Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. III (2004), pp. 871-896

Rectifiability and Parameterization of Intrinsic Regular Surfaces in the Heisenberg Group

BERND KIRCHHEIM – FRANCESCO SERRA CASSANO

Abstract. We construct an intrinsic regular surface in the first Heisenberg group $\mathbb{H}^1 \equiv \mathbb{R}^3$ equipped wiht its Carnot-Carathéodory metric which has Euclidean Hausdorff dimension 2.5. Moreover we prove that each intrinsic regular surface in this setting is a 2-dimensional topological manifold admitting a $\frac{1}{2}$-Hölder continuous parameterization.

Mathematics Subject Classification (2000): 28A75 (primary); 28A78, 22E25 (secondary).

1. – Introduction

In this paper we investigate Euclidean rectifiability and existence of Hölder parameterization for $\mathbb{H}$-*regular surfaces*, a class of intrinsically regular surfaces in the Heisenberg group $\mathbb{H}^1$, which can be represented as $\mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3$ endowed with a left invariant metric d equivalent to its Carnot-Carathéodory metric (see Section 2 for a precise definition). This notion of intrinsically regular surface was introduced in order to study in the setting of Carnot groups, of which $\mathbb{H}^1$ is the simplest example, the classical problem in Geometric Measure Theory (GMT) of defining regular hypersurfaces (i.e. topological submanifold of codimension 1) and different reasonable surface measures on them (see [48], [47], [46], [58], [10] [34], [36] [31], [14], [25], [30], [13], [1], [2], [26], [49], [44], [27], [38], [29] and [39]). Throughout this paper, we shall denote the points of $\mathbb{H}^1$ by $P = [z, t] = [x + iy, t]$, $z \in \mathbb{C}$, $x, y \in \mathbb{R}$, $t \in \mathbb{R}$. If $P = [z, t]$, $Q = [\zeta, \tau] \in \mathbb{H}^1$ and $r > 0$, following the notations of [57], where the reader can find an exhaustive introduction to the Heisenberg group, we define the group operation

$$P \cdot Q := [z + \zeta, t + \tau + 2\Im m(z\bar{\zeta})] \tag{1}$$

The second author was supported by University of Trento, Italy, by GNAMPA, Italy and by MURST, Italy.
Pervenuto alla Redazione il 7 gennaio 2004 e in forma definitiva il 22 ottobre 2004.

and a family of non isotropic dilations

$$\delta_r(P) := [rz, r^2 t], \ \text{for } r > 0 . \tag{2}$$

Moreover $\mathbb{H}^1$ can be endowed with the homogeneous norm

$$\|P\|_\infty := \max\{|z|, |t|^{1/2}\} \tag{3}$$

and the distance d we shall deal with will be defined as

$$d(P, Q) := \|P^{-1} \cdot Q\|_\infty . \tag{4}$$

It is known that $\mathbb{H}^1$ is a Lie group of topological dimension 3, whereas the Hausdorff dimension of $(\mathbb{H}^1, d)$ is 4 (see Proposition 2.1). This phenomenon is already evident from the intrinsic isoperimetric inequality in $\mathbb{H}^1$ proved first by P. Pansu (see [48] and [47]), and then in a different form but in the general framework of Carnot-Carathéodory spaces by several authors (see, e.g., [60], [61], [11], [23], [10], [30], [32] and for a general discussion on the geometry of Carnot-Carathéodory spaces consult also [31] and [7]).

$\mathbb{H}^1$ provides the simplest example of a metric space that is not Euclidean, even locally, but is still endowed with a sufficiently rich underlying structure, due to the existence of intrinsic families of translations and dilations. Indeed, the geometry of $\mathbb{H}^1$ is noneuclidean at every scale, since it was proved by S. Semmes ([54]) that there are no bilipschitz maps from $\mathbb{H}^1$ to any Euclidean space. This fact relies on deep interlacing algebraic and metric properties related to the non-commutativity of $\mathbb{H}^1$ through a Rademacher type theorem due to P. Pansu ([46]). Our interest can be viewed in the framework of the general project meant to develop GMT in the setting of metric spaces. Such a project, already embrionically contained in Federer's book [22], has been explicitly formulated and carried on in the last few years by De Giorgi ([19], [20], [21]), Preiss and Tisěr ([50]), Kirchheim ([34]), David & Semmes ([14]), Ambrosio & Kirchheim ([1], [2]) and Lorent ([37]).

It is well known that the Lie algebra of left invariant vector fields in $\mathbb{H}^1$ is (linearly) generated by

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t} , \tag{5}$$

the only non-trivial commutator relations being

$$[X, Y] = -4T . \tag{6}$$

Throughout this paper, we shall identify vector fields and associated first order differential operators; thus the vector fields X, Y generate a vector bundle on $\mathbb{H}^1$, the so called horizontal vector bundle $\mathrm{H}\mathbb{H}^1$ according to the notation of Gromov, (see [31] and [36]), that is a vector subbundle of $\mathrm{T}\mathbb{H}^1$, the tangent vector bundle of $\mathbb{H}^1$. Since each fiber of $\mathrm{H}\mathbb{H}^1$ can be canonically identified with a vector subspace of $\mathbb{R}^3$, each section ϕ of $\mathrm{H}\mathbb{H}^1$ can be identified with a map $\phi : \mathbb{H}^1 \to \mathbb{R}^3$. At each point $P \in \mathbb{H}^1$ the horizontal fiber is indicated as $\mathrm{H}\mathbb{H}^1_P$ and each fiber can be endowed with the scalar product $\langle \cdot, \cdot \rangle_P$ and the associated norm $|\cdot|_P$ that make the vector fields X, Y orthonormal. Hence we shall also identify a section of $\mathrm{H}\mathbb{H}^1$ with its canonical coordinates with respect to this moving frame. In this way, a section ϕ will be identified with a function $\phi = (\phi_1, \phi_2) : \mathbb{H}^1 \to \mathbb{R}^2$. Analogously, if f is a real function defined in an open subset $\Omega \subset \mathbb{H}^1$, its $\mathbb{H}$-*gradient* is the section of $\mathrm{H}\mathbb{H}^1$ defined by $\nabla_{\mathbb{H}} f = (Xf, Yf)$.

To introduce our results, let us start by recalling some related notions already existing in the literature.

