Boundedness of Global Solutions
for Nonlinear Parabolic Equations
Involving Gradient Blow-up Phenomena

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Abstract. We consider a one-dimensional semilinear parabolic equation with a
gradient nonlinearity. We provide a complete classification of large time behavior
of the classical solutions $u$: either the space derivative $u_x$ blows up in finite time
(with $u$ itself remaining bounded), or $u$ is global and converges in $C^1$ norm to the
unique steady state.
The main difficulty is to prove $C^1$ boundedness of all global solutions. To do so, we
explicitly compute a nontrivial Lyapunov functional by carrying out the method of
Zelenyak. After deriving precise estimates on the solutions and on the Lyapunov
functional, we proceed by contradiction by showing that any $C^1$ unbounded global
solution should converge to a singular stationary solution, which does not exist.
As a consequence of our results, we exhibit the following interesting situation:
– the trajectories starting from some bounded set of initial data in $C^1$ describe an
unbounded set, although each of them is individually bounded and converges to
the same limit;
– the existence time $T^*$ is not a continuous function of the initial data.

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1. – Introduction and main results

We consider the problem

$$
\begin{cases}
  u_t = u_{xx} + |u_x|^p, & t > 0, \quad 0 < x < 1, \\
  u(t, 0) = 0, \quad u(t, 1) = M, & t > 0, \\
  u(0, x) = u_0(x), & 0 < x < 1.
\end{cases}
$$

(1.1)

Here $p > 2$, $M \geq 0$ and $u_0 \in X$, where $X = \{v \in C^1([0, 1]); \ v(0) = 0, \ v(1) = M\}$, endowed with the $C^1$ norm. The problem (1.1) admits a unique maximal

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classical solution \( u = u(u_0; t, .) \), whose existence time will be denoted by 
\( T^* = T^*(u_0) \in (0, \infty] \). Note that we make no restriction on the signs of \( u \) or \( u_x \).

The equation (1.1) possesses both mathematical and physical interest. It can serve as a typical model-case in the theory of parabolic PDEs. Indeed, it is the simplest example of a parabolic equation with a nonlinearity depending on the first order spatial derivatives of \( u \), and it can be considered as an analogue of the extensively studied equation with zero order nonlinearity \( u_t - u_{xx} = u^p \).

On the other hand, the equation (1.1) (and its \( N \) dimensional version) arises in the viscosity approximation of Hamilton-Jacobi type equations from stochastic control theory [21] and in some physical models of surface growth [18]. A rather up-to-date list of references can be found in [6] (especially for the Cauchy problem, which has been more studied). For studies concerning the Dirichlet and Neumann problems, we also refer to [2], [9], [1], [28], [4], [5], [10].

The aim of this paper is to provide a complete classification of large time behavior of the solutions of (1.1). A basic fact about (1.1) is that the solutions satisfy a maximum principle:

\[
\min_{[0,1]} u_0 \leq u(t, x) \leq \max_{[0,1]} u_0, \quad 0 \leq t < T^*, \quad 0 \leq x \leq 1.
\]

Since (1.1) is well-posed in \( C^1 \), therefore only three possibilities can occur:

(I) \( u \) exists globally and is bounded in \( C^1 \):
\[
T^* = \infty \quad \text{and} \quad \sup_{t \geq 0} |u_x(t, .)|_\infty < \infty.
\]

Moreover, due to the results in [30] (see the last part of this Introduction for more details), \( u \) has to converge in \( C^1 \) to a steady state (which is actually unique when it exists);

(II) \( u \) blows up in finite time in \( C^1 \) norm (finite time gradient blow-up):
\[
T^* < \infty \quad \text{and} \quad \lim_{t \to T^*} |u_x(t, .)|_\infty = \infty;
\]

(III) \( u \) exists globally but is unbounded in \( C^1 \) (infinite time gradient blow-up):
\[
T^* = \infty \quad \text{and} \quad \limsup_{t \to \infty} |u_x(t, .)|_\infty = \infty.
\]

For simplicity, let us first consider the case \( M = 0 \). When \( |u_0|_{C^1} \) is sufficiently small, then it is known that (I) occurs (and \( u \) converges to the unique steady state \( V_0 \equiv 0 \)). If, on the contrary, \( u_0 \) is suitably large, then (II) occurs (see [28]).

