then the matrix $[(X_a \phi^i)(x)]$ has maximal rank, hence ϕ is a C^∞ submersion provided that $\{X_1, \cdots, X_m\}$ are independent at any $x \in \Omega$.

When $\Omega = \mathbf{H}_n$, the Heisenberg group, and $X = \{X_\alpha, Y_\alpha : 1 \leq \alpha \leq n\}$ where $X_\alpha = (1/2)\partial/\partial x^\alpha + y^\alpha \partial/\partial t$ and $Y_\alpha = J X_\alpha$, we relate subelliptic harmonic morphisms to harmonic morphisms (from a certain Lorentzian manifold), in the spirit of [1] where subelliptic harmonic maps were related to harmonic maps (with respect to the Fefferman metric). Here J is given by $J L_\alpha = i L_\alpha$ and $J L_{\overline{\alpha}} = -i L_{\overline{\alpha}}$. We may state

THEOREM 2. *Let $\mathbf{H}_n = \mathbf{C}^n \times \mathbf{R}$ be the Heisenberg group endowed with the standard strictly pseudoconvex CR structure and the contact form $\theta_0 = dt + i \sum_{\alpha=1}^{n} (z^\alpha d\overline{z}^\alpha - \overline{z}^\alpha dz^\alpha)$. Consider the Fefferman metric*

$$F_{\theta_0} = \pi^* G_{\theta_0} + \frac{2}{n+2} (\pi^* \theta_0) \odot (d\gamma)$$

on $C(\mathbf{H}_n) = \left(\Lambda^{n+1,0}(\mathbf{H}_n) \setminus \{0\}\right)/\mathbf{R}_+$, *where* $\pi : C(\mathbf{H}_n) \to \mathbf{H}_n$ *is the projection and* γ *a fibre coordinate on* $C(\mathbf{H}_n)$. *Then a smooth map* $\phi : \mathbf{H}_n \to N$ *is a subelliptic harmonic morphism, with respect to the system of vector fields* $X = \{X_\alpha, Y_\alpha\}$, *if and only if* $\phi \circ \pi : (C(\mathbf{H}_n), F_{\theta_0}) \to N$ *is a harmonic morphism.*

The main ingredient is to relate the Laplace-Beltrami operator $\Box$ of the Fefferman metric F_{θ_0} to the Hörmander operator H on $\mathbf{H}_n$. This is rather well known in CR geometry (cf. J.M. Lee, [10], where $\Box$ is related to the sublaplacian Δ_b of the given strictly pseudoconvex CR manifold) yet not presented in the literature on PDEs. We emphasize on the relationship between subelliptic and hyperbolic PDEs by providing a short direct proof that, for the Heisenberg group, $\pi_* \Box = -2H$ where

$$\Box f = \frac{1}{2} \sum_{\alpha=1}^{n} \left(\frac{\partial^2 f}{\partial (u^\alpha)^2} + \frac{\partial^2 f}{\partial (u^{\alpha+n})^2} \right) + 2(|z|^2 \circ \pi) \frac{\partial^2 f}{\partial (u^{2n+1})^2}$$
$$+ 2u^{\alpha+n} \frac{\partial^2 f}{\partial u^\alpha \partial u^{2n+1}} - 2u^\alpha \frac{\partial^2 f}{\partial u^{\alpha+n} \partial u^{2n+1}} + 2(n+2) \frac{\partial^2 f}{\partial u^{2n+1} \partial u^{2n+2}} ,$$

for any $f \in C^2(\mathbf{H}_n)$. Here $u^A = x^A \circ \pi$, $1 \leq A \leq 2n+1$, and $u^{2n+2} = \gamma$, where $(x^A) = (z^\alpha = x^\alpha + i y^\alpha, t)$ are coordinates on $\mathbf{H}_n$.

Acknowledgements. The Author is grateful to Sorin Dragomir for carefully explaining to her the construction of the Fefferman metric within pseudohermitian geometry. The Author has profited from the many suggestions of the (anonymous) Referee.

2. – Hörmander systems

Let $\Omega \subseteq \mathbf{R}^n$ be an open set and $X = \{X_1, \cdots, X_m\}$ a system of C^∞ vector fields on Ω. We say X satisfies the *Hörmander condition* (or that X is a *Hörmander system*) on Ω, if the vector fields $X_1, \cdots, X_m$ together with their commutators up to some fixed length r span the tangent space $T_x(\Omega)$, at each $x \in \Omega$. If $X_a = b_a^A(x)\partial/\partial x^A$ then we set $X_a^* f = -\partial(b_a^A f)/\partial x^A$, for any $f \in C_0^1(\Omega)$. Our convention as to the range of indices is $a, b, \cdots \in \{1, \cdots, m\}$ and $A, B, \cdots \in \{1, \cdots, n\}$. The *Hörmander operator* is

$$Hu = \sum_{a=1}^{m} X_a^* X_a u = - \sum_{A,B=1}^{n} \frac{\partial}{\partial x^A} \left(a^{AB}(x) \frac{\partial u}{\partial x^B} \right) ,$$

where $a^{AB}(x) = \sum_{a=1}^{m} b_a^A(x) b_a^B(x)$. The matrix a^{AB} is symmetric and positive semi-definite, yet it may fail to be definite, hence in general H is not elliptic (H is a degenerate elliptic operator).

EXAMPLE 1 (CF. [9], P. 4634). The system of vector fields

$$X_1 = \partial/\partial x^1\,,\ X_2 = \partial/\partial x^2 + (x^1)^k \partial/\partial x^3\ (k \geq 0) \tag{3}$$

satisfies the Hörmander system on $\mathbf{R}^3$ with $r = k+1$. We have $X_a^* = -X_a$, $a \in \{1, 2\}$, hence the Hörmander operator is

$$Hu = -\frac{\partial^2 u}{\partial(x^1)^2} - \frac{\partial^2 u}{\partial(x^2)^2} - (x^1)^{2k}\frac{\partial^2 u}{\partial(x^3)^2} - 2(x^1)^k \frac{\partial^2 u}{\partial x^2 \partial x^3}\,. \tag{4}$$

As we shall see later, there is a CR structure $\mathcal{H}(k)$ on $\Omega = \mathbf{R}^3 \setminus \{x^1 = 0\}$ such that the (rank 2) distribution $\mathcal{D}$ spanned by the X_a's is precisely the Levi (or maximally complex) distribution of $(\Omega, \mathcal{H}(k))$.

