The Extended Future Tube Conjecture for SO(1, n)

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Abstract. Let $C$ be the open upper light cone in $\mathbb{R}^{1+n}$ with respect to the Lorentz product. The connected linear Lorentz group $SO_{\mathbb{R}}(1, n)^0$ acts on $C$ and therefore diagonally on the $N$-fold product $T^N$ where $T = \mathbb{R}^{1+n} + iC \subset \mathbb{C}^{1+n}$. We prove that the extended future tube $SO_{\mathbb{C}}(1, n) \cdot T^N$ is a domain of holomorphy.

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For $K \in \{\mathbb{R}, \mathbb{C}\}$ let $K^{1+n}$ denote the $(1+n)$-dimensional Minkowski space, i.e., on $K^{1+n}$ we have given the bilinear form

$$(x, y) \mapsto x \bullet y := x_0 y_0 - x_1 y_1 - \cdots - x_n y_n$$

where $x_j$ respectively $y_j$ are the components of $x$ respectively $y$ in $K^{1+n}$. The group $O_K(1, n) = \{g \in Gl_K(1+n); gx \bullet gy = x \bullet y \text{ for all } x, y \in K^{1+n}\}$ is called the linear Lorentz group. For $n \geq 2$ the group $O_{\mathbb{R}}(1, n)$ has four connected components and $O_{\mathbb{C}}(1, n)$ has two connected components. The connected component of the identity $O_K(1, n)^0$ of $O_K(1, n)$ will be called the connected linear Lorentz group. Note that $SO_{\mathbb{R}}(1, n) = \{g \in O_{\mathbb{R}}(1, n); \det(g) = 1\}$ has two connected components and $O_{\mathbb{R}}(1, n)^0 = SO_{\mathbb{R}}(1, n)^0$. In the complex case we have $SO_{\mathbb{C}}(1, n) = O_{\mathbb{C}}(1, n)^0$.

The forward cone $C$ is by definition the set $C := \{y \in \mathbb{R}^{1+n}; y \bullet y > 0 \text{ and } y_0 > 0\}$ and the future tube $T$ is the tube domain over $C$ in $\mathbb{C}^{1+n}$, i.e., $T = \mathbb{R}^{1+n} + iC \subset \mathbb{C}^{1+n}$. Note that $T^N = T \times \cdots \times T$ is the tube domain in the space of complex $(1+n) \times N$-matrices $\mathbb{C}^{(1+n)\times N}$ over $C^N = C \times \cdots \times C \subset \mathbb{R}^{(1+n)\times N}$. The group $SO_{\mathbb{C}}(1, n)$ acts by matrix multiplication on $\mathbb{C}^{(1+n)\times N}$ and the subgroup $SO_{\mathbb{R}}(1, n)^0$ stabilizes $T^N$. In this note we prove the

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Extended future tube conjecture:

\[ \text{SO}_C(1, n) \cdot T^N = \bigcup_{g \in \text{SO}_C(1, n)} g \cdot T^N \]

is a domain of holomorphy.

This conjecture arise in the theory of quantized fields for about 50 years. We refer the interested reader to the literature ([HW], [J], [SV], [StW], [W]). There is a proof of this conjecture in the case where \( n = 3 \) ([He2]), [Z]. The proof there uses essentially that \( T \) can be realized as the set \( \{ Z \in \mathbb{C}^{2 \times 2}; \frac{1}{2} (Z - t \bar{Z}) \text{ is positive definite} \} \). Moreover the proof for \( n = 3 \) is unsatisfactory. It does not give much information about \( \text{SO}_C(1, n) \cdot T^N \) except for holomorphic convexity.

Here we prove that more is true. Roughly speaking, we show that the basic Geometric Invariant Theory results known for compact groups (see [He1]) also holds for \( X : = T^N \) and the non compact group \( \text{SO}_R(1, n)^0 \). More precisely this means \( \text{SO}_C(1, n) \cdot X = Z \) is a universal complexification of the \( G \)-space \( X, G = \text{SO}_R(1, n)^0 \) in the sense of [He1]. There exists complex analytic quotients \( X//G \) and \( Z//G^C \), \( G^C = \text{SO}_C(1, n) \), given by the algebra of invariant holomorphic functions and there is a \( G \)-invariant strictly plurisubharmonic function \( \rho : X \to \mathbb{R} \), which is an exhaustion on \( X//G \). Let

\[ \mu : X \to g^*, \quad \mu(z)(\xi) = \left. \frac{d}{dt} \right|_{t=0} (t \to \rho(\exp it\xi \cdot z)), \]

be the corresponding moment map. Then the diagram

\[ \begin{array}{ccc}
\mu^{-1}(0) & \hookrightarrow & X \\
\downarrow & & \downarrow \pi \\
\mu^{-1}(0)/G & \equiv & X//G \equiv Z//G^C
\end{array} \]

where all maps are induced by inclusion is commutative, \( X//G, X, Z \) and \( Z//G^C \) are Stein spaces and \( \rho|\mu^{-1}(0) \) induces a strictly plurisubharmonic exhaustion on \( \mu^{-1}(0)/G = X//G = Z//G^C \). Moreover the same statement holds if we replace \( X = T^N \) with a closed \( G \)-stable analytic subset \( A \) of \( X \).

1. Geometric Invariant Theory of Stein spaces

Let \( Z \) be a Stein space and \( G \) a real Lie group acting as a group of holomorphic transformations on \( Z \). A complex space \( Z//G \) is said to be an analytic Hilbert quotient of \( Z \) by the given \( G \)-action if there is a \( G \)-invariant surjective holomorphic map \( \pi : Z \to Z//G \), such that for every open Stein subspace \( Q \subset Z//G \).
i. its inverse image $\pi^{-1}(Q)$ is an open Stein subspace of $Z$ and 
ii. $\pi^*O_{Z//G}(Q) = O(\pi^{-1}(Q))^G$, where $O(\pi^{-1}(Q))^G$ denotes the algebra of $G$-invariant holomorphic functions on $\pi^{-1}(Q)$ and $\pi^*$ is the pull back map.

Now let $G^c$ be a linearly reductive complex Lie group. A complex space $Z$ endowed with a holomorphic action of $G^c$ is called a holomorphic $G^c$-space.