Our primary goal is therefore to rule out (III), that is, infinite time gradient blow-up.
For $M > 0$, the situation is slightly more involved. There exists a critical value

$$M_c = \frac{(p - 1)^{p-2}}{p - 2}$$

such that (1.1) has a unique (regular) steady state $V_M$ if $M < M_c$ and no steady state if $M > M_c$ (the explicit formula for $V_M$ is recalled at the beginning of Section 2). In the critical case $M = M_c$, there still exists a steady state $V_{M_c}$, but it is singular, satisfying $V_{M_c} \in C([0, 1]) \cap C^2((0, 1])$ with $V_{M_c, x}(0) = \infty$. Restricting to initial data $u_0$ such that $u_0$ and $u_{0,x} \geq 0$, it was shown among other things in [2] that:

(i) if $M > M_c$, then all solutions of (1.1) satisfy (II);
(ii) if $0 < M < M_c$, then both (I) and (II) occur. However the possibility of (III) remained as an open problem.

Therefore, our main interest lies in the case $M < M_c$. Our main result is the following.

**Theorem 1.** Assume $0 \leq M < M_c$. Then all global solutions of (1.1) are bounded in $C^1$. In other words, (III) cannot occur. Moreover, they converge in $C^1$ norm to $V_M$.

For the case $M > M_c$, we improve the result of [2] by removing the restrictions $u_0 \geq 0$ and $u_{0,x} \geq 0$ on the initial data.

**Proposition 2.** Assume $M > M_c$. Then all solutions of (1.1) blow up in finite time in $C^1$ norm.

**Remarks 1.** (a) In the critical case $M = M_c$, all solutions have to blow up in $C^1$ in either finite or infinite time. Moreover, if (III) occurs, then the solution will converge in $C([0, 1])$ to the singular steady state $V_{M_c}$, as $t \to \infty$. This follows from Proposition 3.2 below. However, the possibility of (III) remains an open problem in this case. We conjecture that this could occur$^{(1)}$.

(b) For $0 < p \leq 2$, the situation is quite different. It is known from the local theory (cf. [19, 20]) that a bound on $u$ implies a bound on $u_x$. Therefore all solutions are global and bounded in $C^1$, and they converge to the unique steady state.

(c) For results on the asymptotic behavior of solutions of (1.1) with finite time gradient blow-up, we refer to [9] (see also [3], [13] for variants of (1.1)).

As a consequence of our results, we exhibit the following interesting situation: although $C^1$ boundedness of global solutions is true, the global solutions of (1.1) do not satisfy a *uniform a priori estimate*, i.e., the supremum in (I) cannot be estimated in terms of the norm of the initial data. In other words, there exists a bounded, even compact, subset $S \subset X$, such that the trajectories

$^{(1)}$After the completion of this paper, a positive answer to this conjecture was given to us by Juan-Luis Vázquez (personal communication).
starting from \( S \) describe an unbounded subset of \( X \), although each of them is individually bounded and converges to the same limit. As a further consequence, the existence time \( T^* \), defined as a function from \( X \) into \((0, \infty]\), is not (upper semi-)continuous.

**Proposition 3.** Assume \( 0 \leq M < M_c \). There exists \( u_0 \in X \) and a sequence \((u_{0,n})\) in \( X \) with the following properties.

(i) \( u_{0,n} \to u_0 \) in \( C^1 \);
(ii) \( T^*(u_{0,n}) = \infty \) for each \( n \), and \( T^*(u_0) < \infty \);
(iii) \( \sup_{t \geq 0} |u_{n,x}(t, \cdot)|_\infty =: K_n \to \infty \).

**Remarks 2.** (a) The questions of boundedness and a priori estimates of global solutions have been intensively studied for the zero order reaction-diffusion equation

\[(1.3) \quad u_t = u_{xx} + |u|^{p-1} u, \]

with \( p > 1 \) and homogeneous Dirichlet boundary conditions. The situation is different from ours: all global solutions are not only bounded in \( L^\infty \) [7], but they also satisfy a uniform a priori estimate depending only on \( |u_0|_\infty \) [17], [23]. Moreover, the function \( T^* : X = C_0(0, 1) \to (0, \infty] \) is continuous (see [25] and the references therein). Recently, for global positive solutions, it was even shown [15], [24], [26] that at any given \( t > 0 \), the bound is actually universal, i.e. independent of the solution. These results remain true for the \( n \) dimensional version of (1.3) for subcritical \( p \) (even in one space dimension), this equation possesses some global solutions which become unbounded in \( L^\infty \), as \( t \to \infty \). In some cases, it is even known [11] that these solutions coexist with global bounded and finite time blow-up solutions.

(b) Set \( G = \{ u_0 \in X ; \ T^*(u_0) = \infty \} \) and \( B = \{ u_0 \in X ; \ T^*(u_0) < \infty \} \). Then, we have \( G \cup B = X \) for both (1.1) and (1.3) (with e.g. \( X = C_0(0, 1) \) in the latter case). However, \( G \) is closed for (1.3) (cf. [7]), while \( G \) is open for (1.1) (this follows easily from the proof of Proposition 3).