The first key point we want to stress here deals with the meaning of rectifiability in $\mathbb{H}^1$. Basically, in the Euclidean framework, a set $F \subset \mathbb{R}^n$ is (countably) $(n-1)$-rectifiable (from now on, we shall say only 'Euclidean rectifiable') if, roughly speaking, it is, up to a $\mathcal{H}^{n-1}$-negligible set, a countable union of compact subsets K_j of good hypersurfaces (i.e. Lipschitz or continously differentiable hypersurfaces) where $\mathcal{H}^m$ denotes the Euclidean m-dimensional measure on $\mathbb{R}^n$. Looking for a similar statement in the setting of the Heisenberg group (or, in general, of a metric space), we must ask preliminarily what are the good hypersurfaces in $\mathbb{H}^1$. In fact, there is a classical notion of rectifiability in a metric space that goes back to Federer (see [22], 3.2.14) that has been recently used by Ambrosio & Kirchheim (see [1], [2]) in the framework of a theory of currents in metric spaces (as for the rectifiability in metric spaces see for instance [34], [50] and also the monograph [42] and the references therein). According to this notion, a good surface in a metric space should be the image of an open subset of an Euclidean space via a Lipschitz map. Unfortunately, such a notion does not fit the geometry of the Heisenberg group, that indeed would be, according with this definition, purely unrectifiable (see [1]). On the other hand, in the Euclidean setting $\mathbb{R}^n$, a $\mathbf{C}^1$-hypersurface can be equivalently viewed as the (local) set of zeros of a function $f : \mathbb{R}^n \to \mathbb{R}$ with non-vanishing gradient. Such a notion can be easily transposed to the Heisenberg group, since there is an intrinsic notion of $\mathbf{C}^1_{\mathbb{H}}$-functions: we can say that a continuous real function f on $\mathbb{H}^1$ belongs to $\mathbf{C}^1_{\mathbb{H}}$ if $\nabla_{\mathbb{H}} f$ (in the sense of distributions) is a continuous vector-valued function. Thus, an $\mathbb{H}$-regular surface S will be locally defined as the set of points $P \in \mathbb{H}^1$ such that $f(P) = 0$, provided that $\nabla_{\mathbb{H}} f \neq 0$ on S (see Definition 2.21). A few comments are now in place to point out similar geometric properties (in the measure theoretical sense) of the $\mathbb{H}$-regular surfaces and classical (Euclidean) regular surfaces and to mention some of their applications.

First of all, we emphasise that the class of $\mathbb{H}$-regular surfaces is different from the class of Euclidean regular surfaces, in the sense that there are $\mathbb{H}$-regular surfaces that are not Euclidean continuously differentiable submanifolds, and conversely there are continuously differentiable 2-submanifolds in $\mathbb{R}^3$ that are not $\mathbb{H}$-regular hypersurfaces (see [26], Remark 6.2 and Example 2). We notice that Euclidean continuously differentiable 2-manifolds are $\mathbb{H}$-regular surfaces provided they do not contain characteristic points, i.e. points P such that the Euclidean tangent space at P coincides with the horizontal fiber $H\mathbb{H}^1_P$ at P. Frobenius theorem yields that, for a general smooth manifold, the set of characteristic points has empty interior; in fact there are few characteristic points ([13], [4], [39]). On the other hand the boundary of a smooth bounded set with trivial topology in $\mathbb{H}^1$ does always contain characteristic points. The fact that these points should not be allowed in $\mathbb{H}$-regular submanifolds is not surprising: for example it is already well known from the theory of subelliptic pde's that characteristic points of the boundary can behave like cusps for the Laplace operator.

Another important point supporting the choice of the notion of $\mathbb{H}$-regular surfaces is the fact that this definition fits with an Implicit Function Theorem, proved in [26] for the Heisenberg group and in [27] for a general Carnot group, so that a $\mathbb{H}$-regular manifold S has a local continuous parameterization

$$\Phi : I \subset (\mathbb{R}^2, |\cdot|) \to (S, d) \tag{7}$$

for a suitable rectangle $I \subset \mathbb{R}^2$ (see Theorem 2.23 below). In general, such a parameterization is not continuously differentiable or even Lipschitz continuous (see [26], Example 3), but from Φ we see that S is a topological submanifold of dimension 2 in $(\mathbb{H}^1, d)$. On the other hand, by using again the Implicit Function Theorem and the Blow-Up theorem (see Theorem 2.24), an area type formula for the 3-dimensional spherical Hausdorff measure $\mathcal{S}^3_d$ in $(\mathbb{H}^1, d)$ and the existence of the tangent group in the sense of GMT for $\mathbb{H}$-regular surfaces were established (see [26] and [27]).

More precisely, a local representation of $\mathcal{S}^3_d(S)$ was given in terms of the parameterization defined in (7) and with respect to the 2-dimensional Lebesgue measure on $\mathbb{R}^2$ (see Theorems 2.23 (vi) and 2.24 (ii)). In particular, we infer that the Hausdorff dimension of S in $(\mathbb{H}^1, d)$ equals 3. Moreover, if we define the *tangent group* $T^g_{\mathbb{H}}S(P)$ to $S = \{f = 0\}$ at P as

$$T^g_{\mathbb{H}}S(P) := \{[x + iy, t] \in \mathbb{H}^1 : Xf(P)x + Yf(P)y = 0\},$$

then it is a proper subgroup of $\mathbb{H}^1$ and

$$\lim_{r\to 0} \frac{\mathcal{S}^3_d(S \cap U(P, r))}{r^3} = \mathcal{H}^2(T^g_{\mathbb{H}}S(P) \cap U(0, 1)) = 4$$

exists for every $P \in S$ being $U(P, r)$ the open ball centered at P with radius $r > 0$ with respect to the distance d (see Theorem 2.24).

Based on this, also the notion of $\mathbb{H}$-rectifiability was introduced: a set $\Gamma \subset \mathbb{H}^1$ is said 3-*dimensional* $\mathbb{H}$-*rectifiable* if there exists a sequence of $\mathbb{H}$-regular surfaces $(S_i)_i$ in $\mathbb{H}^1$ such that $\mathcal{S}^3_d(\Gamma \setminus \cup_{i\in\mathbb{N}} S_i) = 0$. This intrinsic notion of rectifiability has been proven particularly useful to obtain in [26] an analog of De Giorgi's structure theorem for sets of intrinsic finite perimeter in the setting of Heisenberg group, and more recently in the setting of a general Carnot group of step 2 ([28] and [29]). The notions of Euclidean and $\mathbb{H}$-rectifiability have been compared in [5], generalizations of this notion of rectifiability recently have been studied by V. Magnani in [39] for general Carnot groups.

In this paper we will stress another aspect of the deep difference between the Euclidean and Carnot-Carathéodory geometry from GMT's point of view. In fact, we will exhibit an $\mathbb{H}$-regular surface $S_0 \subset \mathbb{H}^1 \equiv \mathbb{R}^3$ which looks as an Euclidean fractal set. Indeed it has Hausdorff dimension 2.5 in $(\mathbb{R}^3, |\cdot|)$ and, consequently, is not Euclidean 2-rectifiable (see Theorem 3.1). Notice that an example of a set $F \subset \mathbb{H}^1$ having Hausdorff dimension 2 in $(\mathbb{H}^1, d)$ which looks like an Euclidean fractal set was already constructed by R. Strichartz ([58]) but it is not a topological surface set because its topological dimension cannot be 2 (see Remark 3.2 and also [6]).