**Theorem 1.1.** Let $Z$ be a holomorphic $G^c$-space, where $G^c$ is a linearly reductive complex Lie group.

i. If $Z$ is a Stein space, then the analytic Hilbert quotient $Z//G^c$ exists and is a Stein space.

ii. If $Z//G^c$ exists and is a Stein space, then $Z$ is a Stein space.

**Proof.** Part i. is proven in [He1] and part ii. in [HeMP].

**Remark 1.1.**

i. If the analytic Hilbert quotient $\pi: Z \to Z//G^c$ exists, then every fiber $\pi^{-1}(q)$ of $\pi$ contains a unique $G^c$-orbit $E_q$ of minimal dimension. Moreover, $E_q$ is closed and $\pi^{-1}(q) = \{z \in Z; E_q \subset \overline{G^c.z}\}$. Here $\overline{\cdot}$ denotes the topological closure.

ii. Let $X$ be a subset of $Z$, such that $G^c \cdot X := \bigcup_{g \in G^c} g \cdot X = Z$ and assume that $Z//G^c$ exists. Then $G^c \cdot X$ is a Stein space if and only if $Z//G^c = \pi(X)$ is a Stein space.

iii. Let $V^c$ be a finite dimensional complex vector space with a holomorphic linear action of $G^c$. Then the algebra $\mathbb{C}[V^c]^G^c$ of invariant polynomials is finitely generated (see e.g. [Kr]).

In particular, the inclusion $\mathbb{C}[V^c]^G^c \hookrightarrow \mathbb{C}[V^c]$ defines an affine variety $V^c//G^c$ and an affine morphism $\pi^c: V^c \to V^c//G^c$. If we regard $V^c//G^c$ as a complex space, then $\pi^c: V^c \to V^c//G^c$ gives the analytic Hilbert quotient of $V^c$ (see e.g. [He1]).

**Remark 1.2.** For a non-connected linearly reductive complex group $G$ let $G^0$ denote the connected component of the identity and let $Z$ be a holomorphic $G$-space. The analytic Hilbert quotient $Z//G$ exists if and only if the quotient $Z//G^0$ exists. Moreover, the quotient map $\pi_G: Z \to Z//G$ induces a map $\pi_{G/G^0}: Z//G^0 \to Z//G$ which is finite. In fact the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\pi_{G^0}} & Z//G^0 \\
\downarrow & & \downarrow \\
Z//G^0 & \longrightarrow & Z//G \\
\pi_{G/G^0} & &
\end{array}
$$

commutes and $\pi_{G/G^0}$ is the quotient map for the induced action of the finite group $G/G^0$ on $Z//G^0$. 
2. – The geometry of the Minkowski space

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ or $\mathbb{C}$ and $(e_0, \ldots, e_n)$ the standard orthonormal basis for $\mathbb{K}^{1+n}$. The space $\mathbb{K}^{1+n}$ together with the quadratic form $\eta(z) = z_0^2 - z_1^2 - \cdots - z_n^2$, where $z_j$ are the components of $z$, is called the $(1+n)$-dimensional linear Minkowski space. Let $<,>_L$ denote the symmetric non-degenerated bilinear form which corresponds to $\eta$, i.e., $z \cdot w := < z, w >_L = \langle zJw \rangle_\mathbb{C}$ where $z^t$ denotes the transpose of $z$ and $J = (e_0, -e_1, \ldots, -e_n)$ or equivalently $z \cdot w = < z, Jw >_E$ where $<,>_E$ denotes the standard Euclidean product on $\mathbb{R}^{1+n}$, respectively its $\mathbb{C}$-linear extension to $\mathbb{C}^{1+n}$.

Let $O_\mathbb{K}(1, n)$ denote the subgroup of $\text{GL}_\mathbb{K}(1+n)$ which leave $\eta$ fixed, i.e., $O_\mathbb{K}(1, n) = \{ g \in \text{GL}_\mathbb{K}(1+n); g \cdot gw = z \cdot w \text{ for all } z, w \in \mathbb{K}^{1+n} \}$. Note that $SO_\mathbb{K}(1, n) = \{ g \in O_\mathbb{K}(1, n); \det g = 1 \}$ is an open subgroup of $O_\mathbb{K}(1, n)$. For $\mathbb{K} = \mathbb{C}$, $SO_\mathbb{C}(1, n)$ is connected. But in the real case $SO_\mathbb{R}(1, n)$ consists of two connected components ($n \geq 2$). The connected component $SO_\mathbb{R}(1, n)^0 = O_\mathbb{R}(1, n)^0$ of the identity is called the connected linear Lorentz group. Note that $SO_\mathbb{R}(1, n)^0$ is not an algebraic subgroup of $SO_\mathbb{R}(1, n)$ but is Zariski dense in $SO_\mathbb{R}(1, n)$. We have $\mathbb{K}^{1+n} = \mathbb{K}[\eta] = \mathbb{K}[\mathbb{K}^{1+n}]^{SO_\mathbb{C}(1, n)} = \mathbb{K}[\mathbb{K}^{1+n}]^{O_\mathbb{C}(1, n)}$.

Now let $\mathbb{C}^{(1+n)\times N} = \mathbb{C}^{1+n} \times \cdots \times \mathbb{C}^{1+n}$ be the $N$-fold product of $\mathbb{C}^{1+n}$, i.e., the space of complex $(1+n) \times N$-matrices. The group $O_\mathbb{C}(1, n)$ acts on $\mathbb{C}^{(1+n)\times N}$ by left multiplication. A classical result in Invariant Theory says that $\mathbb{C}^{[\mathbb{C}^{(1+n)\times N}]^{O_\mathbb{C}(1, n)}}$ is generated by the polynomials $p_k(z_1, \ldots, z_N) = z_k \cdot z_j$ where $z = (z_1, \ldots, z_N) \in \mathbb{C}^{(1+n)\times N}$.

**Remark 2.1.** The (algebraic) Hilbert quotient $\mathbb{C}^{(1+n)\times N} / O_\mathbb{C}(1, n)$ can be identified with the space $\text{Sym}_N(\text{min}(1+n, N))$ of symmetric $N \times N$-matrices of rank smaller or equal $\text{min}(1+n, N)$.