**Remarks 3.** (a) Let us mention some results related to ours, regarding other parabolic equations with gradient terms. The semilinear equation \( u_t - \Delta u = u^p - \lambda |\nabla u|^q \) (\( p, q > 1, \lambda > 0 \)) has been studied by several authors (see [27] for a recent survey). In particular, it was proved in [11], [29] that for certain \( p \) (even in one space dimension), this equation possesses some global solutions which become unbounded in \( L^\infty \), as \( t \to \infty \). In some cases, it is even known [11] that these solutions coexist with global bounded and finite time blow-up solutions. For quasilinear equations involving mean curvature type operators, like \( u_t = ((1 + u_x^2)^{-1/2} u_x)_x + \lambda u \) (\( \lambda > 0 \)), a phenomenon of infinite time gradient blow-up (with \( u \) remaining bounded) has been exhibited in [8], thus showing a situation opposite to ours. For the convective-reactive problem
The question of boundedness of global solutions is still open, but partial results in this direction may be found in [14].

(b) For singular problems of the form $u_t - u_{xx} = \lambda (1-u)^{-k}$ ($k > 1, \lambda > 0$), it has been proved in [12], for increasing solutions, that quenching, i.e. blow-up of $u_t$ (when $u$ reaches 1), can occur only in finite time.

To explain the ideas of our proof, let us first recall that in a classical paper [30], Zelenyak showed that any one-dimensional quasilinear uniformly parabolic equation possesses a (strict) Lyapunov functional, of the form

$$
\mathcal{L}(u(t)) = \int_0^1 \phi(u(t, x), u_x(t, x)) \, dx.
$$

The construction of $\phi$ is in principle explicit, although too complicated to be completely computed in most situations. As a consequence, for any solution $u$ of (1.1) which is global and bounded in $C^1$, the (non-empty) $\omega$–limit set of $u$ consists of equilibria. Since (1.1) admits at most one equilibrium $V$, such $u$ has to converge to $V$. (In fact, it was also proved in [30] that whether or not equilibria are unique, any bounded solution of a one-dimensional uniformly parabolic equation converges to an equilibrium; but this need not concern us here.) For $M > 0$, our proof proceeds by contradiction and makes essential use of the Zelenyak construction (for the case $M = 0$, the proof is actually simpler and does not require the Lyapunov functional – see the end of Section 2). It consists of three steps:

– Assuming that a $C^1$ unbounded global solution would exist, we analyze its possible final singularities (along a sequence $t_n \to \infty$). We shall show that $u_x$ remains bounded away from the left boundary and describe the shape of $u_x$ near the boundary (cf. Section 2).

– We shall carry out the Zelenyak construction in a sufficiently precise way to determine the density $\phi(u, v)$ of the Lyapunov functional. It will turn out that, whenever $u$ remains in a bounded set of $\mathbb{R}$ (as it does here in view of the estimate (1.2)), $\phi(u, v)$ remains bounded from below uniformly w.r.t. $v$ (see Proposition 3.1).

– Using this property of $\phi$ in the classical Lyapunov argument, together with the fact that singularities may occur only near the boundary, it will be possible to prove the following convergence result: any global solution, even unbounded in $C^1$, has to converge in $C([0, 1])$ to a stationary solution $W$ of (1.1) with $W(0) = 0, W(1) = M$ (see Proposition 3.2). On the other hand, if $u$ were unbounded, then our estimates would imply $W_x(0) = \infty$. But such a $W$ is not available if $M \neq M_c$, leading to a contradiction.

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2. – Preliminary estimates and proof of Theorem 1 for $M = 0$

Let us recall that the only solutions $v \in C([0, 1]) \cap C^2(0, 1)$ of
\[
\begin{aligned}
    &v_{xx} + |v_x|^p = 0, \quad 0 < x < 1, \\
    &v(0) = 0,
\end{aligned}
\]
are given by $v = 0$ or $v = v_k(x) := M_c((x + k)\frac{p-2}{p-1} - k\frac{p-2}{p-1})$, $k \in [0, \infty)$. Moreover, $v_k(1)$ (resp., $v_k(x(0))$) increases from 0 to $M_c$ (resp., 0 to $\infty$), as $k$ decreases from $\infty$ to 0. In particular, for each $M \in (0, M_c)$, $V_M = v_k(M)$ for a unique $k(M) \in (0, \infty)$.

We start with some preliminary estimates. They are collected in Lemmas 2.1–2.6.