Finally, we will improve the regularity of the parameterization in (7) given by the Implicit Function Theorem. We will actually prove it is locally Hölder continuous of order $1/2$ and this result is in some sense sharp. Indeed, there are $\mathbb{H}$-regular surfaces not admitting Hölder parameterization of order better than $1/2$ (see Theorem 4.1).

Let us recall that the problem of good parameterizations of hypersurface-type set in Euclidean spaces was studied in [51], [52], [53], [59], [15] (see also [54] and [55] for the problem in a general metric space). In particular the problem of the best Hölder parameterization for an Ahlfors regular subset of $\mathbb{R}^n$ has been studied in [40] and [41], while for a (Euclidean) submanifold in a Carnot group it arose in [31]. Eventually, the problem of characterizing $\mathbb{H}$-regular surfaces as images under Lipschitz maps of a suitable "sample" metric space having 3-dimensional positive and finite Hausdorff measure has been proposed in [26] and it is essentially open. A partial answer has been given in [49] by S. Pauls for some hypersuperfaces of special Carnot groups but it does not apply to the Heisenberg group $\mathbb{H}^1$.

ACKNOWLEDGMENTS. We wish to thank R. Peirone for some hints on the construction of a prelimary version of the example in Theorem 3.1. We also thank Z. Balogh, B. Franchi and R. Serapioni for useful discussions on the subject. Finally we are grateful to the referee for valuable comments and suggestions. In particular for the proposed conjecture in Remark 4.3 we were not able to answer.

2. – Notations and preliminary results

In this section we introduce the basic notation and recall some known results. We denote by $\tau_P : \mathbb{H}^1 \to \mathbb{H}^1$ the left-translation by P defined as

$$Q \mapsto \tau_P(Q) := P \cdot Q$$

for any fixed $P \in \mathbb{H}^1$ where "$\cdot$" denotes the group law defined in (1). We denote as $P^{-1} := [-z, -t]$ the inverse of P and as 0 the origin of $\mathbb{R}^3$. For further reference, we explicitely state that

PROPOSITION 2.1. *The function d defined by* (4) *is a distance in $\mathbb{H}^1$ and the usual invariance and scaling properties related to translations and dilations hold, i.e.* $\forall P, Q, Q' \in \mathbb{H}^1$ *and* $\forall r > 0$

$$d(\tau_P Q, \tau_P Q') = d(Q, Q') \quad \text{and} \quad d(\delta_r Q, \delta_r Q') = r\, d(Q, Q')\,. \tag{8}$$

In addition, for any bounded subset Ω of $\mathbb{H}^1$ there exist positive constants $c_1(\Omega)$, $c_2(\Omega)$ such that

$$c_1(\Omega)|P - Q|_{\mathbb{R}^3} \le d(P, Q) \le c_2(\Omega)|P - Q|_{\mathbb{R}^3}^{1/2} \tag{9}$$

for $P, Q \in \Omega$. In particular, the topologies defined by d and by the Euclidean distance coincide on $\mathbb{H}^1$.

REMARK 2.2. We stress that, because the topologies defined by d and by the Euclidean distance coincide, the topological dimension of $\mathbb{H}^1$ is 3. On the contrary, the Hausdorff dimension of $(\mathbb{H}^1, d)$ is 4.

From now on, $U(P, r)$ will be the open ball with centre P and radius r with respect to the distance d.

It is well-known that the 3-dimensional Lebesgue measure $\mathcal{L}^3$ on $\mathbb{H}^1 \equiv \mathbb{R}^3$ is left (and right) invariant and it is the Haar measure of the group. If $E \subset \mathbb{H}^1$ then we write $|E|$ for its Lebesgue measure.

DEFINITION 2.3. We shall denote respectively by $\mathcal{H}^m$ and $\mathcal{S}^m$ the m-dimensional Hausdorff and the spherical Hausdorff measure obtained from the Euclidean distance $|\cdot|$ in $\mathbb{R}^3 \equiv \mathbb{H}^1$ according to their classical definitions (see [22]). Instead of, we shall denote respectively by $\mathcal{H}_d^m$ and $\mathcal{S}_d^m$ the m-dimensional Hausdorff and the spherical Hausdorff measure obtained from the distance d in $\mathbb{H}^1$ according to the definition given in [42] for a general metric space.

Translation invariance and homogeneity under dilations of the Hausdorff measure follow as usual from (8), more precisely we have

PROPOSITION 2.4. *Let $A \subseteq \mathbb{H}^1$, $P \in \mathbb{H}^1$ and $m, r \in (0, \infty)$. Then*

$$\mathcal{H}_d^m(\tau_P A) = \mathcal{H}_d^m(A)$$
$$\mathcal{H}_d^m(\delta_r(A)) = r^m \mathcal{H}_d^m(A)\,.$$

In the following we shall identify the vector fields and the associated first order differential operators. The vector fields X, Y define a vector bundle on $\mathbb{H}^1$ (the horizontal vector bundle $H\mathbb{H}^1$) that can be canonically identified with a vector subbundle of the tangent vector bundle of $\mathbb{R}^3$. Since each fiber of $H\mathbb{H}^1$ can be in a canonic way understood as a vector subspace of $\mathbb{R}^3$, each section

ϕ of $H\mathbb{H}^1$ is associated with a map $\phi : \mathbb{H}^1 \to \mathbb{R}^3$. At each point $P \in \mathbb{H}^1$ the horizontal fiber is indicated as $H\mathbb{H}^1_P$ and each fiber can be endowed with the scalar product $\langle\cdot,\cdot\rangle_P$ and the norm $|\cdot|_P$ that make the vector fields X, Y orthonormal. Hence we shall also identify a section of $H\mathbb{H}^1$ with its canonical coordinates with respect to this moving frame. In this way, a section ϕ will be identified with a function $\phi = (\phi_1, \phi_2) : \mathbb{H}^1 \to \mathbb{R}^2$. As it is common in Riemannian geometry, when dealing with two sections ϕ and ψ whose argument is not explicitely written, we shall drop the index P in the scalar product writing $\langle\psi, \phi\rangle$ for $\langle\psi(P), \phi(P)\rangle_P$. The same convention shall be adopted for the norm.

For sake of completness, let us recall here the definition of the Carnot–Carathéodory metric associated with X, Y. In fact, this definition has been developed in a much more general setting (see, e.g., [45]).