With this identification the quotient map $\pi_\mathbb{C} : \mathbb{C}^{(1+n)\times N} \rightarrow \mathbb{C}^{(1+n)\times N} / O_\mathbb{C}(1, n)$ is given by $\pi_\mathbb{C}(Z) = '{ZJZ}$ where '{Z denotes the transpose of Z and J is as above. For the group $SO_\mathbb{C}(1, n)$ the situation is slightly more complicated. If $N \geq 1+n$ additional invariants appear, but they are not relevant for our considerations, since the induced map $\mathbb{C}^{(1+n)\times N} / SO_\mathbb{C}(1, n) \rightarrow \mathbb{C}^{(1+n)\times N} / O_\mathbb{C}(1, n)$ is finite.

There is a well known characterization of closed $O_\mathbb{C}(1, n)$-orbits in $\mathbb{C}^{(1+n)\times N}$. In order to formulate this we need more notations. Let $z = (z_1, \ldots, z_N) \in \mathbb{C}^{(1+n)\times N}$ and $L(z) := Cz_1 + \cdots + Cz_N$ be the subspace of $\mathbb{C}^{1+n}$ spanned by $z_1, \ldots, z_N$. The Lorentz product $<,>_L$ restricted to $L(z)$ is in general degenerated. Thus let $L(z)^0 = \{ w \in L(z); < w, v >_L = 0 \text{ for all } v \in L(z) \}$. It follows that $\text{dim} L(z)/L(z)^0 = \text{rank}(zJz) = \text{rank} \pi_\mathbb{C}(z)$. Elementary consideration show the following.

**Lemma 2.1.** The orbit $O_\mathbb{C}(1, n) \cdot z$ through $z \in \mathbb{C}^{(1+n)\times N}$ is closed if and only if the orbit $SO_\mathbb{C}(1, n) \cdot z$ is closed and this is the case if and only if $L(z)^0 = \{0\}$, i.e., $\text{dim} L(z) = \text{rank} \pi_\mathbb{C}(z)$. 

The light cone $N := \{ y \in \mathbb{R}^{1+n}; \eta(y) = 0 \}$ is of codimension one and its complement $\mathbb{R}^{1+n} \setminus N$ consists of three connected components (here of course we assume $n \geq 2$). By the forward cone $C$ we mean the connected component which contains $e_0$. It is easy to see that $C = \{ y \in \mathbb{R}^{1+n}; y \cdot e_0 > 0 \}$ and $\eta(y) > 0 = \{ y \in \mathbb{R}^{1+n}; y \cdot x > 0 \text{ for all } x \in N^+ \}$ where $N^+ = \{ x \in N; x \cdot e_0 > 0 \}$. In particular, $C$ is an open convex cone in $\mathbb{R}^{1+n}$. Since $J$ has only one positive Eigenvalue, the following version of the Cauchy-Schwarz inequality holds.

**Lemma 2.2.** If $\eta(y) > 0$, then $\tilde{x} \cdot y \leq 0$ for $\tilde{x} := x - \frac{x \cdot y}{\eta(y)^2} y$ and all $x \in \mathbb{R}^{1+n}$.

In particular

$$\eta(x) \cdot \eta(y) \leq (x \cdot y)^2$$

and equality holds if and only if $x$ and $y$ are linearly dependent.

The elementary Lemma has several consequences which are used later on. For example,

- if $y_1, y_2 \in C^\pm := C \cup (-C) = \{ y \in \mathbb{R}^{1+n}; \eta(y) > 0 \}$, then $y_1 \cdot y_2 \neq 0$.
- Moreover,
- if $y_1, y_2 \in N = \{ y \in \mathbb{R}^{1+n}; \eta(y) = 0 \}$, and $y_1 \cdot y_2 = 0$, then $y_1$ and $y_2$ are linearly dependent.

The tube domain $T = \mathbb{R}^{1+n} + iC \subset \mathbb{C}^{1+n}$ over $C$ is called the future tube. Note that $\text{SO}_{\mathbb{R}}(1, n)^0$ acts on $T$ by $g \cdot (x + iy) = gx + igy$ and therefore on the $N$-fold product $T^N = T \times \cdots \times T \subset \mathbb{C}^{(1+n) \times N}$ by matrix multiplication.

**Remark 2.2.** It is easy to show that the $\text{SO}_{\mathbb{R}}(1, n)^0$-action on $C$ and consequently also on $T^N$ is proper. In particular $T^N / \text{SO}_{\mathbb{R}}(1, n)^0$ is a Hausdorff space.

The complexified group $\text{SO}_{\mathbb{C}}(1, n)$ does not stabilize $T^N$. The domain

$$\text{SO}_{\mathbb{C}}(1, n) \cdot T^N = \bigcup_{g \in \text{SO}_{\mathbb{C}}(1, n)} g \cdot T^N$$

is called the extended future tube.

### 3. Orbit connectedness of the future tube

Let $G$ be a Lie group acting on $Z$. A subset $X \subset Z$ is called orbit connected with respect to the $G$-action on $Z$ if $\Sigma(z) = \{ g \in G; g \cdot z \in X \}$ is connected for all $z \in X$.

In this section we prove the following

**Theorem 3.1.** The $N$-fold product $T^N$ of the future tube is orbit connected with respect to the $\text{SO}_{\mathbb{C}}(1, n)$-action on $\mathbb{C}^{(1+n) \times N}$. 
We first reduce the proof of this Theorem for the $SO_C(1, n)$-action to the proof of the related statement about the Cartan subgroups of $SO_C(1, n)$. For this we use the results of Bremigan in [B]. For the convenience of the reader we briefly recall those parts, which are relevant for the proof of Theorem 3.1.

Starting with a simply connected complex semisimple Lie group $G^C$ with a given real form $G$ defined by an anti-holomorphic group involution, $g \mapsto \tilde{g}$, there is a subset $S$ of $G^C$ such that $GSG$ contains an open $G \times G$-invariant dense subset of $G^C$. The set $S$ is given as follows.

Let $\text{Car}(G^C) = \{H_1, \ldots, H_\ell\}$ be a complete set of representatives of the Cartan subgroups of $G^C$, which are defined over $\mathbb{R}$. Associated to each $H \in \text{Car}(G^C)$ are the Weyl group $W(H) := N_{G^C}(H)/H$, the real Weyl group $W_\mathbb{R}(H) := \{gH \in W(H); \tilde{g}H = gH\}$ and the totally real Weyl group $W_{\mathbb{R}}(H) := \{gH \in W_\mathbb{R}(H); \tilde{g} = g\}$. Here $N_{G^C}(H)$ denotes the normalizer of $H$ in $G^C$.