**Lemma 2.1.** Let $u$ be a maximal solution of (1.1). For all $t_0 \in (0, T^*)$, there exists $C_1 > 0$ such that
\[
|u_t| \leq C_1, \quad t_0 \leq t < T^*, \quad 0 \leq x \leq 1.
\]

**Proof.** The function $h = u_t$ satisfies
\[
\begin{aligned}
    &h_t = h_{xx} + a(t, x)h_x, \quad t_0 < t < T^*, \quad 0 < x < 1, \\
    &h(t, 0) = h(t, 1) = 0, \quad t_0 < t < T^*, \\
    &h(t_0, x) = u_{xx}(t_0, x) + |u_x(t_0, x)|^p, \quad 0 < x < 1,
\end{aligned}
\]
where $a(t, x) = p|u_x|^{p-2}u_x$. It follows from the maximum principle that $|h| \leq |h(t_0)|_{\infty}$ in $[t_0, T^*) \times [0, 1]$. \qed

The following two lemmas give upper and lower bounds on $u_t$ which show, in particular, that $u_t$ remains bounded away from the boundary.

**Lemma 2.2.** Let $u$ be a maximal solution of (1.1). For all $t_0 \in (0, T^*)$, there exists $C_1 > 0$ such that, for all $t_0 \leq t < T^*$ and $0 \leq x \leq 1$,

\[
(2.1) \quad u_x(t, x) \leq C_1 x + ((p - 1)x)^{-\frac{1}{p-1}}
\]

and

\[
(2.2) \quad u_x(t, 1 - x) \geq -C_1 x - ((p - 1)x)^{-\frac{1}{p-1}}.
\]

**Proof.** Fix $t \in [t_0, T^*)$ and let $y(x) = (u_x(t, x) - C_1 x)^+$, where $C_1$ is given by Lemma 2.1. The function $y$ satisfies
\[
y' + y^p = (u_{xx} - C_1)1_{[u_x > C_1 x]} + (u_x - C_1 x)^p.
\]
For each $x$ such that $u_x(t, x) > C_1 x$, we have $(y' + y^p)(x) \leq (u_{xx} - C_1 + |u_x|^p)(x) \leq 0$ by Lemma 2.1. Therefore, we have $y' + y^p \leq 0$ on $(0, 1)$. By integration, it follows that $y(x) \leq ((p - 1)x)^{-\frac{1}{p-1}}$, hence (2.1).

As for (2.2), it follows similarly by considering $y(x) = (-u_x(t, 1 - x) - C_1 x)^+$. \qed
LEMMA 2.3. Let $u$ be a maximal solution of (1.1). There exists $C_2 > 0$ such that, for all $T \in (0, T^*)$,

\begin{equation}
\max_{Q_T} u_x(t, x) \leq \max(C_2, \max_{0 \leq t \leq T} u_x(t, 0)),
\end{equation}

where $Q_T = [0, T] \times [0, 1]$, and

\begin{equation}
\min_{Q_T} u_x(t, x) \geq \min(-C_2, \min_{0 \leq t \leq T} u_x(t, 1)).
\end{equation}

PROOF. The function $w = u_x$ satisfies $w_t = w_{xx} + a(t, x)w_x$ in $(0, T^*) \times (0, 1)$, where $a(t, x) = p|u_x|^p - 2u_x$. Therefore, $w$ attains its extrema in $Q_T$ on the parabolic boundary of $Q_T$.

Since, by Lemma 2.2, we have $u_x(t, 1) \leq C$ and $u_x(t, 0) \geq -C$ for all $t \in [0, T^*)$, the conclusion follows.

The following lemma will provide a useful lower bound on the blow-up profile of $u_x$ in case $u_x(t, 0)$ or $u_x(t, 1)$ becomes unbounded.

LEMMA 2.4. Let $u$ be a maximal solution of (1.1). For all $t_0 \in (0, T^*)$, there exists $C_3 > 0$ such that, for all $0 \leq t < T^*$ and $0 \leq x \leq 1$,

\begin{equation}
[u_x^+(t, x) + C_3]^{1-p} \leq [u_x^+(t, 0) + C_3]^{1-p} + (p - 1)x
\end{equation}

and

\begin{equation}
[(u_x^-)^+(t, 1 - x) + C_3]^{1-p} \leq [(u_x^-)^+(t, 1) + C_3]^{1-p} + (p - 1)x.
\end{equation}

PROOF. Fix $t \in [t_0, T^*)$ and let $z(x) = u_x^+(t, x) + C_1^{1/p}$, where $C_1$ is given by Lemma 2.1. The function $z$ satisfies

\[z' + z^p = u_{xx}1_{[u_x > 0]} + (u_x^+(t, x) + C_1^{1/p})^p \geq (u_{xx} + |u_x|^p)1_{[u_x > 0]} + C_1 \geq 0\]

on $[0, 1]$ by Lemma 2.1. By integration, it follows that $z^{1-p}(x) \leq z^{1-p}(0) + (p - 1)x$, that is, (2.5) with $C_3 = C_1^{1/p}$.