EXAMPLE 2. Let $\mathbf{H}_n = \mathbf{C}^n \times \mathbf{R}$ be the *Heisenberg group* with coordinates $(z, t) = (z^1, \cdots, z^n, t)$ and set $z^\alpha = x^\alpha + i y^\alpha$, $1 \leq \alpha \leq n$. Consider the *Lewy operators*

$$L_{\overline{\alpha}} = \frac{\partial}{\partial \overline{z}^\alpha} - i z^\alpha \frac{\partial}{\partial t}\,,\ 1 \leq \alpha \leq n\,,$$

and the system of vector fields

$$X := \{X_\alpha, X_{\alpha+n} : 1 \leq \alpha \leq n\}\,,\ X_{\alpha+n} = J X_\alpha\,,\ X_\alpha = \frac{1}{2}\left(L_\alpha + L_{\overline{\alpha}}\right)\,, \tag{5}$$

where $L_\alpha = \overline{L_{\overline{\alpha}}}$. The Heisenberg group is thought of as a CR manifold (of hypersurface type) with the standard CR structure

$$T_{1,0}(\mathbf{H}_n)_x = \sum_{\alpha=1}^{n} \mathbf{C} L_{\alpha,x}\,,\ x \in \mathbf{H}_n\,.$$

Then J is the complex structure in the (real rank $2n$) distribution $H(\mathbf{H}_n) := \mathrm{Re}\left(T_{1,0}(\mathbf{H}_n) \oplus T_{0,1}(\mathbf{H}_n)\right)$, i.e. $J(Z + \overline{Z}) := i(Z - \overline{Z})$, for any $Z \in T_{1,0}(\mathbf{H}_n)$. As $[L_\alpha, L_{\overline{\alpha}}] = -2i\delta_{\alpha\beta} T$ (with $T = \partial/\partial t$), (5) is a Hörmander system on $\mathbf{H}_n$, with $r = 1$. Next $X_a^* = -X_a$ and the corresponding Hörmander operator is

$$Hu = -\frac{1}{4}\sum_{\alpha=1}^{n}\left\{\frac{\partial^2 u}{\partial(x^\alpha)^2} + \frac{\partial^2 u}{\partial(y^\alpha)^2}\right\} - y^\alpha \frac{\partial^2 u}{\partial x^\alpha \partial t} + x^\alpha \frac{\partial^2 u}{\partial y^\alpha \partial t} - |z|^2 \frac{\partial^2 u}{\partial t^2}\,. \tag{6}$$

3. – Subelliptic harmonic morphisms

First we see that a weak subelliptic harmonic morphism $\phi : \Omega \to N$ is actually smooth. Indeed, let $x \in \Omega$ and $p = \phi(x) \in N$. As ϕ is localizable, we may consider an open neighborhood U of x and a local system (V, y^i) of harmonic coordinates at p (cf. e.g. [3], p. 143, i.e. $p \in V$ and $\Delta_N y^i = 0$ in V, where Δ_N is the Laplace-Beltrami operator of N) such that $\phi(U) \subset V$. Then $y^i \circ \phi \in L^1_{loc}(U)$ and $H(y^i \circ \phi) = 0$. Moreover, it is a well known fact that H is hypoelliptic, i.e. if $Hu = f$ in distributional sense, and f is smooth, then u is smooth, too. Hence $y^i \circ \phi \in C^\infty(U)$. To show that ϕ is a subelliptic harmonic map we need the following

LEMMA 1 (T. Ishihara, [7]). *Let N be a ν-dimensional Riemannian manifold and C_i, $C_{ij} \in \mathbf{R}$ a system of constants such that $C_{ij} = C_{ji}$ and $\sum_{i=1}^{\nu} C_{ii} = 0$. Let $p \in N$. Then there is a normal coordinate system (V, y^i) in p and a harmonic function $v : V \to \mathbf{R}$ such that*

$$\frac{\partial v}{\partial y^i}(p) = C_i\,,\ v_{i,j}(p) = C_{ij}\,.$$

Here $v_{i,j}$ are the second order covariant[(1)] derivatives

$$v_{i,j} = \frac{\partial^2}{\partial y^i \partial y^j} - \left| \begin{matrix} k \\ ij \end{matrix} \right| \frac{\partial v}{\partial y^k}\,.$$

PROOF OF THEOREM 1. Let $i_0 \in \{1, \cdots, \nu\}$ be a fixed index and consider the constants $C_i = \delta_{ii_0}$ and $C_{ij} = 0$. By Ishihara's lemma there is a local harmonic function $v : V \to \mathbf{R}$ such that

$$\frac{\partial v}{\partial y^i}(p) = \delta_{ii_0}\,,\ v_{i,j}(p) = 0.$$

A calculation shows that

$$X_a(v \circ \phi) = \frac{\partial v}{\partial y^j} X_a(\phi^j)\,,$$

$$H(v \circ \phi) = (H\phi^j)\frac{\partial v}{\partial y^j} - \sum_{a=1}^{m} (X_a\phi^j)(X_a\phi^k) \left\{ v_{j,k} + \left| \begin{matrix} i \\ jk \end{matrix} \right| \frac{\partial v}{\partial y^i} \right\}. \tag{7}$$

Then (by (7))

$$0 = H(v \circ \phi)(x) = (H\phi^{i_0})(x) - \sum_{a=1}^{m} (X_a\phi^j)(x)(X_a\phi^k)(x) \left| \begin{matrix} i_0 \\ jk \end{matrix} \right| (p)\,.$$

(1) Here $\left| \begin{matrix} k \\ ij \end{matrix} \right|$ are the Christoffel symbols (of the second kind) associated to the Riemannian metric on N.