For $H \in \text{Car}(G^C)$ let $R(H)$ be a complete set of representatives of the double coset space $W_{\mathbb{R}}(H) \backslash W_\mathbb{R}(H)/W_{\mathbb{R}}(H)$ chosen in such a way that $\epsilon = \epsilon^{-1}$ holds for all $\epsilon \in R(H)$. Then $S := \bigcup H \epsilon$ has the claimed properties.

Although $SO_C(1, n)$ is not simply connected, the results above remain true for $G := SO_\mathbb{R}(1, n)^0$ and $G^C := SO_C(1, n)$, as one can see by going over to the universal covering.

**Remark 3.1.** Using the classification of the $SO_\mathbb{R}(1, n)^0 \times SO_\mathbb{R}(1, n)^0$-orbits in $SO_C(1, n)$ as presented in [J], the same result can be obtained for $G^C = SO_C(1, n)$.

Since $T^N$ is $SO_\mathbb{R}(1, n)^0$-stable, $SO_\mathbb{R}(1, n)^0$ is connected and $SO_\mathbb{R}(1, n)^0 \cdot S \cdot SO_\mathbb{R}(1, n)^0$ is dense in $SO_C(1, n)$, Theorem 3.1 follows from

**Proposition 3.1.** The set $\Sigma_S(w) := \{g \in S; g \cdot w \in T^N\}$ is connected for all $w \in T^N$.

In the case $n = 2m - 1$ we may choose $\text{Car}(SO_C(1, n)) = \{H_0\}$ where

$$H_0 = \left\{ \begin{pmatrix} \sigma & 0 & \cdots & 0 \\ 0 & \tau_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tau_{m-1} \end{pmatrix}; \sigma \in SO_C(1, 1), \tau_j \in SO_C(2) \right\} \text{ and } R(H_0) = \{\text{Id}\}.$$

In the even case $n = 2m$ we make the choice $\text{Car}(SO_C(1, n)) = \{H_1, H_2\}$ where

$$H_1 = \left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}; h \in H_0 \right\}, H_2 = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \tau_1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tau_m \end{pmatrix}; \tau_j \in SO_C(2) \right\}.$$

$$R(H_1) = \{\text{Id}\} \text{ and } R(H_2) = \{\text{Id}, \epsilon\} \text{ with } \epsilon = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{Id}_{2m-3} \end{pmatrix}.$$
Observe that in the case $H_2$, where $\epsilon$ is present, $S$ is not connected. But the “$\epsilon$-part” of $S$ is not relevant, since any $h \in H_2$ does not change the sign of the first component of the imaginary part of $z_j \in T$ and therefore $\Sigma_{H_2\epsilon}(z)$ is empty for all $z \in T^N$. Thus it is sufficient to prove the following

**Proposition 3.2.** For every possible $H \in \{H_0, H_1, H_2\}$ and every $w \in T^N$ the set $\Sigma_H(w) = \{h \in H; h \cdot w \in T^N\}$ is connected.

**Proof.** We will carry out the proof in the case where $n = 2m - 1$ and $H = H_0$. The proof in the other cases is analogous. Note that $H$ splits into its real and imaginary part, i.e., $H = H_\mathbb{R} \cdot H_1 \cong H_\mathbb{R} \times H_1$ where $H_\mathbb{R}$ denotes the connected component of the identity of $SO_\mathbb{R}(1, n)^0 \cap H = \{h \in H; h = h\}$ and $H_1 = \exp iH_\mathbb{R}$. Thus the $2 \times 2$ blocks appearing for $h \in H_1$ are given by

$$
\sigma_k = \begin{pmatrix} a & ib \\ ib & a \end{pmatrix} \quad \text{where } a^2 + b^2 = 1 \quad \text{and} \quad \tau_j = \begin{pmatrix} c_j & -id_j \\ id_j & c_j \end{pmatrix} \quad \text{where } c_j^2 - d_j^2 = 1, c_j > 0.
$$

Let $S^1 := \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$, $\mathcal{H} := \{(x, y) \in \mathbb{R}^2; x^2 - y^2 = 1 \quad \text{and} \quad x > 0\}$, identify $H_1$ with $S^1 \times H \times \cdots \times H \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2m}$ and let

$$
\tilde{\psi} : \mathbb{R}^{2m} \to \mathbb{R}^{(1+n)\times(1+n)}, \quad \tilde{\psi}(a, b, c_1, d_1, \ldots, c_{m-1}, d_{m-1}) = \begin{pmatrix} \sigma & 0 & \cdots & 0 \\ 0 & \tau_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \tau_{m-1} \end{pmatrix}
$$

where $\sigma = \begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ and $\tau_j = \begin{pmatrix} c_j & -id_j \\ id_j & c_j \end{pmatrix}$. The restriction $\psi$ of $\tilde{\psi}$ to $S^1 \times \mathcal{H} \times \cdots \times \mathcal{H}$ is a diffeomorphism onto its image $H_1$.

For every $w_k \in T$, $k = 1, \ldots, N$ we get the linear map $\tilde{\varphi}_k : \mathbb{R}^{2m} \to \mathbb{R}^{1+n}$, $p \mapsto \text{Im}(\tilde{\psi}(p) \cdot w_k)$. Note that

- If $p = (p_1, \ldots, p_m) \in \tilde{\varphi}_k^{-1}(C)$, then $(p_1, \ldots, rp_j, \ldots, p_m) \in \tilde{\varphi}_k^{-1}(C)$ for all $0 < r \leq 1$ and $j = 2, \ldots, m$.
- If $p = (p_1, \ldots, p_m), p_j \in \tilde{\varphi}_k^{-1}(C)$, then $(s \cdot p_1, p_2, \ldots, p_m) \in \tilde{\varphi}_k^{-1}(C)$ for all $s > 1$.

where $p_1 = (a, b), p_j = (c_j, d_j) \in \mathbb{R}^2, j = 2, \ldots, m$.

It remains to show that $\tilde{\Sigma}_{H_1}(w)$ is connected for all $w \in T^N$.