The estimate (2.6) follows similarly by considering $z(x) = (u_x^-)^+(t, 1 - x) + C_1^{1/p}$.

The following Lemma enables us to rule out infinite time gradient blow-up to $-\infty$ i.e., at $x = 1$.

LEMMA 2.5. Let $u$ be a global solution of (1.1). Then it holds

\[\inf_{[0, \infty) \times [0, 1]} u_x > -\infty.\]

The proof of Lemma 2.5 relies on the following property.
Lemma 2.6. Let $u$ be a global solution of (1.1). Then we have
\[
\lim_{t \to \infty} \left( \max_{x \in [0,1]} u(t,x) \right) = M.
\]

**Proof.** Fix $w_0 \in C^c_\infty(\mathbb{R})$, $w_0 \geq 0$, such that $w_0 \geq u_0 - M$ on $[0, 1]$ and let $w$ be the classical solution of
\[
\begin{cases}
  w_t = w_{xx} + |w_x|^p, & 0 < t < T, \quad -\infty < x < \infty, \\
  w(0, x) = w_0(x), & -\infty < x < \infty,
\end{cases}
\]
with $T$ its maximal existence time. By the maximum principle, we have
\[
0 \leq w \leq |w_0|_\infty \quad \text{on} \quad (0, T) \times \mathbb{R}.
\]
Also, since $u - M$ satisfies the differential equation in (1.1) and since $w \geq 0 \geq u - M$ at $x = 0, 1$ for all $t \in (0, T)$, the comparison principle implies that
\[
w \geq u - M \quad \text{on} \quad (0, T) \times \mathbb{R}.
\]
On the other hand, the function $z := w_x$ satisfies $z_t = z_{xx} + a(t, x)z_x$ in $(0, T) \times \mathbb{R}$, where $a(t, x) = p|w_x|^{p-2}w_x$. By a further application of the maximum principle, we deduce that
\[
|w_x| \leq |w_0,x|_\infty \quad \text{on} \quad (0, T) \times \mathbb{R}.
\]
This, along with (2.7), implies in particular that $T = \infty$.

Let now $A = |w_{0,x}|_\infty^{p-2}$. Due to (2.9), we have $w_t \leq w_{xx} + A|w_x|^2$ and the function $y := e^{Aw} - 1 \geq 0$ thus satisfies $y_t \leq y_{xx}$ in $(0, \infty) \times \mathbb{R}$. It follows that $y(t) \leq G_t \ast y(0)$, where $G_t(x) = (4\pi t)^{-1/2} \exp[-x^2/4t]$. Therefore, for all $t > 0$, we get
\[
A|w(t)|_\infty \leq |e^{Aw(t)} - 1|_\infty \leq (4\pi t)^{-1/2}|e^{Aw_0} - 1|_{L^1(\mathbb{R})},
\]
hence $|w(t)|_\infty \leq C_t^{-1/2}$. In view of (2.8), this yields the Lemma. \hfill \Box

**Proof of Lemma 2.5.** Assume that the Lemma is false. Then, by Lemma 2.3, there exists a sequence $t_n \to \infty$ such that $u_x(t_n, 1) \to -\infty$.

Fix $\varepsilon > 0$. By (2.6) in Lemma 2.4, for $n \geq n_0(\varepsilon)$ large enough, we have
\[
\left[(-u_x)^+(t_n, 1 - x) + C_3 \right]^{1-p} \leq p\varepsilon, \quad 0 \leq x \leq \varepsilon
\]
hence,
\[
(-u_x)^+(t_n, 1 - x) \geq (p\varepsilon)^{-1/(p-1)} - C_3, \quad 0 \leq x \leq \varepsilon.
\]
By choosing $\varepsilon = \varepsilon(p, C_3) > 0$ small, we deduce that $u_x(t_n, 1 - x) \leq -1$ on $[0, \varepsilon]$, hence
\[
u(t_n, 1 - x) \geq M + x, \quad 0 \leq x \leq \varepsilon
\]
for all $n \geq n_0(\varepsilon)$. But this contradicts Lemma 2.6. \hfill \Box

Lemma 2.5 is already enough to conclude in the case $M = 0$.