To prove (2) in Theorem 1 consider the constants $C_{ij} \in \mathbf{R}$ such that $C_{ij} = C_{ji}$ and $\sum_{i=1}^{\nu} C_{ii} = 0$. Let $x \in \Omega$ and $p = \phi(x) \in N$. By Ishihara's lemma there is a normal coordinate system (V, y^i) in p and a local harmonic function v on V such that

$$\frac{\partial v}{\partial y^i}(p) = 0, \quad v_{i,j}(p) = C_{ij}\,.$$

As ϕ is a subelliptic harmonic morphism (again by (7))

$$0 = H(v \circ \phi)(x) = -\sum_{a=1}^{m} (X_a \phi^j)(x)(X_a \phi^k)(x) C_{jk}$$

that is

$$C_{jk}\, X^{jk}(x) = 0\,, \tag{8}$$

where

$$X^{jk} := \sum_{a=1}^{m} (X_a \phi^j)(X_a \phi^k)\,.$$

The identity (8) may be also written as

$$\sum_{i \neq j} C_{ij} X^{ij}(x) + \sum_{i} C_{ii} \left\{ X^{ii}(x) - X^{11}(x) \right\} = 0\,. \tag{9}$$

Now let us choose the constants C_{ij} such that $C_{ij} = 0$ for any $i \neq j$ and

$$C_{ii} = \begin{cases} 1, & i = i_0 \\ -1, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

where $i_0 \in \{2, \cdots, \nu\}$ is a fixed index. Then (9) gives

$$X^{i_0 i_0}(x) - X^{11}(x) = 0\,,$$

that is

$$X^{11}(x) = X^{22}(x) = \cdots = X^{\nu\nu}(x)$$

and (9) becomes

$$\sum_{i \neq j} C_{ij} X^{ij}(x) = 0\,. \tag{10}$$

Let us fix $i_0, j_0 \in \{1, \cdots, \nu\}$ such that $i_0 \neq j_0$, otherwise arbitrary, and set

$$C_{ij} = \begin{cases} 1, & i = i_0\,,\ j = j_0 \quad \text{or} \quad i = j_0\,,\ j = i_0 \\ 0, & \text{otherwise} \end{cases}$$

Then (10) implies that $X^{i_0 j_0}(x) = 0$. Let us set

$$\lambda := X^{11} = \sum_{a=1}^{m} (X_a \phi^1)^2 \in C^\infty(U) \,,$$

where $U = \phi^{-1}(V) \subset \Omega$. Summing up the results obtained so far, we have

$$\sum_{a=1}^{m} (X_a \phi^i)(x)(X_a \phi^j)(x) = \lambda(x)\delta^{ij} \,,$$

which is (2), and in particular

$$\nu\, \lambda(x) = \sum_{a,i} (X_a \phi^i)(x)^2 \,.$$

Therefore, we built a global C^∞ function $\lambda : \Omega \to [0, +\infty)$. Indeed, if $(V, \varphi = (y^1, \cdots, y^\nu))$ and $(V', \varphi' = (y'^1, \cdots, y'^\nu))$ are two normal coordinate systems at $p = \phi(x)$ and $F = \varphi' \circ \varphi^{-1}$, then the identities

$$X_a \phi'^i = \frac{\partial F^i}{\partial \xi^j} X_a \phi^j \,, \; \sum_k \frac{\partial F^k}{\partial \xi^i}(p) \frac{\partial F^k}{\partial \xi^j}(p) = \delta_{ij}$$

yield

$$\sum_i (X_a \phi'^i)(x)^2 = \sum_j (X_a \phi^j)(x)^2 \,.$$

Assume there is $x_0 \in \Omega$ such that $\lambda(x_0) \neq 0$ and consider

$$v^i := \Big((X_1 \phi^i)(x_0), \cdots, (X_m \phi^i)(x_0) \Big) \in \mathbf{R}^m \,, \; 1 \leq i \leq \nu \,.$$

Clearly $v^i \neq 0$, for any i, and $v^i \cdot v^j = 0$, for any $i \neq j$. Consequently $rank[(X_a \phi^i)(x_0)] = \nu$, hence $\nu \leq m$. Thus, whenever $\nu > m$ it follows that $\lambda = 0$, i.e. $X_a \phi^i = 0$, and then the commutators of the X_a's, up to the length r, annihilate ϕ^i. As $X = \{X_1, \cdots, X_m\}$ is a Hörmander system and Ω is connected, it follows that $\phi^i = const$. Theorem 1 is proved.

4. – The relationship to hyperbolic PDEs

A smooth map $\Phi : M \to N$ of semi-Riemannian manifolds is a *harmonic morphism* if for any local harmonic function $v : V \to \mathbf{R}$ on N, the pullback $v \circ \Phi$ is harmonic on M, i.e. $\Delta_M(v \circ \Phi) = 0$ in $U = \Phi^{-1}(V)$ (cf. e.g. [13]). In the context of Example 2, we shall relate the subelliptic harmonic morphisms $\phi : \mathbf{H}_n \to N$ to harmonic morphisms from the Lorentzian manifold $(C(\mathbf{H}_n), F_{\theta_0})$. We need to recollect a few notions of CR and pseudohermitian geometry. A *CR structure* (of *CR dimension n*) on a C^∞ manifold M, of real dimension $2n+1$, is a complex rank n complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbf{C}$ of the complexified tangent bundle such that $T_{1,0}(M) \cap T_{0,1}(M) = (0)$, where $T_{0,1}(M) := \overline{T_{1,0}(M)}$ is the complex conjugate of $T_{1,0}(M)$, and $[Z, W] \in \Gamma^\infty(T_{1,0}(M))$, for any $Z, W \in \Gamma^\infty(T_{1,0}(M))$ (the *formal integrability* property). A pair $(M, T_{1,0}(M))$ is a *CR manifold* (of CR dimension n) and $H(M) := \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ (a real rank $2n$ distribution on M) its *Levi distribution*. The Levi distribution carries the complex structure J defined by $J(Z + \overline{Z}) = i(Z - \overline{Z})$, for any $Z \in T_{1,0}(M)$. If $(M, T_{1,0}(M))$ is an orientable CR manifold the conormal bundle $H(M)^\perp := \{\omega \in T^*(M) : Ker(\omega) \supseteq H(M)\}$ is a trivial line bundle, hence admits globally defined nowhere zero sections $\theta \in \Gamma^\infty(H(M)^\perp)$, each of which is a *pseudohermitian structure* on M (a term coined in [12]). The *Levi form* is $L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W})$, $Z, W \in T_{1,0}(M)$. Often the following real version of the Levi form is used. Set $G_\theta(X, Y) := (d\theta)(X, JY)$, $X, Y \in H(M)$; then L_θ and the $\mathbf{C}$-linear extension of G_θ coincide (on $T_{1,0}(M) \otimes T_{0,1}(M)$). The CR manifold M is *nondegenerate* (respectively *strictly pseudoconvex*) if L_θ is nondegenerate (respectively positive definite) for some θ. If M is nondegenerate then each pseudohermitian structure θ is a *contact form*, i.e. $\theta \wedge (d\theta)^n$ is a volume form on M. For a fixed contact form θ, there is a unique vector field T on M such that $T \rfloor d\theta = 0$ and $\theta(T) = 1$ (the *characteristic direction* of $d\theta$). A complex p-form η on a CR manifold $(M, T_{1,0}(M))$ is a form of type $(p, 0)$ if $T_{0,1}(M) \rfloor \eta = 0$. Let $\Lambda^{p,0}(M) \to M$ be the vector bundle of all forms of type $(p, 0)$ and set