Let $e := ((1, 0), (1, 0), \ldots, (1, 0)) = \psi^{-1}(\text{Id}) \in \psi^{-1}(\Sigma_{H_1}\epsilon(w))$ and $p = (p_1, \ldots, p_m) := \psi^{-1}(h) \in \psi^{-1}(\Sigma_{H_1}(w))$. From the convexity of $C$ and the linearity of $\tilde{\varphi}_k$ it follows that $q(t) = (q_1(t), \ldots, q_m(t)) = e + t(p - e)$ is contained in $\bigcap_{k=1}^N \tilde{\varphi}_k^{-1}(C)$ for $t \in [0, 1]$. Thus

$$
\tilde{\gamma}_p(t) := \begin{pmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_m(t) \end{pmatrix} \in \psi^{-1}(\Sigma_{H}(w))
$$

for $t \in [0, 1]$. Here $\| \cdot \|_E$ denotes the standard Euclidean norm. Thus $\gamma_h(t) := \psi(\tilde{\gamma}_p(t))$ gives a curve which connects $\text{Id}$ with $h$. \qed
Since $\text{SO}_R(1, n)^0$ is a real form of $\text{SO}_C(1, n)$, orbit connectness implies the following (see [He1])

**Corollary 3.1.** Let $Y$ be a complex space with a holomorphic $\text{SO}_C(1, n)$-action. Then every holomorphic $\text{SO}_R(1, n)^0$-equivariant map $\varphi : T^N \to Y$ extends to a holomorphic $\text{SO}_C(1, n)$-equivariant map $\Phi : \text{SO}_C(1, n) \cdot T^N \to Y$.

In the terminology of [He1] Corollary 3.1 means that $\text{SO}_C(1, n) \cdot T^N$ is the universal complexification of the $\text{SO}_R(1, n)^0$-space $T^N$.

### 4. The strictly plurisubharmonic exhaustion of the tube

Let $X, Q, P$ be topological spaces, $q : X \to Q$ and $p : X \to P$ continuous maps. A function $f : X \to \mathbb{R}$ is said to be an exhaustion of $X$ mod $p$ along $q$ if for every compact subset $K$ of $Q$ and $r \in \mathbb{R}$ the set $p(q^{-1}(K) \cap f^{-1}((-\infty, r]))$ is compact.

The characteristic function of the forward cone $C$ is up to a constant given by the function $\tilde{\rho} : C \to \mathbb{R}, \tilde{\rho}(y) = \eta(y) \frac{n+1}{2}$. It follows from the construction of the characteristic function, that $\log \tilde{\rho}$ is a $\text{SO}_R(1, n)^0$-invariant strictly convex function on $C$ (see [FK] for details). In particular

$$\rho : T^N \to \mathbb{R}, \quad (x_1 + iy_1, \ldots, x_N + iy_N) \mapsto \frac{1}{\eta(y_1)} + \cdots + \frac{1}{\eta(y_N)}$$

is a $\text{SO}_R(1, n)^0$-invariant strictly plurisubharmonic function on $T^N$. Of course this may also be checked by direct computation.

Let $\pi_C : \mathbb{C}^{(1+n) \times N} \to \mathbb{C}^{(1+n) \times N} // \text{SO}_C(1, n)$ be the analytic Hilbert quotient and $\pi_R : T^N \to T^N / \text{SO}_R(1, n)^0$ the quotient by the $\text{SO}_R(1, n)^0$-action. In the following we always write $z = x + iy$, i.e., $z_j = x_j + iy_j$ where $x_j$ denote the real and $y_j$ the imaginary part of $z_j$. For example $z_j \cdot z_k = x_j \cdot x_k - y_j \cdot y_k + i(x_j \cdot y_k + x_k \cdot y_j)$.

The main result of this section is the following

**Theorem 4.1.** The function $\rho : T^N \to \mathbb{R}$, is an exhaustion of $T^N$ mod $\pi_R$ along $\pi_C$.

We do the case of one copy first.

**Lemma 4.1.** Let $D_1 \subset T$ and assume that $\pi_C(D_1) \subset \mathbb{C}$ is bounded. Then $\{(x \cdot y, \eta(x), \eta(y)) \in \mathbb{R}^3 ; z = x + iy \in D_1\}$ is bounded.

**Proof.** The condition on $D_1$ means, that there is a $M \geq 0$ such that

$$|\eta(x) - \eta(y)| \leq M \quad \text{and} \quad |x \cdot y| \leq M$$

for all $z = x + iy \in D_1$. Since $\eta(x)\eta(y) \leq (x \cdot y)^2$ and $\eta(y) \geq 0$, this implies that $\{(x \cdot y, \eta(x), \eta(y)) \in \mathbb{R}^3 ; z \in D_1\}$ is bounded. $\square$
**Lemma 4.2.** Let $D_2 \subset T \times T$ be such that $\pi_C(D_2)$ is bounded. Then $\{(\eta(x_1), \eta(y_1), \eta(x_2), \eta(y_2), x_1 \cdot x_2, y_1 \cdot y_2) \in \mathbb{R}^6; (z_1, z_2) \in D_2\}$ is bounded.

**Proof.** Lemma 4.1 implies that there is a $M_1 \geq 0$ such that $|\eta(x_j)| \leq M_1$, $|\eta(y_j)| \leq M_1$ and $|x_j \cdot y_j| \leq M_1$, $j = 1, 2$, for all $(z_1, z_2) \in D_2$. Now $\eta(z_1 + z_2) = \eta(z_1) + \eta(z_2) + 2 \cdot z_1 \cdot z_2$ shows that $\{(\eta(z_1), \eta(z_2)) \in \mathbb{R}; (z_1, z_2) \in D_2\}$ is bounded. But $z_1 + z_2 \in T$, thus Lemma 4.1 implies $|\eta(x_1 + x_2)| \leq M_2$ and $|\eta(y_1 + y_2)| \leq M_2$ for some $M_2 \geq 0$ and all $(z_1, z_2) \in D_2$. This gives

$$|x_1 \cdot x_2| \leq \frac{3}{2} \max \{M_1, M_2\} \quad \text{and} \quad |y_1 \cdot y_2| \leq \frac{3}{2} \max \{M_1, M_2\}. \quad \square$$

**Remark 4.1.** Based on the following we only need, that the set $\{(\eta(y_1), \eta(y_2), y_1 \cdot y_2) \in \mathbb{R}^3; (z_1, z_2) \in D_2\}$ is bounded. We apply this to points $y_j + iy_1$ where $\pi_C(y_j + iy_1) = \eta(y_j) - \eta(y_1) + 2iy_j \cdot y_1$.

**Remark 4.2.** For every compact sets $B \subset C$ and $K \subset \mathbb{C}$ the set

$$M(B, K) := \{x \in \mathbb{R}^{1+n}; \pi_C(x + iy) \in K \text{ for some } y \in B\}$$

is compact.