DEFINITION 2.5. We say that an absolutely continuous curve $\gamma : [0, T] \to \mathbb{H}^1$ is a sub-unit curve with respect to X, Y if there exist real measurable functions $a_1(s), a_2(s)$, $s \in [0, T]$ such that $a_1^2 + a_2^2 \leq 1$ and

$$\dot\gamma(s) = a_1(s)X(\gamma(s)) + a_2(s)Y(\gamma(s)), \qquad \text{for a.e. } s \in [0, T].$$

If $P_1, P_2 \in \mathbb{H}^1$, their Carnot–Carathéodory distance $d_C(P_1, P_2)$ is

$$\begin{aligned} &d_C(P_1, P_2)\\ &\quad= \inf\{T > 0 : \text{ there is a subunit curve } \gamma : [0, T] \to \mathbb{H}^1, \gamma(0) = P_1,\ \gamma(T) = P_2\}. \end{aligned}$$

Notice that the above set of curves joining P_1 and P_2 is not empty, by Chow's theorem, since by (6) the rank of the Lie algebra generated by X, Y is 3, and hence d_C is a distance on $\mathbb{H}^1$.

REMARK 2.6. Alternatively, sub-unit curves can be defined as absolutely continuous functions γ such that $\dot\gamma$ is a measurable section of $H\mathbb{H}^1$ such that $|\dot\gamma(s)|_{\gamma(s)} \leq 1$ for a.e. s.

The following results are well known: see, for instance, [7], [61].

PROPOSITION 2.7. *The Carnot–Carathéodory distance d_C is (globally and bilipschitzly) equivalent to the distance d defined in* (4).

PROPOSITION 2.8.

$$\mathcal{L}^3 = c_S(d)\mathcal{S}^4_d = c_H(d)\mathcal{H}^4_d. \tag{10}$$

In particular (as proved in [43] *and* [48]) *the Hausdorff dimension of* $(\mathbb{H}^1, d)$ *and* $(\mathbb{H}^1, d_C)$ *is* 4.

Due to its definition and normalization, the Q-dimensional spherical measure always has on a homogeneous group of dimension Q as the Heisenberg group density 1, so it easily follows that $c_S(d) = \mathcal{L}^3(U_d(0, 1))$. It is also well known that $c_H(d) > c_S(d)$.

If Ω is an open subset of $\mathbb{H}^1$ and $k \geq 0$ is a non negative integer, the symbols $\mathbf{C}^k(\Omega)$, $\mathbf{C}^\infty(\Omega)$ indicate the usual spaces of real valued functions which are (sufficiently often) continuously differentiable in the Euclidean sense. We denote by $\mathbf{C}^k(\Omega, \mathrm{H}\mathbb{H}^1)$ the set of all C^k-sections of $\mathrm{H}\mathbb{H}^1$ where the C^k regularity is understood as regularity between smooth manifolds. The notions of $\mathbf{C}^k_0(\Omega, \mathrm{H}\mathbb{H}^1)$, $\mathbf{C}^\infty(\Omega, \mathrm{H}\mathbb{H}^1)$ and $\mathbf{C}^\infty_0(\Omega, \mathrm{H}\mathbb{H}^1)$ are defined analogously.

The similar structure of some statements in $\mathbb{H}^1$ with others in $\mathbb{R}^3$ becomes transparent using the intrinsic notions of a gradient for functions $\mathbb{H}^1 \to \mathbb{R}$ and of divergence for sections of $\mathrm{H}\mathbb{H}^1$.

Definition 2.9. If Ω is an open subset of $\mathbb{H}^1$, $f \in \mathbf{C}^1(\Omega)$ and $\phi = (\phi_1, \phi_2) \in \mathbf{C}^1(\Omega, \mathrm{H}\mathbb{H}^1)$, define

$$\nabla_{\mathbb{H}} f := (Xf, Yf) \tag{11}$$

and

$$\mathrm{div}_{\mathbb{H}}\, \phi := X\phi_1 + Y\phi_2 \,. \tag{12}$$

Alternatively $\nabla_{\mathbb{H}} f$ can be defined as the section of $\mathrm{H}\mathbb{H}^1$

$$\nabla_{\mathbb{H}} f := Xf\, X + Yf\, Y$$

whose canonical coordinates are (Xf, Yf). This is consistent with the the already mentioned identification of sections and their coordinates.

A natural definition of functions of bounded variation and of sets of finite perimeter in $\mathbb{H}^1$ was the first time introduced in [10]. There are, however, several ways to define functions of bounded variation associated with a vector subbundle of $\mathrm{T}\mathbb{R}^d$ generated by a family of vector fields; these definitions have been proposed independently over the last few years by different authors (see [9], [8], [30], [24]). All these definitions are in fact equivalent, as it is proved in [24]: see in particular the beginning of Section 2 in [24] for a discussion. Following one of these definitions we shall say that $E \subset \mathbb{H}^1$ has *locally finite* $\mathbb{H}$*-perimeter* (or, following De Giorgi, E is a $\mathbb{H}$*-Caccioppoli set*) if for any bounded open set $\Omega \subseteq \mathbb{H}^1$

$$|\partial E|_{\mathbb{H}}(\Omega) := \sup\left\{ \int_E \mathrm{div}_{\mathbb{H}}\, \phi \; d\mathcal{L}^3 \,:\, \phi \in \mathbf{C}^1_0(\Omega, \mathrm{H}\mathbb{H}^1),\ |\phi(P)|_P \leq 1 \right\} < \infty \,. \tag{13}$$

In such a way, $|\partial E|_{\mathbb{H}}$ defines a Radon measure in $\mathbb{H}^1$. If ∂E is an Euclidean regular manifold with outward unit normal n, then

$$|\partial E|_{\mathbb{H}} = \Big(\langle X, n\rangle^2 + \langle Y, n\rangle^2 \Big)^{1/2}\ \mathcal{H}^2 \llcorner \partial E \,,$$

see [10] and [24].

Now, Riesz' representation theorem yields the existence of a $|\partial E|_{\mathbb{H}}$-measurable section ν_E of $\mathrm{H}\mathbb{H}^1$ such that $|\nu_E(P)|_P = 1$ for $|\partial E|_{\mathbb{H}}$-a.e. P and for all $\phi \in \mathbf{C}_0^1(\mathbb{H}^1, \mathrm{H}\mathbb{H}^1)$ (13) ? we have

$$-\int_E \mathrm{div}_{\mathbb{H}}\ \phi\ d\mathcal{L}^3 = \int_{\mathbb{H}^1} \langle \nu_E, \phi\rangle\ d|\partial E|_{\mathbb{H}}\,.$$

We shall call ν_E the *generalized inward normal* to E.

DEFINITION 2.10. Let $[z,t]$, $P_0 \in \mathbb{H}^1$ with $z = x + iy$ be given. We set

$$\pi_{P_0}([z,t]) = xX(P_0) + yY(P_0)\,.$$

The map $P_0 \to \pi_{P_0}([z,t])$ is a smooth section of $\mathrm{H}\mathbb{H}^1$.