$$C(M) = \Big(\Lambda^{n+1,0}(M) \setminus \{\text{zero section}\}\Big) \Big/ \mathbf{R}_+ ,$$

where $\mathbf{R}_+$ is the multiplicative group of the positive reals. When M is nondegenerate $C(M) \to M$ is a principal S^1-bundle and $C(M) \approx M \times S^1$ (the trivial bundle) when M is embeddable (i.e. CR isomorphic to a real hypersurface in $\mathbf{C}^{n+1}$, e.g. the boundary of a domain in $\mathbf{C}^{n+1}$). If M is strictly pseudoconvex, with each contact form θ (such that L_θ is positive definite) one may associate (cf. [10]) a Lorentzian metric F_θ on $C(M)$ (the *Fefferman metric* of (M, θ)) which can be computed in terms of the connection 1-forms and (pseudohermitian) scalar curvature of a canonical connection ∇ on M, known as the *Tanaka-Webster connection* of (M, θ). The Tanaka-Webster connection obeys to the axioms i) $H(M)$ is ∇-parallel, ii) $\nabla J = 0$, $\nabla g_\theta = 0$, and iii) the torsion of ∇ is pure (e.g. in sense of [2], p. 65) and its existence and uniqueness is actually guaranteed when M is merely nondegenerate (cf. [12] and [11]). Here g_θ is the Webster metric (cf. e.g. [2], p. 65) of (M,θ), i.e. $g_\theta = G_\theta$ on $H(M)\otimes H(M)$, $g_\theta(X,T)=0$ for any $X\in H(M)$, and $g_\theta(T,T)=1$. We only recall the construction of the Fefferman metric for the case of the Heisenberg group (for the general case of an arbitrary strictly pseudoconvex CR manifold, cf. [10]). As the Heisenberg group is embeddable (the map $f:\mathbf{H}_n\to\partial\Omega_{n+1}$, $f(z,t) := (z, t+i|z|^2)$, $(z,t)\in\mathbf{H}_n$, is a CR isomorphism of $\mathbf{H}_n$ onto the boundary of the Siegel domain $\Omega_{n+1} = \{(z, u+iv)\in\mathbf{C}^{n+1} : v > |z|^2\}$) the bundle $C(\mathbf{H}_n)\to\mathbf{H}_n$ is of course trivial. $\theta_0 = dt + i\sum_{\alpha=1}^n (z^\alpha d\overline{z}^\alpha - \overline{z}^\alpha dz^\alpha)$ is a contact form on $\mathbf{H}_n$ (with the corresponding Levi form positive definite). An element $[\omega]\in C(\mathbf{H}_n)$ is a class of a $(n+1,0)$-form $\omega = \lambda(\theta_0\wedge\theta^1\wedge\cdots\wedge\theta^n)_x$, for some $\lambda\in\mathbf{C}^* = \mathbf{C}\setminus\{0\}$ and $x\in\mathbf{H}_n$. Here $\theta^\alpha = dz^\alpha$. We shall use the local fibre coordinate γ on $C(\mathbf{H}_n)$ given by

$$\gamma([\omega]) := \arg(\lambda/|\lambda|),$$

where $\arg : S^1\to[0,2\pi)$. Let us extend the Levi form G_{θ_0} to the whole of $T(\mathbf{H}_n)$ by requesting that $G_{\theta_0}(V,T)=0$, for every V (the characteristic direction of $d\theta_0$ is $T=\partial/\partial t$). Consider the (globally defined) 1-form $\sigma = (1/(n+2))d\gamma$ on $C(\mathbf{H}_n)$ and set

$$F_{\theta_0} = \pi^* G_{\theta_0} + 2(\pi^*\theta_0)\odot\sigma\,,$$

where $\odot$ denotes the symmetric tensor product. Then F_{θ_0} is a Lorentz metric on $C(\mathbf{H}_n)$ (the Fefferman metric of $(\mathbf{H}_n,\theta_0)$, cf. [10]).

Proof of Theorem 2. With respect to the local coordinates $(u^a) = (u^A,\gamma)$, where $u^A = x^A\circ\pi$, the Fefferman metric may be written as

$$\begin{aligned} F_{\theta_0} &= 2\sum_{\alpha=1}^n \left[(du^\alpha)^2 + (du^{\alpha+n})^2\right] \\ &\quad + \frac{2}{n+2}\left[du^{2n+1} + 2\sum_{\alpha=1}^n (u^\alpha du^{\alpha+n} - u^{\alpha+n}du^\alpha)\right]\odot du^{2n+2}\,. \end{aligned} \tag{11}$$

We wish to compute the Laplace-Beltrami operator

$$\Box f = \frac{1}{\sqrt{|F|}}\frac{\partial}{\partial u^a}\left(\sqrt{|F|}F^{ab}_{\theta_0}\frac{\partial f}{\partial u^b}\right)\,,\ f\in C^2(C(\mathbf{H}_n))\,.$$