**Proof.** Since $B$ and $K$ are compact, $M(B, K)$ is closed. We have to show that it is bounded. First note that $B_1 \subset B_2$ implies $M(B_1, K) \subset M(B_2, K)$. Using the properness of the $SO_{\mathbb{R}}(1, n)^0$-action on $C$, we see, that there is an interval $I = [t \cdot e_0; a \leq t \leq b]$, $a > 0$ in $\mathbb{R} \cdot e_0$ and a compact subset $N$ in $SO_{\mathbb{R}}(1, n)^0$, such that $N \cdot I := \bigcup_{g \in N} g \cdot I \supset B$. Thus $M(B, K) \subset M(N \cdot I, K) = N \cdot M(I, K) := \bigcup_{g \in N} g \cdot M(I, K)$.

It remains to show that $M(I, K)$ is bounded. For $x \in M(I, K), x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$, there exists a $M_1 \geq 0$ such that $|x \cdot (y_0 \cdot e_0)| = |x_0 \cdot y_0| \leq M_1$ for all $y_0 \cdot e_0 \in I$. Since $a \leq y_0 \leq b$ and $a > 0$, this implies $|x_0^2| \leq \frac{M_1^2}{|y_0|^2} \leq \frac{M_1^2}{a^2}$. There also exists a $M_2 \geq 0$ such that $|\eta(x)| = |x_0^2 - x_1^2 - \cdots x_n^2| \leq M_2$, so we get $x_1^2 + \cdots x_n^2 \leq \frac{M_1^2}{a^2} + M_2$. \quad \square

**Corollary 4.1.** For every $r > 0$ the set $M(B, K) \cap \{y \in \mathbb{R}^{1+n}; r \leq \eta(y)\}$ is compact.
Proof of Theorem 4.1. Using Remark 4.2 it is sufficient to prove that the set
\[ S := (\pi_C^{-1}(K) \cap \{ \rho \leq r \}) \cap ((\mathbb{R}^{1+n} + i(\mathbb{R}^{>0} \cdot e_0)) \times T^{N-1}) \]
is compact. For \( z = (z_1, \ldots, z_N) \in S \) let \( z_j = x_j + iy_j \), where \( x_j \) denotes the real part and \( y_j \) the imaginary part of \( z_j \). By the definition of \( S \) we have \( y_1 = y_{10} \cdot e_0 \) where \( y_{10} = y_1 \cdot e_0 \). Moreover, we get \( t^2 \leq \eta(y_1) = (y_{10})^2 \leq M \). Therefore the set \( \{ y_1 \in \mathbb{R}^{1+n}; (z_1, \ldots, z_N) \in S \} = \{ t \cdot e_0; t^2 \in [\frac{1}{2}, M], t > 0 \} \) is compact.

By Remark 4.1 we get that the sets \( \{ (\eta(y_1), \eta(y_j), y_1 \cdot y_j) \in \mathbb{R}^3; (z_1, \ldots, z_N) \in S \} \) are bounded for \( j = 2, \ldots, N \). Therefore we get the boundedness of \( \{ \pi_C(y_j + iy_1) \in \mathbb{C}; (z_1, \ldots, z_N) \in S \} \). Thus the \( y_j, j = 2, \ldots, N, \) with \( (z_1, \ldots, z_N) \in S \) are lying in the sets \( M(I, B_j) = \{ y \in \mathbb{R}^{1+n}; r \leq \eta(y) \}, \) where \( I := \{ t \cdot e_0; t^2 \in [\frac{1}{2}, M], t > 0 \} \) and \( B_j \) are compact subsets of \( \mathbb{C} \), containing \( \pi_C(y_j + iy_1) \in \mathbb{C}; (z_1, \ldots, z_N) \in S \}. \) By Corollary 4.1 these sets are compact, which implies that the set \( \{ y_1, \ldots, y_N \} \in \mathbb{R}^{(1+n)\times N}; (z_1, \ldots, z_N) \in S \) is compact. Hence using Lemma 4.3 it follows that \{ \{x_1, \ldots, x_N \} \in \mathbb{R}^{(1+n)\times N}; (z_1, \ldots, z_N) \in S \} \) is bounded. Thus \( S \) is bounded and therefore compact. \( \square \)

5. – Saturatedness of the extended future tube

We call \( A \subset X \) saturated with respect to a map \( \rho : X \to Y \) if \( A \) is the inverse image of a subset of \( Y \).

Let \( \pi_C : \mathbb{C}^{(1+n)\times N} \to \mathbb{C}^{(1+n)\times N}/\text{SO}_C(1, n) \) be the analytic Hilbert quotient, which is given by the algebra of \( \text{SO}_C(1, n) \)-invariant polynomials functions on \( \mathbb{C}^{(1+n)\times N} \) (see Section 1) and let \( U_r \) denote the set \( \{ z \in T^N; \rho(z) < r \} \) for some \( r \in \mathbb{R} \cup \{+\infty\} \), where \( \rho \) is the strictly plurisubharmonic exhaustion function, which we defined in Section 4.

Theorem 5.1. The set \( \text{SO}_C(1, n) \cdot U_r = \text{SO}_C(1, n) \cdot \{ z \in T^N; \rho(z) < r \} \) is saturated with respect to \( \pi_C \).

It is well known, that each fiber of \( \pi_C \) contains exactly one closed orbit of \( \text{SO}_C(1, n) \) (see Section 1). Moreover, every orbit contains a closed orbit in its closure. Therefore it is sufficient to prove

Proposition 5.1. If \( z \in U_r \) and \( \text{SO}_C(1, n) \cdot u \) is the closed orbit in \( \text{SO}_C(1, n) \cdot z \), then \( \text{SO}_C(1, n) \cdot u \cap U_r \neq \emptyset \).

The idea of proof is to construct a one-parameter group \( \gamma \) of \( \text{SO}_C(1, n) \), such that \( \gamma(t)z \in U_r \) for \( |t| \leq 1 \) and \( \lim_{t \to 0} \gamma(t)z = \text{SO}_C(1, n) \cdot u \).