**Proof of Theorem 1 for $M = 0$.** Let $u$ be a global solution of (1.1). Since $M = 0$, we observe that $v(t,x) := u(t,1-x)$ solves (1.1) with $u_0$ replaced by $u_0(1-x)$. Therefore, Lemma 2.5 implies that both $u_x$ and $v_x$ are bounded below on $[0, \infty) \times [0, 1]$, which means that $u_x$ is bounded. (For the convergence of $u$ to $V_0 = 0$, see the end of the proof of the case $M > 0$ below.) \hfill \Box
3. – Lyapunov functional and proof of Theorem 1 for $M > 0$

As a main step, we now carry out the argument of Zelenyak to construct a Lyapunov functional. The key point here is that the Lyapunov functional enjoys nice properties on any global trajectory of (1.1), even if it were unbounded in $C^1$.

**Proposition 3.1.** Fix any $K > 0$ and let $D_K = [-K, K] \times \mathbb{R}$. There exist functions $\phi \in C^1(D_K; \mathbb{R})$, and $\psi \in C(D_K; (0, \infty))$ with the following property. For any solution $u$ of (1.1) with $|u| \leq K$, defining

$$L(u(t)) := \int_0^1 \phi(u(t, x), u_x(t, x)) \, dx,$$

it holds

$$\frac{d}{dt} L(u(t)) = -\int_0^1 \psi(u(t, x), u_x(t, x)) u_t^2(t, x) \, dx, \quad 0 < t < T^*.$$

Furthermore, we have

$$\phi \geq 0.$$

As a consequence of Proposition 3.1 and of the estimates of Section 2, we shall obtain the following convergence result. Of course, the main point here is that we do not assume $u$ to be bounded, but only global (since otherwise the statement is nothing more but the result of [30], and the convergence even holds in $C^1$).

**Proposition 3.2.** Let $u$ be a global solution of (1.1). Then, as $t \to \infty$, $u(t)$ converges in $C([0, 1])$ to a stationary solution of (1.1), i.e. a function $W \in C([0, 1]) \cap C^2(0, 1)$ of

$$W_{xx} + |W_x|^p = 0, \quad 0 < x < 1,$$

$$W(0) = 0, \quad W(1) = M.$$

Moreover, the convergence also holds in $C^1([\varepsilon, 1])$ for all $\varepsilon > 0$.

**Proof of Proposition 3.1.** For a given function $\varphi(u, v)$, let us denote

$$H = \varphi_u + |v|^p \varphi_{vv} - v \varphi_{vu}.$$ 

Here we assume that $\varphi, \varphi_u, \varphi_v, \varphi_{uv}$ are continuous and $C^1$ in $v$ in $D_K$, and that $\varphi_{uv}$ is continuous in $D_K$ and, except perhaps at $v = 0$, $C^1$ in $v$. We observe that $H$ is continuous and differentiable in $v$ in $D_K$ and satisfies

$$H_v = |v|^p \varphi_{vvv} + p|v|^{p-2} v \varphi_{vv} - v \varphi_{vuv} \quad (H_v = 0 \text{ for } v = 0).$$
Now suppose that $\psi := \varphi_{vv}$ satisfies

\begin{equation}
\psi_u - |v|^{p-2}v \psi_v - p|v|^{p-2} \psi = 0, \quad |u| \leq K, \quad v \neq 0.
\end{equation}

It follows that $H_v = 0$, hence

$$H = H(u) = \varphi_u(u, 0).$$

Let then

$$\phi(u, v) = \varphi(u, v) - \int_0^u H(s) \, ds = \varphi(u, v) - \varphi(u, 0) + \varphi(0, 0).$$

We compute, using integration by parts and $u_t(t, 0) = u_t(t, 1) = 0$,

$$\frac{d}{dt} L(u(t)) = \int_0^1 \left\{ (\varphi_u(u, u_x) - H(u))u_t + \varphi_v(u, u_x)u_{xt} \right\}(t, x) \, dx$$

$$= \int_0^1 \left\{ (\varphi_u(u, u_x) - H(u) - \varphi_{uu}(u, u_x)u_x - \varphi_{vv}(u, u_x)u_{xx}) \right\} u_t(t, x) \, dx.$$ 

Using the definition of $H$ and $u_{xx} = u_t - |u_x|^p$, we deduce that

$$\frac{d}{dt} L(u(t)) = - \int_0^1 \psi(u(t, x), u_x(t, x)) u_t^2(t, x) \, dx.$$

We have thus obtained (3.1), provided (3.4) is true.

Now, the equation (3.4) can be solved by the method of characteristics. For each $K > 0$, one finds that the function $\psi$ defined by

$$\psi(u, v) = [1 + (p - 2)|v|^{p-2}(K + 1 - u)]^{-p/(p-2)} > 0$$

is a solution of (3.4) on $[-K, K] \times (\mathbb{R} - \{0\})$.