A calculation shows that

$$F := \det[(F_{\theta_0})_{ab}] = -\left(\frac{2^n}{n+2}\right)^2\,, \tag{12}$$

$$F^{ab}_{\theta_0} : \begin{pmatrix} 1/2 & \cdots & 0 & 0 & \cdots & 0 & u^{n+1} & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 1/2 & 0 & \cdots & 0 & u^{2n} & 0 \\ 0 & \cdots & 0 & 1/2 & \cdots & 0 & -u^1 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1/2 & -u^n & 0 \\ u^{n+1} & \cdots & u^{2n} & -u^1 & \cdots & -u^n & 2|z|^2 \circ \pi & n+2 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & n+2 & 0 \end{pmatrix}, \tag{13}$$

hence

$$\begin{aligned} \Box f = & \frac{1}{2}\sum_{\alpha=1}^{n}\left[\frac{\partial^2 f}{\partial(u^\alpha)^2} + \frac{\partial^2 f}{\partial(u^{\alpha+n})^2}\right] + 2(n+2)\frac{\partial^2 f}{\partial u^{2n+1}\partial u^{2n+2}} \\ & + 2u^{\alpha+n}\frac{\partial^2 f}{\partial u^\alpha \partial u^{2n+1}} - 2u^\alpha \frac{\partial^2 f}{\partial u^{\alpha+n}\partial u^{2n+1}} + 2(|z|\circ\pi)^2 \frac{\partial^2 f}{\partial (u^{2n+1})^2} . \end{aligned} \tag{14}$$

Let $c \in \pi^{-1}(0) \subset C(\mathbf{H}_n)$. $[F^{ab}_{\theta_0}(c)]$ has spectrum $\{1/2,\ n+2,\ -n-2\}$ (with multiplicities $\{2n,\ 1,\ 1\}$, respectively). The corresponding eigenspaces are

$$Eigen(1/2) = \sum_{j=1}^{2n} \mathbf{R} e_j\ ,\ Eigen(\pm(n+2)) = \mathbf{R}(0, \cdots, 0, 1, \pm 1)\,,$$

(where $\{e_1, \cdots, e_{2n+2}\} \subset \mathbf{R}^{2n+2}$ is the canonical linear basis). Consequently, under the coordinate transformation

$$\begin{cases} w^j = \sqrt{2}\, u^j\,, & 1 \le j \le 2n \\ w^{2n+1} = \dfrac{1}{\sqrt{2(n+2)}}\left(u^{2n+1} + \gamma\right) \\ w^{2n+2} = \dfrac{1}{\sqrt{2(n+2)}}\left(u^{2n+1} - \gamma\right) \end{cases}$$

(14) goes over to the canonical hyperbolic form

$$(\Box f)(c) = \sum_{A=1}^{2n+1} \frac{\partial^2 f}{\partial (w^A)^2}(c) - \frac{\partial^2 f}{\partial (w^{2n+2})^2}(c)\,.$$

The unit circle S^1 acts freely on $C(\mathbf{H}_n)$ by $R_w([\omega]) = [\omega]\cdot w := [w\omega]$, $w \in S^1$. Then $u^A \circ R_w = u^A$ and $\gamma \circ R_w = \gamma + \arg(w) + 2k\pi$, for some $k \in \mathbf{Z}$, hence $R_w^* F_{\theta_0} = F_{\theta_0}$, i.e. $S^1 \subset Isom(C(\mathbf{H}_n), F_{\theta_0})$. As well known, this yields $\Box^{R_w} = \Box$, where we set $\Box^\psi f := (\Box f^{\psi^{-1}})^\psi$ and $f^{\psi^{-1}} := f \circ \psi$, for any diffeomorphism ψ of $C(\mathbf{H}_n)$ in itself (we adopt the conventions in [6], p. 241). Therefore,

$$\pi_* \Box : C^\infty(\mathbf{H}_n) \to C^\infty(\mathbf{H}_n),\quad (\pi_*\Box)u := (\Box(u\circ\pi))\tilde{}\,,$$

is well defined, where, for a given S^1 invariant function f on $C(\mathbf{H}_n)$, $\tilde{f}$ denotes the corresponing base map. Finally, a calculation based on (6) and (14) leads to

$$(\pi_* \square) u = -2\, H u, \; u \in C^2(\mathbf{H}_n)\,. \tag{15}$$

At this point, Theorem 2 is proved. For given a local harmonic function $v : V \to \mathbf{R}$ on N and $\phi : \mathbf{H}_n \to N$ a subelliptic harmonic morphism then (by (15)) $0 = -2H(v \circ \phi) = (\pi_* \square)(v \circ \phi)$ hence $\square(v \circ \Phi) = 0$, i.e. $\Phi = \phi \circ \pi$ is a harmonic morphism.

EXAMPLE 1 (continued). Theorem 2 applies, with only minor modifications, to subelliptic harmonic morphisms from $\Omega = \mathbf{R}^3 \setminus \{x^1 = 0\}$, with respect to the Hörmander system (5). $\mathbf{R}^3$ is a CR manifold with the CR structure $\mathcal{H}(k)$ spanned by

$$Z := 2\,\frac{\partial}{\partial z} - i \left(\frac{z + \overline{z}}{2}\right)^k \frac{\partial}{\partial t}\,,$$

where $z = x^1 + i x^2$ and $t = x^3$. Next

$$\theta(k) = dt + \frac{i}{2}\left(\frac{z + \overline{z}}{2}\right)^k (dz - d\overline{z})$$

is a pseudohermitian structure on $\mathbf{R}^3$ with the Levi form

$$L_{\theta(k)}(Z, \overline{Z}) = -k \left(\frac{z + \overline{z}}{2}\right)^{k-1}\,,$$