Let us give now some elementary definitions and results concerning intrinsic differentiability in the Heisenberg group. These results are basically due to P. Pansu ([46]), or are inspired by his ideas. All proofs of the results below can be found in [26]. Extensions of these intrinsic differentiability's results to Carnot groups have been carried out in [34], [62], [38], [35], [3] and [29].

DEFINITION 2.11. We shall say that a map L from $\mathbb{H}^1$ to $\mathbb{R}$ is $\mathbb{H}$-linear if it is a homomorphism and if it is positively homogeneous of degree 1 with respect to the dilations of $\mathbb{H}^1$.

DEFINITION 2.12. Let Ω be an open set in $\mathbb{H}^1$. We shall say $f : \Omega \to \mathbb{R}$ is Pansu-differentiable (differentiable in the sense of Pansu: see [46] and [36]) at P_0 if there exists a $\mathbb{H}$-linear map L from $\mathbb{H}^1$ to $\mathbb{R}$ such that

$$\lim_{P\to P_0} \frac{f(P) - f(P_0) - L(P_0^{-1}\cdot P)}{d(P,P_0)} = 0\,.$$

REMARK 2.13. The above definition is equivalent to the following one: there exists a homomorphism L from $\mathbb{H}^n$ to $\mathbb{R}$ such that

$$\lim_{\lambda\to 0+} \frac{f(\tau_{P_0}(\delta_\lambda v)) - f(P_0)}{\lambda} = L(v)$$

locally uniformly in $\mathbb{H}^1$. In particular, L is unique and we shall write $L = d_{\mathbb{H}} f(P_0)$.

PROPOSITION 2.14. *A map L from $\mathbb{H}^1$ to $\mathbb{R}$ is $\mathbb{H}$-linear if and only if there exists $(a,b) \in \mathbb{R}^2$ such that, if $v = [x+iy, t] \in \mathbb{H}^1$, then $L(v) = \langle (a,b), (x,y)\rangle_{\mathbb{R}^2}$.*

Definition 2.15. With the notations of Definition 2.12 we shall say that f is differentiable along X (Y) at P_0 if the map $\lambda \mapsto f(\tau_{P_0}(\delta_\lambda e_1))$ (respectively: $\lambda \mapsto f(\tau_{P_0}(\delta_\lambda e_2))$) is differentiable at $\lambda = 0$, where e_k is the k-th vector of the canonical basis of $\mathbb{R}^3$.

Clearly, if $f \in \mathbf{C}^1(\Omega)$ then f is differentiable along X and Y at all points of Ω. Hence, if we set for each f differentiable along X and Y at P_0 the horizontal gradient to be

$$\nabla_{\mathbb{H}} f = Xf\, X + Yf\, Y \tag{14}$$

then this definition naturally extends the one given for (classically differentiable functions) in (11) of Definition 2.9.

Proposition 2.16. *With the notations of Definition* 2.12 *and Proposition* 2.14, *if f is Pansu-differentiable at P_0, then it is differentiable along X and Y at P_0, and*

$$d_{\mathbb{H}} f(P_0)(v) = \langle \nabla_{\mathbb{H}} f, \pi_{P_0}(v)\rangle_{P_0}\,. \tag{15}$$

Definition 2.17. If $\Omega \subset \mathbb{H}^1$, we shall denote by $\mathbf{C}^1_{\mathbb{H}}(\Omega)$ the set of continuous real functions in Ω such that $\nabla_{\mathbb{H}} f$ is continuous in Ω. Moreover, we shall denote by $\mathrm{Lip}_{\mathbb{H}}(\Omega)$ the set of all Lipschitz functions $f : (\Omega, d) \to \mathbb{R}$. Analogously, the space $\mathrm{Lip}_{\mathbb{H},\mathrm{loc}}(\Omega)$ is defined in the usual way.

Proposition 2.18. *With the notations of Definition* 2.17, *a continuous function belongs to $\mathbf{C}^1_{\mathbb{H}}(\Omega)$ if and only if its distributional derivatives Xf, Yf are continuous in Ω.*

Remark 2.19. $\mathbf{C}^1(\Omega) \subset \mathbf{C}^1_{\mathbb{H}}(\Omega)$, and the inclusion is strict (see [26], Remark 5.9).

Theorem 2.20. *If $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ then f is Pansu-differentiable at any point $P_0 \in \Omega$. Moreover $\mathbf{C}^1_{\mathbb{H}}(U) \subset \mathrm{Lip}_{\mathbb{H},\mathrm{loc}}(\Omega)$.*

Definition 2.21. We shall say that $S \subset \mathbb{H}^1$ is an $\mathbb{H}$-*regular hypersurface* if for every $P \in S$ there exist an open ball $U(P, r)$ and a function $f \in \mathbf{C}^1_{\mathbb{H}}(U(P, r))$ such that

$$S \cap U(P, r) = \{Q \in U(P, r) : f(Q) = 0\}\,; \tag{i}$$

$$\nabla_{\mathbb{H}} f(P) \neq 0\,. \tag{ii}$$

Definition 2.22. If $S \subset \mathbb{H}^1$ is a $\mathbb{H}$-regular hypersurface and $P \in S$, we define the *tangent group* $T^g_{\mathbb{H}} S(P)$ to S at P as follows

$$T^g_{\mathbb{H}} S(P) := \{Q : \langle \nabla_{\mathbb{H}}(f \circ \tau_P)(0), \pi_0(Q)\rangle_0 = 0\}\,.$$

By (ii) of Definition 2.21, $T^g_{\mathbb{H}}S(P)$ is a proper subgroup of $\mathbb{H}^1$. Then the *tangent plane* to S at P is the lateral

$$T_{\mathbb{H}}S(P) := P \cdot T^g_{\mathbb{H}}S(P)\,.$$

Once more, observe that this is a good definition. Indeed the tangent plane does not depend on the particular function f defining the surface S because of points (i) and (iv) of Theorem 2.23 below.

Finally, let us recall three useful results on $\mathbb{H}$-regular surfaces proved in [26] in the setting of the Heisenberg group and in [27] and [29] in for general Carnot group.