Define $\varphi$ by

$$\varphi(u, v) = \int_0^v \int_0^z \psi(u, s) \, ds \, dz \geq 0.$$ 

It is easy to check that $\varphi$ enjoys the regularity properties assumed at the beginning of the proof and $\phi = \varphi$, hence (3.2).
Proof of Proposition 3.2. Fix any sequence $t_n \to \infty$ and let $u_n = u(t_n + \ldots)$. Denote $Q := [0, \infty) \times (0, 1]$ and $Q_\varepsilon := [0, \infty) \times (\varepsilon, 1]$, for all $\varepsilon > 0$.

From (1.2) and Lemma 2.1, we know that
\begin{equation}
|u| + |u_t| \leq C \quad \text{in} \quad [1, \infty) \times [0, 1].
\end{equation}

Also, using (2.1), $p > 2$ and Lemma 2.5, we obtain
\begin{equation}
|\partial_x u_n|_{L^\infty([1, \infty) \times [\varepsilon, 1])} \leq C\|. 
\end{equation}

It follows from (3.5) and (3.6) that
\begin{equation}
\text{(3.7) the sequence $(u_n)$ is relatively compact in $C([0, T] \times [0, 1])$ for each $T > 0$.}
\end{equation}

On the other hand, using (2.1), (2.2) and (3.5), we have
\begin{equation}
|u_x| \leq C(\varepsilon) \quad \text{and hence} \quad |u_{xx}| \leq C(\varepsilon) \quad \text{in} \quad [1, \infty) \times (\varepsilon, 1].
\end{equation}

Since $w := u_x$ satisfies $w_t - w_{xx} = p|u_x|^{p-2}u_xu_{xx}$, parabolic regularity estimates then imply that for each $q > 1$,
\begin{equation}
|w_t(t_n + \ldots)|_{L^q([0, T) \times (\varepsilon, 1))} \leq C(\varepsilon, T, q), \quad T > 0.
\end{equation}

It follows from (3.8) and (3.9) that the sequence $(\partial_x u_n)$ is relatively compact in $C([0, T] \times [\varepsilon, 1])$ for each $\varepsilon, T > 0$. This, together with (3.7), implies that some subsequence $(u_{n_k})$ converges to a function $W \in C(\overline{Q})$, with $W_x \in C(Q)$, which satisfies
\begin{equation}
\begin{cases}
W_t - W_{xx} = |W_x|^p & \text{in} \quad Q, \\
W(t, 0) = 0, \quad W(t, 1) = M, & t \geq 0.
\end{cases}
\end{equation}

(The convergence of $u_{n_k}$ is uniform in each set $[0, T] \times [0, 1]$ and the convergence of $\partial_x u_{n_k}$ is uniform in each set $[0, T] \times [\varepsilon, 1].$)

Now, by (1.2) we may find $K > 0$ such that
\begin{equation}
|u| \leq K \quad \text{on} \quad [0, \infty) \times [0, 1].
\end{equation}

Since $\psi$, given by Proposition 3.1, is positive continuous, we have
\begin{equation}
\eta(K, R) := \inf \{ \psi(u, v); \quad |u| \leq K, \quad |v| \leq R \} > 0, \quad \text{for all} \quad R > 0.
\end{equation}
Fix any \( \varepsilon \in (0, 1) \). By (3.1), (3.8), (3.10) and (3.2), we get, for all \( T > 1 \),

\[
\eta(K, C(\varepsilon)) \int_1^T \int_1^1 u_1^2(t, x) \, dx \, dt \leq \int_1^T \int_1^1 \psi(u, u_x) u_1^2(t, x) \, dx \, dt \leq \mathcal{L}(u(1)) - \mathcal{L}(u(T)) \leq \mathcal{L}(u(1)).
\]

This implies that \( \int_1^\infty \int_1^1 u_1^2(t, x) \, dx \, dt < \infty \), hence

\[
\int_0^\infty \int_1^1 (\partial_t u_{nk})^2(t, x) \, dx \, dt = \int_0^\infty \int_1^1 u_1^2(t, x) \, dx \, dt \to 0, \quad k \to \infty.
\]

Since \( \partial_t u_{nk} \to W_t \) in \( D'((0, \infty) \times (0, 1)) \) and since \( \varepsilon \in (0, 1) \) is arbitrary, it follows that \( W_t \equiv 0 \). Therefore, \( W = W(x) \in C([0, 1]) \cap C^2(0, 1) \) satisfies (3.3).