hence $(\mathbf{R}^3, \mathcal{H}(0))$ is Levi flat while $(\Omega, \mathcal{H}(k))$ is nondegenerate, for any $k \geq 1$. Moreover, if $k \geq 1$ each connected component $\Omega^+ = \{x^1 > 0\}$ and $\Omega^- = \{x^1 < 0\}$ is strictly pseudoconvex. If ∇ is the Tanaka-Webster connection of a nondegenerate CR manifold M (on which a contact form θ has been fixed) and $\{T_\alpha : 1 \leq \alpha \leq n\}$ is a (local) frame in $T_{1,0}(M)$ then we set $\nabla_{T_A} T_B = \Gamma^C_{AB} T_C$, where $A, B, \cdots \in \{0, 1, \cdots, n, \overline{1}, \cdots, \overline{n}\}$. Also $T_{\overline{\alpha}} := \overline{T_\alpha}$ and $T_0 := T$. Let T_∇ be the torsion tensor field of ∇. Then $T_\nabla(T, T_\alpha) = A^{\overline{\beta}}_{\alpha} T_{\overline{\beta}}$ is the *pseudohermitian torsion* (cf. also [4] for the properties of T_∇). Let R^∇ be the curvature tensor field of ∇ and set $R^\nabla(T_B, T_C) T_A = R_A{}^D{}_{BC} T_D$. The (*pseudohermitian*) *Ricci tensor* is $R_{\lambda\overline{\mu}} = R_\lambda{}^\alpha{}_{\alpha\overline{\mu}}$ and the (*pseudohermitian*) *scalar curvature* is $R = R_\lambda{}^\lambda$ (one uses the local coefficients of the Levi form $g_{\alpha\overline{\beta}} = L_\theta(T_\alpha, T_{\overline{\beta}})$ and their inverse $[g^{\alpha\overline{\beta}}] := [g_{\alpha\overline{\beta}}]^{-1}$ to raise and lower indices). The Tanaka-Webster connection of (Ω, θ_k) is given by

$$\Gamma^{\overline{1}}_{1\overline{1}} = 0,\; \Gamma^1_{11} = \frac{2(k-1)}{z + \overline{z}}\,,\; \Gamma^1_{01} = 0\,.$$

In particular, the Tanaka-Webster connection of $(\Omega, \theta(k))$ has pseudohermitian scalar curvature $R = -\frac{k-1}{k}\,(2/(z + \overline{z}))^{k+1}$, and one may explicitely compute the Fefferman metric $F_{\theta(k)}$ of $(\Omega^{\pm}, \theta(k))$. Also the pseudohermitian torsion vanishes (i.e. $A_1^{\overline{1}} = 0$). Note that $(\Omega, \theta(1))$ is Webster flat. Set $\nabla^H u = \pi_{\mathcal{D}} \nabla u$, where ∇u is the gradient of $u \in C^\infty(\Omega)$ with respect to the Webster metric and $\pi_H : T(\Omega) \to \mathcal{D}$ the projection with respect to the direct sum decomposition $T(\Omega) = \mathcal{D} \oplus \mathbf{R}\partial/\partial t$. That is $\nabla^H u = u^1 Z + u^{\overline{1}} \overline{Z}$, where $u^1 = h^{1\overline{1}} u_1$ and $u_1 = Z(u)$. The sublaplacian is $\Delta_b u = -div(\nabla^H u)$, where the divergence is taken with respect to the volume form $\theta(k) \wedge (d\theta(k))^n$. As

$$div(Z) = \frac{k-1}{x_1}\ , \ h^{1\overline{1}} = -\frac{1}{k} x_1^{1-k} \ \ (x_1 = x^1)$$

one has $\Delta_b = \frac{1}{k} x_1^{1-k} (Z\overline{Z} + \overline{Z}Z)$, hence (by (4)) $\Delta_b = -\frac{1}{k} x_1^{1-k} H$ on C^∞ functions. By a result in [10], the Laplace-Beltrami operator $\square$ of the Fefferman metric of $(\Omega^{\pm}, \theta(k))$ is related to the sublaplacian by $\pi_* \square = \frac{1}{2}\Delta_b$ hence

$$(\pi_* \square) u = -\frac{1}{k} x_1^{1-k} H u, \ \ u \in C^\infty(\Omega^{\pm})\,. \tag{16}$$

Consequently

PROPOSITION 1. *For any subelliptic harmonic morphism $\phi : \Omega^{\pm} \to N$, with respect to the Hörmander system* (4)*, the map $\Phi = \phi \circ \pi : C(\Omega^{\pm}) \to N$ is a harmonic morphism, with respect to the Fefferman metric of* $(\Omega^{\pm}, \theta(k))$.

For the converse, cf. our Section 5.

5. – Weak harmonic maps from $C(\mathbf{H}_n)$

From now on, we assume that N is covered by one coordinate chart $\varphi = (y^1, \cdots, y^\nu) : N \to \mathbf{R}^\nu$ and $\Phi^i := y^i \circ \Phi$. A map $\Phi : C(\mathbf{H}_n) \to N$ satisfies *weakly* the harmonic map system

$$\square \Phi^i + F_{\theta_0}^{ab} \left(\left| \begin{matrix} i \\ jk \end{matrix} \right| \circ \pi \right) \frac{\partial \Phi^j}{\partial u^a} \frac{\partial \Phi^k}{\partial u^b} = 0, \ \ 1 \le i \le \nu\ , \tag{17}$$

if Φ^i and their first derivatives (in distributional sense) are square integrable and

$$\sum_{i=1}^{\nu} \left\{ \int_{C(\mathbf{H}_n)} \Phi^i\, \square \varphi^i\, d\,vol(F_{\theta_0}) + \int_{C(\mathbf{H}_n)} F_{\theta_0}^{ab} \left(\left| \begin{matrix} i \\ jk \end{matrix} \right| \circ \pi \right) \frac{\partial \Phi^j}{\partial u^a} \frac{\partial \Phi^k}{\partial u^b}\, \varphi^i\, d\,vol(F_{\theta_0}) \right\} = 0\,,$$

for any $\varphi \in C_0^\infty(C(\mathbf{H}_n), \mathbf{R}^\nu)$. Given a smooth vector field X on $\mathcal{U} \subseteq C(\mathbf{H}_n)$ open, the function $g_X^i \in L^2(\mathcal{U})$ defined a.e. by

$$\int_{\mathcal{U}} g_X^i \varphi \, d\, vol(F_{\theta_0}) = \int_{\mathcal{U}} \Phi^i \, X^* \varphi \, d\, vol(F_{\theta_0}),$$