THEOREM 2.23 [Implicit Function Theorem]. *Let Ω be an open set in $\mathbb{H}^1$, $0 \in \Omega$, and let $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ be such that $|Xf(0)| = Xf(0) > 0$, $f(0) = 0$. Then, if we put*

$$E = \{[z,t] \in \Omega : f([z,t]) < 0\}, \quad S = \{[z,t] \in \Omega : f([z,t]) = 0\}\,,$$

there exists a connected open neighbourhood $\mathcal{U}$ of 0, such that

(i) $$E \cap \mathcal{U} \quad \text{is connected};$$

(ii) $$E \text{ has finite } \mathbb{H}\text{-perimeter in } \mathcal{U}\,;$$

(iii) $$\partial E \cap \mathcal{U} = S \cap \mathcal{U}\,;$$

(iv) $$\nu_E(P) = \nabla_{\mathbb{H}} f(P)/|\nabla_{\mathbb{H}} f(P)|_P \ \text{ for all } \ P \in S \cap \mathcal{U}\,.$$

If we put now $I = [-\delta,\delta] \times [-\delta^2,\delta^2]$, $J = [-h,h]$, then there exists a unique continuous function

$$\phi = \phi(\eta,\tau) : I \to J$$

such that the following parameterization of S and integral representation of the perimeter hold

(v) $$S \cap \bar{\mathcal{U}} = \{[x+iy,t] \in \bar{\mathcal{U}} : y = \eta,\ x = \phi(\eta,\tau)\,, t = 2\phi(\eta,\tau)\eta + \tau,\ (\eta,\tau) \in I\};$$

(vi) $$|\partial E|_{\mathbb{H}}(\mathcal{U}) = \int_I \frac{|\nabla_{\mathbb{H}} f|}{Xf}(\Phi(\eta,\tau))\,d\eta\,d\tau\,,$$

where

(16) $$\Phi(\eta,\tau) = (\phi(\eta,\tau), \eta, 2\phi(\eta,\tau)\eta + \tau)\,.$$

THEOREM 2.24 [Blow-up Theorem]. *Let Ω be an open set in $\mathbb{H}^1$, let $E \subset \mathbb{H}^1$ be such that $\partial E \cap \Omega = S \cap \Omega$ where $S \subset \mathbb{H}^1$ is a $\mathbb{H}$-regular surface. If $P_0 \in \mathbb{H}^1$ and $r > 0$ denote*

$$E_{P_0,r} := \left\{ P \in \mathbb{H}^1 : \ P_0 \cdot \delta_r(P_0^{-1} \cdot P) \in E \right\}\,.$$

Then

$$\text{(i)}\quad \lim_{r\to 0}\frac{|\partial E_{P_0,r}|_{\mathbb{H}}(U(0,1))}{r^3} = \lim_{r\to 0}\frac{|\partial E|_{\mathbb{H}}(U(P_0,r))}{r^3} = \mathcal{H}^2(T^g_{\mathbb{H}}S(P_0)\cap U(0,1)) = 4 \textit{ if } P_0\in S\cap\Omega\,;$$

$$\text{(ii)}\qquad |\partial E|_{\mathbb{H}}\llcorner\,\Omega = 4\,\mathcal{S}^3_d\llcorner\,(S\cap\Omega)\,.$$

THEOREM 2.25 [Whitney Extension Theorem]. *Let $F\subset\mathbb{H}^1$ be a closed set, and assume $f: F\to\mathbb{R}$, $k: F\to \mathrm{H}\mathbb{H}^1$ are continuous functions. We set*

$$R(P',P) := \frac{f(P')-f(P)-\langle k(P),\pi_P(P^{-1}\cdot P')\rangle_P}{d(P,P')}\,,$$

and, if $K\subset F$ is a compact set,

$$\rho_K(\delta) := \sup\{|R(P',P)|:\ P,P'\in K,\ 0<d(P,P')<\delta\}\,. \tag{17}$$

If $\rho_K(\delta)\to 0$ as $\delta\to 0$ for every compact set $K\subset F$, then there exist $\tilde f:\mathbb{H}^1\to\mathbb{R}$, $\tilde f\in\mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^1)$ such that

$$\tilde f_{|F}\equiv f,\quad \nabla_{\mathbb{H}}\tilde f_{|F}\equiv k\,.$$

3. – $\mathbb{H}$-Regular surfaces as Euclidean fractal sets

In this section we construct an example of an $\mathbb{H}$-regular surface in $\mathbb{H}^1$ which has the Euclidean Hausdorff dimension $5/2$ and, hence, is more of a fractal structure.

THEOREM 3.1. *There exists an $\mathbb{H}$-regular surface $S\subset\mathbb{H}^1$ such that*

$$\mathcal{H}^{(5-\varepsilon)/2}(S)>0 \textit{ for all } \varepsilon\in(0,1)\,. \tag{18}$$

In particular, S is not 2- Euclidean rectifiable.

REMARK 3.2. An interesting example of Euclidean fractal set F in $(\mathbb{H}^1,d)$ with Hausdorff dimension 2 was constructed by R. Strichartz in [58]. However it cannot be a $\mathbb{H}$-regular surface or even a topological surface, i.e. a submanifold of topological dimension 2 in $(\mathbb{H}^1,d)$ (see also [6] for a simpler computation). In fact Gromov proved that a topological surface in $(\mathbb{H}^1,d)$ always has Hausdorff dimension larger or equal than 3 (see [31], Section 2).

REMARK 3.3. By Theorem 1.1 in [5] which states that for every $\alpha\geq 0$

$$\mathcal{H}^{\min\{\alpha,1+\frac{\alpha}{2}\}}\ll\mathcal{H}^\alpha_d\quad\text{on}\quad\mathbb{R}^3$$

it follows $\frac{1}{2}$ is the smallest possible jump between the Euclidean and Carnot-Carathéodory Hausdorff dimension of such subsets of $\mathbb{R}^3$.

In the proof of Theorem 3.1 we will use two auxiliary results. Before its proof we will need two preliminary technical lemmas. The first is contained in a paper by Z. Balogh (see [4], Theorem 4.1), where it was used to construct euclidean surfaces with large sets of characteristic points. The more precise modulus of continuity of the gradient which we state and use here can, however, only be found at the end of the proof of this theorem in [4]. It states that the "pointwise curl" of almost twice differentiable functions can be nonzero on a quite large set.

LEMMA 3.4. *There is a C^1-function $g = g : Q = [0, 1]^2 \to \mathbb{R}$ and a constant $K < \infty$ such that*

$$\mathcal{L}^2(A_g) > 1/2 \ \ \textit{where} \ \ A_g := \{(x, y) \in Q : \nabla g((x, y)) = (2y, -2x)\} \tag{19}$$

$$|\nabla g(z) - \nabla g(w)| \le K(1 + |\log(|z - w|)|)^K |z - w| \ \ \textit{for all} \ \ z, w \in Q\,. \tag{20}$$

The other ingredient is a construction of functions of a prescribed Hölder type continuity which have all level sets of maximal Hausdorff dimension.