But we know (cf. the beginning of Section 2) that the solution of (3.3) is unique whenever it exists. Since the sequence \( t_n \to \infty \) was arbitrary, this readily implies that the whole solution \( u(t) \) actually converges to \( W \). The Proposition is proved. \( \Box \)

**Proof of Theorem 1 for \( 0 < M < M_c \).** Assume that \( u \) is a global solution of (1.1) which is unbounded in \( C^1 \). By Proposition 3.2, as \( t \to \infty \), \( u(t) \) converges to \( W = V_M \), with convergence in \( C([0, 1]) \) and in \( C^1([\varepsilon, 1]) \) for all \( \varepsilon > 0 \).

Since \( u \) is unbounded, by Lemmas 2.3 and 2.5, there exists a sequence \( t_n \to \infty \) such that

\[
(3.11) \quad u_x(t_n, 0) \to \infty.
\]

Using Lemma 2.5, (2.5) and (3.11), we deduce that \( W_x(x) \geq -C \) and \( [W_x^+(x) + C_3]^{1-p} \leq (p-1)x \) in \( (0, 1) \), This easily implies that

\[ W_x(x) \geq ((p-1)x)^{-\frac{1}{p-1}} - C' \quad \text{in } (0, 1). \]

But this is a contradiction, since \( W = V_M \in C^1([0, 1]) \). We have thus proved that all global solutions are bounded in \( C^1 \).

Finally, once boundedness is known, the convergence of global solutions to \( V_M \) in \( C^1 \) (also for \( M = 0 \)) is a standard consequence of the existence of a Lyapunov functional, the uniqueness of the steady-state, and compactness properties of the semi-flow associated with (1.1). The proof of Theorem 1 is complete. \( \Box \)

**Proof of Proposition 2.** This is an immediate consequence of Proposition 3.2 and the fact that (3.3) admits no solution for \( M > M_c \). \( \Box \)

**Proof of Proposition 3.** Let

\[ D = \{ u_0 \in X; u(u_0; t, .) \, \text{converges to} \, V_M \, \text{in} \, C^1 \, \text{as} \, t \to \infty \} \]
and fix $\overline{M} \in (M, M_c)$. We claim that:

\begin{equation}
\text{(3.12)} \quad \text{for all } u_0 \in X, u_0 \leq \min(M, V_M) \text{ implies } u_0 \in D.
\end{equation}

Indeed, by the comparison principle, as long as $u := u(u_0; t, \cdot)$ exists, we have $u \leq V_M$, hence $u_x(t, 0) \leq V_M(x)(0)$, and $u \leq M$, hence $u_x(t, 1) \geq 0$. By Lemma 2.3, we deduce that $u$ is global and bounded in $C^1$. It then follows from [30] that $u$ converges in $C^1$ to the unique steady state $V_M$ as $t \to \infty$, which proves the claim.

Let us first consider the case $M \in (0, M_c)$. By [2, Theorem 1.2], there exists $u_0 \in X$ with $u_0(x) \geq 0$, such that $T^*(\overline{u}_0) < \infty$. For each $\lambda \in [0, 1]$, denote $u_{0,\lambda} := V_M + \lambda(\overline{u}_0 - V_M) \in X$ and $u_\lambda := u(u_{0,\lambda}; t, \cdot)$. For $\lambda > 0$ small, we have $u_{0,\lambda} \leq \min(M, V_M)$, hence $u_{0,\lambda} \in D$. Therefore $\lambda^* := \inf\{\lambda \in [0, 1]; u_{0,\lambda} \not\in D\} \in (0, 1)$. By (3.12) and a standard continuous dependence argument, we have $u_{0,\lambda^*} \not\in D$. This implies that $u_{\lambda^*}$ cannot be global and bounded in $C^1$ (since otherwise it would converge to $V_M$ due to [30]). In view of Theorem 1, the only remaining possibility is that $T^*(u_{0,\lambda^*}) < \infty$. Considering $u_{0,\lambda_n}$ for a sequence $\lambda_n \uparrow \lambda^*$, we obtain the conclusions (i) and (ii) of Proposition 3. We also get (iii), since otherwise $u_{\lambda^*}$ would be global by continuous dependence.

In the case $M = 0$, we fix any $\phi \in X$ with $\phi \geq 0$, $\phi \not\equiv 0$, and we set $u_{0,\lambda} = \lambda \phi$. We have $T^*(u_{0,\lambda}) < \infty$ for $\lambda > 0$ large by [28, Theorem 2.1], whereas $u_{0,\lambda} \in D$ for $\lambda > 0$ small (indeed, by comparing with $V_M$ and $V_M(1 - x)$ for some $M \in (0, M_c)$, one sees that $u$ remains bounded in $C^1$).

The rest of the proof is then similar. \hfill \Box

\begin{thebibliography}{99}


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