is denoted by $X(\Phi^i)$. Here X^* is the formal adjoint of X with respect to the L^2-inner product $(u, v)_{L^2} = \int uv \, dvol(F_{\theta_0})$, for $u, v \in C^\infty(C(\mathbf{H}_n))$, at least one of compact support. In [1], given a strictly pseudoconvex CR manifold M, one related smooth harmonic maps from $C(M)$ (with the Fefferman metric corresponding to a fixed choice of contact form on M) to smooth pseudoharmonic maps from M (as argued there, these are locally J. Jost & C-J. Xu's subelliptic harmonic maps). Here we wish to attack the same problem for weak solutions (of the harmonic, respectively subelliptic harmonic, map equations). We recall the Sobolev space $W_X^{1,2}(\Omega) = \{u \in L^2(\Omega) : X_a u \in L^2(\Omega), \ 1 \le a \le m\}$, adapted to a system of vector fields $X = \{X_1, \cdots, X_m\}$ on $\Omega \subseteq \mathbf{R}^n$ (the $X_a u$'s are understood in distributional sense). Then $\phi : \Omega \to N$ is a weak solution to (1) if $\phi^i \in W_X^{1,2}(\Omega)$ and

$$\sum_{i=1}^{\nu} \left\{ \int_\Omega \sum_{a=1}^{m} (X_a \phi^i) \, (X_a \varphi^i) \, dx - \int_\Omega \sum_{a=1}^{m} \left(\left| \begin{matrix} i \\ jk \end{matrix} \right| \circ \phi \right) \, (X_a \phi^j)(X_a \phi^k) \varphi^i \, dx \right\} = 0 \, ,$$

for any $\varphi \in C_0^\infty(\Omega, \mathbf{R}^\nu)$. Cf. e.g. [9], p. 4641. We wish to show

LEMMA 2. *Let $\Phi : C(\mathbf{H}_n) \to N$ be a S^1-invariant map and $\phi = \tilde{\Phi}$ the corresponding base map. If $\Phi^i \in L^2(C(\mathbf{H}_n))$ and $Y(\Phi^i) \in L^2(\mathcal{U})$, for any smooth vector field Y on $\mathcal{U} \subseteq C(\mathbf{H}_n)$, then $\phi^i \in W_X^{1,2}(\mathbf{H}_n)$.*

One has $\|\Phi^i\|_{L^2(C(\mathbf{H}_n))} = 2\pi \|\phi^i\|_{L^2(\mathbf{H}_n)}$, hence $\phi^i \in L^2(\mathbf{H}^n)$. In particular $\phi^i \in L^1_{loc}(\mathbf{H}_n)$ and $\Phi^i \in L^1_{loc}(C(\mathbf{H}_n))$. Let $\varphi \in C_0^\infty(\mathbf{H}_n)$. Then $\varphi \circ \pi \in C_0^\infty(C(\mathbf{H}_n))$ (because S^1 is compact) and

$$\begin{aligned}
&\int_{C(\mathbf{H}_n)} \Phi^i \square(\varphi \circ \pi) \, dvol(F_{\theta_0}) = \qquad \text{(by (15))} \\
&= -2 \int_{C(\mathbf{H}_n)} (\phi^i \, H\varphi) \circ \pi \, dvol(F_{\theta_0}) = -4\pi \int_{\mathbf{H}_n} \phi^i \, H\varphi \, \theta_0 \wedge (d\theta_0)^n \\
&= -4\pi \int_{\mathbf{H}_n} \sum_{a=1}^{2n} (X_a \phi^i)(X_a \varphi) \, \theta_0 \wedge (d\theta_0)^n \, .
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_{\mathbf{H}_n} \phi^i (X_\alpha^* \varphi) dx = - \int_{\mathbf{H}_n} \phi^i (X_\alpha \varphi) dx = -\frac{1}{2\pi} \int_{C(\mathbf{H}_n)} \Phi^i (X_\alpha \varphi) \circ \pi \, dx d\gamma \\
&= -\frac{1}{2\pi} \int \Phi^i \hat{X}_\alpha (\varphi \circ \pi) \, dx d\gamma \\
&= (\text{as } (\partial/\partial u^a)^* = -\partial/\partial u^a) \\
&= \frac{1}{2\pi} \int \Phi^i \, \left(\hat{X}_a \right)^* (\varphi \circ \pi) \, dx d\gamma = \frac{1}{2\pi} \int \left(\hat{X}_a \Phi^i \right) \, (\varphi \circ \pi) \, dx d\gamma \, .
\end{aligned}$$

The notation dx (respectively $dx\, d\gamma$) is short for $\theta_0 \wedge (d\theta_0)^n$ (respectively, for $d\, vol(F_{\theta_0})$). Also, we set

$$\hat{X}_\alpha := \frac{1}{2}\frac{\partial}{\partial u^\alpha} + u^{\alpha+n}\frac{\partial}{\partial u^{2n+1}}\ ,\ \hat{X}_{\alpha+n} = \frac{1}{2}\frac{\partial}{\partial u^{\alpha+n}} - u^\alpha \frac{\partial}{\partial u^{2n+1}}\ .$$

The Jacobian of the right translation R_w with $w \in S^1$ is the unit matrix, hence for any $\psi \in C_0^\infty(C(\mathbf{H}_n))$

$$\int \left(\hat{X}_a \Phi^i\right) \psi dx d\gamma = -\int \Phi^i \hat{X}_a \psi dx d\gamma = -\int (\Phi^i \circ R_w)(\hat{X}_a \psi) \circ R_w dx d\gamma =$$

(as $\hat{X}_a$ is right-invariant)

$$= -\int \left(\Phi^i \circ R_w\right)(x,\gamma)\ \left((d_{(x,\gamma)} R_w)\hat{X}_a\right)(\psi)(x,\gamma) =$$

(as Φ^i is S^1-invariant)

$$= -\int \Phi^i\ \hat{X}_a(\psi \circ R_w) dx d\gamma = \int (\hat{X}_a \Phi^i)\,(\psi \circ R_w)\, dx d\gamma$$

$$= \int \left((\hat{X}_a \Phi^i) \circ R_{w^{-1}}\right)\ \psi\, dx\, d\gamma\ ,$$

hence $\hat{X}_a \Phi^i = (\hat{X}_a \Phi^i) \circ R_{w^{-1}}$, i.e. there is an element of $L^2(\mathbf{H}_n)$, which we denote by $X_a \phi^i$, such that

$$\hat{X}_a \Phi^i = (X_a \phi^i) \circ \pi\ .$$

We may conclude (by Fubini's theorem) that

$$\int \phi^i\ X_a^* \varphi\, dx = \int (X_a \phi^i)\, \varphi\, dx\ ,$$

i.e. $X_a \phi^i$ is indeed the weak derivative of ϕ^i. The Lemma 2 is proved. At this point we may establish the following