LEMMA 3.5. *There is a function $h : \mathbb{R} \to \mathbb{R}$ such that*

(i) *for all $t \in [0, 1]$ is the (Euclidean) Hausdorff dimension of $h^{-1}(t) \cap [0, 1]$ at least $\frac{1}{2}$,*

(ii) *for each $m \ge 1$ we have*

$$\lim_{r \to 0+} \frac{\log\left(\left(\frac{1}{r}\right)\right)^m}{r^{1/2}} \sup\{|h(x) - h(y)|\ ,\ |x - y| \le r\} = 0\,. \tag{21}$$

A variety of similar constructions can be found in the literatur, however, in order to obtain in Theorem 3.1 an example which is indeed of maximal dimension, we need a very precise version of such an construction which seems new. A question concerning the optimality of this construction will be discussed after its presentation and the proof of Lemma 3.5 given below.

CONSTRUCTION 3.6. We consider the following construction whose only parameter is the sequence $\{p_n\}_{n=1}^{\infty}$ of integers satisfying $0 \le p_n \le 2n - 2$. Given this, we set

$$r_n = 2^{2n-2-p_n} \ \text{ for } n \ge 1 \tag{22}$$

$$l_n = \frac{2^{2n-1} - 1}{2^{4n-2}(2r_n + 1)} l_{n-1} \ \text{ for } n \ge 1, \ \text{ with } l_0 = 1 \tag{23}$$

$$v_n = 2^{-n^2} \ \text{ for } n \ge 0\,, \tag{24}$$

and define the families $\mathcal{C}_k$ of "oriented" (closed) rectangles $((x_1, y_1), (x_2, y_2))$ where $x_1, x_2, y_1, y_2 \in [0, 1]$, $x_1 < x_2$ and which represents the rectangle $[x_1, x_2] \times \{y_1, y_2\}^{\text{conv}}$ with "entrance" (x_1, y_1) and "exit" (x_2, y_2). (The intuition behind this notation is that once we have finished our construction, we will have found a function $h : [0, 1] \to [0, 1]$ whose graph enters and leaves these rectangles at the corresponding points.)

We start out with $\mathcal{C}_0 = \{((0, 0), (1, 1))\}$ and suppose that for some $n \geq 1$ we are given a finite family $\mathcal{C}_{n-1}$ of oriented rectangles such that

- $\text{card}([x_1, x_2] \cap [x_1', x_2']) \leq 1$ if $((x_1, y_1), (x_2, y_2)), ((x_1', y_1'), (x_2', y_2')) \in \mathcal{C}_{n-1}$ are different and moreover if $x_i = x_{3-i}'$ then $y_i = y_{3-i}'$ if $i = 1, 2$;
- for $((x_1, y_1), (x_2, y_2)) \in \mathcal{C}_{n-1}$ either $y_1 = y_2$ or $|y_1 - y_2| = v_{n-1}$ and then $x_2 - x_1 = l_{n-1}$.

Now, fixing such an element $\mathcal{R} = ((x_1, y_1), (x_2, y_2))$ of $\mathcal{C}_{n-1}$, we define the next generation to be $\mathcal{C}_n(\mathcal{R}) = \{\mathcal{R}\}$ if $y_1 = y_2$. In the nondegenerate case $y_1 \neq y_2$ we first pick the two degenerate rectangles

$$\mathcal{R}_- = ((x_1, y_1), (x_1 + 2^{-2n} l_{n-1}, y_1)) \text{ and } \mathcal{R}_+ = ((x_2 - 2^{-2n} l_{n-1}, y_2), (x_2, y_2)),$$

and for

$$k = 2^{2n-1} m + q \text{ where } m \in \{0, \dots, 2r_n\}, q \in \{1, \dots, 2^{2n-1}\}$$

we set $\mathcal{R}_k = ((x_1^k, y_1^k), (x_2^k, y_2^k))$ with

$$\begin{aligned} x_1^k &= x_1 + 2^{-2n} l_{n-1} + (k-1) l_n, \\ x_2^k &= x_1^k + l_n, \\ y_1^k &= \begin{cases} (q-1) 2^{-2n+1} (y_2 - y_1) + y_1 & \text{if } m \text{ is even} \\ (q-1) 2^{-2n+1} (y_1 - y_2) + y_2 & \text{if } m \text{ is odd} \end{cases}, \quad \text{and} \\ y_2^k &= y_1^k + (-1)^m 2^{-2n+1} (y_2 - y_1). \end{aligned}$$

(Notice that $|y_2^k - y_1^k| = 2^{-2n+1} v_{n-1} = v_n$ and $y_2^k = y_1^{k+1}$ for all k.) Then we set

$$\mathcal{C}_n(\mathcal{R}) = \{\mathcal{R}_-, \mathcal{R}_+\} \cup \{\mathcal{R}_k : k = 1, \dots, 2^{2n-1}(2r_n + 1)\}.$$

Having this defined for all $\mathcal{R} \in \mathcal{C}_{n-1}$, we put

$$\mathcal{C}_n = \bigcup \{\mathcal{C}_n(\mathcal{R}) : \mathcal{R} \in \mathcal{C}_{n-1}\}$$

and introduce also the compact union of rectangles

$$C_n = \bigcup \{[x_1, x_2] \times \{y_1, y_2\}^{\text{conv}} : ((x_1, y_1), (x_2, y_2)) \in \mathcal{C}_n\}.$$

We observe that for nondegenerate rectangles $\mathcal{R} = ((x_1, y_1), (x_2, y_2)) \in \mathcal{C}_{n-1}$ the family $\mathcal{C}_n(\mathcal{R})$ satisfies the properties stated for $\mathcal{C}_{n-1}$ above and

- $C_n(\mathcal{R}) = \bigcup\{[x_1', x_2'] \times \{y_1', y_2'\}^{\mathrm{conv}} : ((x_1', y_1'), (x_2', y_2')) \in \mathcal{C}_n(\mathcal{R})\}$ is a compact subset of $[x_1, x_2] \times \{y_1, y_2\}^{\mathrm{conv}} \subset C_{n-1}$,
- $\mathrm{proj}_1(C_n(\mathcal{R})) = [x_1, x_2]$ and $[x_1, x_2]$ is covered in a non-overlapping way, note that $x_2^k = x_1^{k+1}$ and that due to (23) the last x_2^k is just the x-coordinate of the entrance into $\mathcal{R}_+$,
- each vertical slice of $C_n(\mathcal{R})$ is of diameter at most $2v_n$.

So $C_n = \bigcup_{\mathcal{R}\in\mathcal{C}_{n-1}} C_n(\mathcal{R})$ is a compact subset of C_{n-1} and $\mathrm{proj}_1(C_n) = [0, 1]$ for each n. From this it is clear that $C_\infty = \bigcap_{n=1}^\infty C_n$ is the graph of a function $h : [0, 1] \to [0, 1]$, which is continuous as its graph is compact. Now, we state the following crucial properties of this function.