In the following, let \( z = (z_1, \ldots, z_N) \in U_r \) and denote by \( L(z) = \mathbb{C}z_1 + \cdots + \mathbb{C}z_N \) the \( \mathbb{C} \)-linear subspace of \( \mathbb{C}^{1+n} \) spanned by \( z_1, \ldots, z_N \). The subspace
of isotropic vectors in $L(z)$ with respect to the Lorentz product is denoted by $L(z)^0$, i.e., $L(z)^0 = \{ w \in L(z); w \cdot v = 0 \text{ for all } v \in L(z) \}$. Let $\overline{L(z)^0}$ be its conjugate, i.e., $L(z)^0 = \{ \overline{v}; v \in L(z)^0 \}$.

**Lemma 5.1.** For all $\omega \neq 0$, $\omega \in L(z)^0$ we have $\eta(\text{Im}(\omega)) < 0$.

**Proof.** Let $\omega = \omega_1 + i\omega_2$ with $\omega_1 = \text{Re}(\omega), \omega_2 = \text{Im}(\omega)$. Assume that $\eta(\text{Im}(\omega)) = \eta(\omega_2) \geq 0$. Since $\omega \in L(z)^0$, we have $0 = \eta(\omega) = \eta(\omega_1) - \eta(\omega_2) + 2i\omega_1 \cdot \omega_2$.

If $\eta(\omega_2) > 0$, i.e., $\omega_2 \in C$ or $\omega_2 \in -C$, then $\omega_1 \cdot \omega_2 = 0$ contradicts $\eta(\omega_2) = \eta(\omega_2) > 0$. Thus assume $\eta(\omega_1) = \eta(\omega_2) = 0$ and $\omega_1 \cdot \omega_2 = 0$. Hence $\omega_1$ and $\omega_2$ are $\mathbb{R}$-linearly dependent and therefore there is a $\lambda \in \mathbb{C}$, $\omega_3 \in \mathbb{R}^{1+n}$ such that $\omega = \lambda \omega_3$ and $\omega_3 \cdot e_0 \geq 0$. We have $\eta(\omega_3) = 0$ and, since $\omega_3 \in L(z)^0, e_0 \cdot \omega_3 \geq 0$ and $z_1 \in T$, we also have $0 = \omega_3 \cdot \text{Im}(z_1)$. This implies by the definition of $C$ that $\omega_3 = 0$.

**Corollary 5.1.** For $\omega \in L(z)^0, \omega \neq 0$, we have $\omega \cdot \overline{\omega} < 0$. In particular, $L(z)^0 \cap \overline{L(z)^0} = \{0\}$ and the complex Lorentz product is non-degenerate on $L(z)^0 \oplus \overline{L(z)^0}$.

**Corollary 5.2.** Let $W := (L(z) \oplus \overline{L(z)})^\perp := \{ v \in \mathbb{C}^{1+n}; v \cdot u = 0 \text{ for all } u \in L(z)^0 \oplus \overline{L(z)^0} \}$. Then

$$L(z) = L(z)^0 \oplus (L(z) \cap W).$$

**Proof of Proposition 5.1.** Let $z \in U_r$. We use the notation of Corollary 5.2. Define

$$\gamma : \mathbb{C}^* \to \text{SO}_C(1,n) \text{ by } \gamma(t)v = \begin{cases} tv & \text{for } v \in L(z)^0 \\ t^{-1}v & \text{for } v \in \overline{L(z)^0} \\ v & \text{for } v \in W \end{cases}.$$ 

Every component $z_j$ of $z$ is of the form $z_j = u_j + \omega_j$ where $u_j \in W$ and $\omega_j \in L(z)^0$ are uniquely determined by $z_j$. Recall that $W$ is the set $\{ v \in \mathbb{C}^{1+n}; v \cdot u = 0 \text{ for all } u \in L(z)^0 \oplus \overline{L(z)^0} \}$. Since $\lim_{t \to 0} \gamma(t)z_j = u_j$ and $L(u)^0 = \{0\}$ for $u = (u_1, \ldots, u_N), w$ lies in the unique closed orbit in $\text{SO}_C(1,n), z$ (see Lemma 2.1). It remains to show that $u \in U_r$. For every $t \in \mathbb{C}$ we have

$$\eta(\text{Im}(u_j + t\omega_j)) = \eta(\text{Im}(u_j)) + |t|^2 \eta(\text{Im}(\omega_j)).$$

Since $\eta(\text{Im}(u_j + \omega_j)) > 0$ and $\eta(\text{Im}(\omega_j)) \leq 0$, this implies $\eta(\text{Im}(u_j + t\omega_j)) \in C^\pm$ for all $t \in [0,1]$. Moreover, $\eta(\text{Im}(z_j)) < \eta(\text{Im}(u_j))$, for every $j$. Thus $\rho(z) > \rho(u)$ and therefore $u \in U_r$.

**Corollary 5.3.** The extended future tube is saturated with respect to $\pi_C$.

**Remark 5.1.** The function $f : \mathbb{R} \to \mathbb{R}, t \mapsto \eta(\text{Im}(u_j + t\omega_j))$, is strictly concave if $\omega_j \neq 0$. The proof shows $u_j + t\omega_j \in T$ for all $t \in \mathbb{R}$.
6. – The Kählerian reduction of the extended future tube

If one is only interested in the statement of the future tube conjecture, one can simply apply the main result in [He2] (Theorem 1 in Section 2). Our goal here is to show that much more is true.

For \( z \in \mathbb{C}^{(1+n) \times N} \) let \( x = \frac{1}{2}(z + \bar{z}) \) be the real and \( y = \frac{1}{2i}(z - \bar{z}) \) the imaginary part of \( z \), i.e., \( z = (z_1, \ldots, z_N) = (x_1, \ldots, x_N) + i(y_1, \ldots, y_N) \) in the obvious sense. The strictly plurisubharmonic function \( \rho : T^N \to \mathbb{R} \), \( \rho(z) = \frac{1}{\eta(y_1)} + \cdots + \frac{1}{\eta(y_N)} \) defines for every \( \xi \in \mathfrak{so}(1, n) = \mathfrak{o}(1, n) \) the function

\[
\mu_\xi(z) = d\rho(z)(i\xi z) = \frac{d}{dt}\Big|_{t=0} \rho(\exp(it\xi \cdot z)).
\]

Here of course \( \mathfrak{so}(1, n) = \mathfrak{o}(1, n) \) denotes the Lie algebra of \( \mathfrak{O}(1, n) \). The real group \( \text{SO}(1, n)^0 \) acts by conjugation on \( \mathfrak{so}(1, n) \) and therefore by duality on the dual vector space \( \mathfrak{so}(1, n)^* \). It is easy to check that the map \( \xi \to \mu_\xi \) depends linearly on \( \xi \). Thus

\[
\mu : T^N \to \mathfrak{so}(1, n)^*; \quad \mu(z)(\xi) := \mu_\xi(z).
\]

is a well defined \( \text{SO}_\mathbb{R}(1, n)^0 \)-equivariant map. In fact \( \mu \) is a moment map with respect to the Kähler form \( \omega = 2i\partial\bar{\partial}\rho \).