THEOREM 3. *Let $\phi : \mathbf{H}_n \to N$ be a map such that $\Phi := \phi \circ \pi$ satisfies weakly the harmonic map system* (17)*. Then ϕ is a weak solution to the subelliptic harmonic map system* (1)*.*

Combining the regularity results in [9] and [14] with Theorem 3 we obtain the following

COROLLARY 1. *Let N be a Riemannian manifold of sectional curvature $\leq \kappa^2$, for some $\kappa > 0$. Let $\Phi : C(\mathbf{H}_n) \to N$ be a bounded S^1-invariant weak solution to the harmonic map equation* (17) *such that $\Phi(C(\mathbf{H}_n))$ is contained in a regular*(2) *ball of N. Then Φ is smooth.*

(2)That is a ball $B(p,\nu) = \{q \in N : d_N(q,p) \leq \nu\}$ such that $\nu < \min\{\pi/(2\kappa)\, ,\, i(p)\}$, where $i(p)$ is the injectivity radius of p (cf. [9], p. 4644).

PROOF OF THEOREM 3. The statement follows from the preceding calculations and the identity

$$\left(\left|\begin{matrix} i \\ jk \end{matrix}\right| \circ \Phi\right) \frac{\partial \Phi^j}{\partial u^a} \frac{\partial \Phi^k}{\partial u^b} F^{ab}_{\theta_0} = \left(\left|\begin{matrix} i \\ jk \end{matrix}\right| \circ \Phi\right) \left\{ 2 \sum_{a=1}^{2n} (\hat{X}_a \Phi^j)(\hat{X}_a \Phi^k) \right.$$
$$\left. + (n+2) \left(\frac{\partial \Phi^j}{\partial \gamma} \frac{\partial \Phi^k}{\partial u^{2n+1}} + \frac{\partial \Phi^j}{\partial u^{2n+1}} \frac{\partial \Phi^k}{\partial \gamma} \right) \right\}$$

(itself a consequence of (13)) provided we show that $\partial \Phi^j / \partial \gamma = 0$, as a distribution. Indeed, given $\varphi \in C_0^\infty(C(\mathbf{H}_n))$ set $C := \sup_{C(\mathbf{H}_n)} |\partial \varphi / \partial \gamma|$, $\Gamma := supp(\varphi)$ and $\Gamma_H = \pi(\Gamma)$. Then, given $\phi^j_\nu \in C_0^\infty(\mathbf{H}_n)$ such that $\phi^j = L^2\text{-}\lim_{\nu \to \infty} \phi^j_\nu$,

$$\left| \int_{C(\mathbf{H}_n)} (\phi^j_\nu \circ \pi) \frac{\partial \varphi}{\partial \gamma} dvol(F_{\theta_0}) - \int_{C(\mathbf{H}_n)} (\phi^j \circ \pi) \frac{\partial \varphi}{\partial \gamma} dvol(F_{\theta_0}) \right|$$
$$\leq 2\pi \, C \, Vol(\Gamma_H)^{1/2} \, \| \phi^j_\nu - \phi^j \|_{L^2(\mathbf{H}_n)}$$

hence (as (12) implies $(\partial / \partial \gamma)^* = -\partial / \partial \gamma$)

$$\frac{\partial \Phi^j}{\partial \gamma}(\varphi) = - \int_{C(\mathbf{H}_n)} (\phi^j \circ \pi) \frac{\partial \varphi}{\partial \gamma} dvol(F_{\theta_0})$$
$$= - \lim_{\nu \to \infty} \int_{C(\mathbf{H}_n)} (\phi^j_\nu \circ \pi) \frac{\partial \varphi}{\partial \gamma} dvol(F_{\theta_0}) = 0,$$

by Green's lemma and $div(\partial / \partial \gamma) = 0$, again as a consequence of (12).

EXAMPLE 1 (continued). As shown in Section 3, as a consequence of the hypoellipticity of the Hörmander operator together with the existence of local harmonic coordinates on the target manifold, there is no notion of *weak* subelliptic harmonic morphism (of course, this is true for harmonic morphisms between Riemannian manifolds, as well). In the context of the Hörmander system (4) we say a localizable map $\Phi : (C(\Omega^\pm), F_{\theta(k)}) \to (N, h)$ is a *weak harmonic morphism* if, for any local harmonic function $v : V \to \mathbf{R}$ on N one has $v \circ \Phi \in L^1_{loc}(\mathcal{U})$, for any $\mathcal{U} \subseteq C(\Omega^\pm)$ open such that $\Phi(\mathcal{U}) \subset V$, and $\Box(v \circ \Phi) = 0$ in distributional sense. Then we may prove the following regularity result (and converse of Proposition 1).

PROPOSITION 2. *If $\Phi : C(\Omega^\pm) \to N$ is S^1-invariant weak harmonic morphism then the base map $\phi = \tilde{\Phi} : \Omega^\pm \to N$ is a smooth subelliptic harmonic morphism* (*in particular, Φ is smooth*).

Let $\varphi \in C_0^\infty(\Omega^\pm)$. Then

$$0 = \Box(v \circ \Phi)(\varphi \circ \pi) = \int_{C(\Omega^\pm)} (v \circ \Phi) \Box(\varphi \circ \pi) \, dvol(F_{\theta(k)}) =$$

(by (16) and Fubini's theorem)

$$= -\frac{2\pi}{k} \int_{\Omega^\pm} (v \circ \phi)(x) x_1^{1-k} (H\varphi)(x) \, dx = -\frac{2\pi}{k} H(x_1^{1-k} \, v \circ \phi)(\varphi),$$

i.e. $H(x_1^{1-k}\ v \circ \phi) = 0$ in distributional sense [here dx is short for $\theta(k) \wedge (d\theta(k))^n$]. Hence there is $f \in C^\infty(U)$ such that $v \circ \phi = x_1^{k-1} f$, i.e. $v \circ \phi$ is smooth (here $U \subseteq \Omega^\pm$ is any open set such that $\phi(U) \subset V$). Then, again by (16), $H(v \circ \phi) = 0$. Q.e.d.

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