In order to emphasize the general ideas, we set \( G := \text{SO}(1, n)^0 \), \( G^C := \text{SO}_\mathbb{C}(1, n) \), \( X := T^N \) and \( Z := G^C \cdot X \). The corresponding analytic Hilbert quotient, induced by \( \pi_C : \mathbb{C}^{(1+n) \times N} \to \mathbb{C}^{(1+n) \times N} / / \text{SO}_\mathbb{C}(1, n) \) are denoted by \( \pi_X : X \to X / G \), \( \pi_Z : Z \to Z / G^C \). Note that, by what we proved, we have \( X / G = Z / G^C \).

**Proposition 6.1.**

i. For every \( q \in Z / G^C \) we have \((\pi_C)^{-1}(q) \cap \mu^{-1}(0) = G \cdot x_0\) for some \( x_0 \in \mu^{-1}(0) \) and \( G^C \cdot x_0 \) is a closed orbit in \( Z \).

ii. The inclusion \( \mu^{-1}(0) \overset{\iota}{\to} X \subset Z \) induces a homeomorphism \( \mu^{-1}(0)/G \overset{\iota}{\to} Z / G^C \).

**Proof.** A simple calculation shows that the set of critical points of \( \rho|G^C \cdot x \cap X \), i.e., \( \mu^{-1}(0) \cap G^C \cdot x \), consists of a discrete set of \( G \)-orbits. Moreover, every critical point is a local minimum (see [He2], Proof of Lemma 2 in Section 2).

On the other hand Remark 5.1 of Section 5 says that if \( \rho|G^C \cdot x \cap X \) has a local minimum in \( x_0 \in G^C \cdot x \cap X \), then \( G^C \cdot x_0 = G^C \cdot x \) is necessarily closed in \( Z \). Moreover, \( \rho|G^C \cdot x \cap X \) is then an exhaustion and therefore \( \mu^{-1}(0) \cap (G^C \cdot x_0 \cap X) = G \cdot x_0 \) (see [He2], Lemma 2 in Section 2). This proves the first part.

The statement i. implies that \( \iota : \mu^{-1}(0) \overset{\iota}{\to} X \subset Z \) induces a bijective continuous map \( \iota : \mu^{-1}(0)/G \to Z / G^C \). Since the \( G \)-action on \( X \) is proper and \( \mu^{-1}(0) \) is closed, the action on \( \mu^{-1}(0) \) is proper. In particular \( \mu^{-1}(0)/G \) is a Hausdorff topological space.
Theorem 5.1 implies that $\bar{\iota}$ is a homeomorphism, since for every sequence $q_\alpha \to q_0$ in $Z//G^C$ we find a sequence $(x_\alpha)$ such that $x_\alpha$ are contained in a compact subset of $\mu^{-1}(0)$ and $\pi_C(x_\alpha) = q_\alpha$. Thus every convergent subsequence of $(x_\alpha)$ has a limit point in $G \cdot x_0$ where $\pi_C(x_0) = q_0$. 

**Proposition 6.2.** The restriction $\rho|\mu^{-1}(0) : \mu^{-1}(0) \to \mathbb{R}$ induces a strictly plurisubharmonic continuous exhaustion $\bar{\rho} : Z//G^C \to \mathbb{R}$.

**Proof.** The exhaustion property for $\bar{\rho}$ follows from Theorem 4.1. The argument that $\bar{\rho}$ is strictly plurisubharmonic is the same as in [HeHuL].

**Theorem 6.1.** The extended future tube $Z$ is a domain of holomorphy.

**Proof.** Proposition 6.2 implies that $Z//G^C$ is a Stein space (see [N] Theorem II). Hence $Z$ is a Stein space.

In fact, much more has been proved here. We would like to comment on this. By definition, an analytic subset of a complex manifold is closed. For the following recall that orbit-connectedness is a condition on the $G^C$-orbits.

**Proposition 6.3.** Every analytic $G$-invariant subset $A$ of $X$ is orbit connected in $Z$ and $G^C \cdot A$ is an analytic subset of $Z$. In particular, $G^C \cdot A$ is a Stein space. Moreover the restriction maps

$$\mathcal{O}(Z)^{G^C} \to \mathcal{O}(G^C \cdot A)^{G^C} \to \mathcal{O}(A)^G$$

are surjective.

**Proof.** If $b \in G^C \cdot A \cap X$, then $b = g \cdot a$ for some $g \in G^C$ and $a \in A$. Hence $g \in \Sigma_G(a) = \{g \in G^C; g \cdot a \in X\}$. The identity principle for holomorphic functions shows that $\Sigma_G(a) \cdot a \in A$. Thus $b \in A$ This shows $G^C \cdot A \cap X = A$. But $\{g \cdot X; g \in G^C\}$ is an open covering of $X$ such that $G^C \cdot A \cap g \cdot X = g \cdot A$. This shows that $G^C \cdot A$ is an analytic subset of $Z$. In particular, it is a Stein space. The last statement follows from orbit connectedness (see [He1]).

**Proposition 6.4.** For every $G$-invariant analytic subset $A$, its saturation $\hat{A} = \pi_X^{-1}(\pi_X(A))$ is an analytic subset of $X$. Moreover, $\hat{A}//G$ is canonically isomorphic to $A//G$ and $\pi_{\hat{A}} : \hat{A} \to \hat{A}//G \subset X//G$ is the Hilbert quotient of $\hat{A}$ whose restriction to $A$ gives the analytic Hilbert quotient of $A$.

**Proof.** We already know that $A^c = G^C \cdot A$ is an analytic subset of $Z$. Its saturation $\hat{A}^c = \pi_Z^{-1}(\pi_Z(A^c)) = \pi_Z^{-1}(\pi_Z(A))$ is an analytic subset of $Z$ and it is easily checked that $\hat{A} = \hat{A}^c \cap X = \pi_X^{-1}(\pi_X(A))$ has the desired properties.

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