Proposition 3.7. *The function $h = h_{\{p_n\}}$ constructed above satisfies*:

(i) *if for some $d > 0$ the bound*

$$\liminf_{n\to\infty} \sum_{k=1}^{n} [(1 - 2d)(2k + 1) - (1 - d)(p_k + 3)] > -\infty$$

holds then $\mathcal{H}^d(h^{-1}(t)) > 0$ for all $t \in [0, 1]$.

(ii) *if $\limsup_{n\to\infty} p_n/n < 2$ then for some $c = c_{\{p_n\}} < \infty$ and all $n \geq 1$ with $p_k \geq 2$ for $k \geq n$ we have the inequality*

$$|h(x)-h(y)| \leq c\, 2^{-\sum_{k=1}^{n}(p_k-2)/2} \sqrt{|x - y|} \ \text{ if } x, y \in [0, 1] \text{ and } |x-y| \leq l_n\,.$$

Proof. Throughout the whole proof, the corners of the rectangles constructed above will play an important rôle, therefore we need some special notations for them. For this purpose let

$$G_n = \bigcup_{((x_1,y_1),(x_2,y_2))\in\mathcal{C}_n} \{(x_1, y_1), (x_2, y_2)\},\ G_n^i = \mathrm{proj}_i(G_n) \text{ for } i = 1, 2\,.$$

Thus $G_k \subset G_{k+1} \subset C_\infty$ and $\bigcup_k G_k$ is dense in C_∞.

We turn to the proof of (i). Given a d satisfying the assumption it is easy to check that $d \leq 1/2$. We set $f_n = r_n(l_n/l_{n-1})^d$, and using (23) we compute

$$\begin{aligned} f_n = r_n \left(\frac{2^{2n-1} - 1}{2^{4n-2}(2r_n + 1)}\right)^d &\geq r_n^{1-d} \left(\frac{2^{2n-1} - 1}{2^{4n}}\right)^d \\ &= 2^{(2n+1)(1-2d)-(p_n+3)(1-d)}(1 - 2^{1-2n})^d\,. \end{aligned}$$

Because $\prod_{n=1}^\infty(1 - 2^{1-2n}) > 0$, we see that our assumption on d and $\{p_n\}_1^\infty$ ensures

$$\liminf_{n\to\infty} \prod_{k=1}^{n} f_k > 0\,. \tag{25}$$

Following the usual pattern of lower estimates for Hausdorff measure, we establish (i) by turning (25) into a Frostman type estimate for a suitably choosen measure on $h^{-1}(t)$ (see, for instance, Theorem 8.8 in [42]). We can of course suppose that $t \notin \bigcup_k G_k^2$ because the construction of h implies that for $t \in G_k^2$ there are $x_1 < x_2$ with $((x_1, t), (x_2, t)) \in \mathcal{C}_l$ for all $l > k$ and hence $(x_1, x_2) \subset h^{-1}(t)$. It is also clear from the way we selected the $\mathcal{C}_k$'s that we can construct a sequence of subsets $C_n(t) \subset \{x \in [0, 1] : (x, t) \in C_n\}$ $(n = 1, 2, \dots)$ such that $h^{-1}(t) \supset \bigcap_{k=0}^{\infty} C_k(t)$ where

a) $C_0(t) = [0, 1] \supset C_1(t) \supset C_2(t) \supset \dots$
b) each $C_n(t)$ is the union of a finite system $\mathcal{C}_n(t)$ of disjoint intervals of length l_n,
c) for each $n \geq 1$ and $I \in \mathcal{C}_{n-1}(t)$ $\mathcal{C}_n(t, I)$ is the family of all $J \in \mathcal{C}_n(t)$ that are contained in I of cardinality r_n and the distance between any two different intervals in $\mathcal{C}_n(t, I)$ is at least $(2^{2n} - 1)l_n$.

We consider the canonical "uniformly" distributed probabilities

$$\mu_n = \left(\prod_{k=1}^{n} r_k\right)^{-1} \sum_{J \in \mathcal{C}_n(t)} \frac{1}{l_n} (\mathcal{H}^1 \llcorner J)$$

and note that $\mu_n(I) = r_n(\prod_{k=1}^{n} r_k)^{-1} = \mu_{n-1}(I)$ for $I \in \mathcal{C}_{n-1}(t)$, so $\mu_n \rightharpoonup^* \mu$, a probability measure living on $h^{-1}(t)$. We will show that

$$\left(\inf_{m \geq n} \prod_{k=1}^{m} f_k\right) \mu(I) \leq 2|I|^d \text{ if } I \text{ is a compact interval with } |I| \leq l_n \,. \tag{26}$$

The considerations of covers approximating the d-dimensional Hausdorff measure of $\bigcap_n C_n(t)$ then gives, using a constant $c_d > 0$ determined by our choice of normalization of the Hausdorff measure, that $\mathcal{H}^d(h^{-1}(t)) \geq c_d \liminf_{n \to \infty} \prod_{k=1}^{n} f_k$, so (i) would follow from (25).

To verify (26), we first note that due to the distance required in c) it is enough to establish this inequality assuming that $I \subset J \in \mathcal{C}_n(t)$. Obviously, we can also suppose the following to hold, as it can always be achieved by modifications making the inequality (26) even sharper

d) $\min I, \max I \in \bigcap_k C_k(t)$
e) $n \geq 1$ is the largest number of all $\tilde{n}$ for which an $\tilde{I} \in \mathcal{C}_{\tilde{n}}(t)$ containing I does exist.

We claim that

$$\mu(I)|I|^{-d} \leq 2\mu_n(J)|J|^{-d} \,.$$

Indeed, d) and e) ensure that for some $q \in \{2, \dots, r_{n+1}\}$ the interval I intersects precisely q intervals from $\mathcal{C}_{n+1}(t, J)$. This yields

- $\mu(I) \leq q\mu_{n+1}(J') = q\mu_n(J)/r_{n+1}$ for any $J' \in \mathcal{C}_{n+1}(t)$

- $\operatorname{diam}(I) \geq (q-1)(2^{2n+2}-1)l_{n+1}$, compare with c) above

  and allows us to estimate that

$$\frac{\mu(I)}{|I|^d} \leq \frac{\mu_n(J)}{|J|^d}\frac{q}{(q-1)^d}\frac{l_n^d}{r_{n+1}((2^{2n+2}-1)l_{n+1})^d}$$

$$\leq \frac{\mu_n(J)}{|J|^d}2r_{n+1}^{1-d}\frac{l_n^d}{r_{n+1}(l_n/r_{n+1})^d} \leq 2\frac{\mu_n(J)}{|J|^d}\,.$$

It remains to observe that

$$\frac{\mu_n(J)}{|J|^d} = \left(\prod_{j=1}^n r_j^{-1}\right)\prod_{j=1}^n\left(\frac{l_{j-1}}{l_j}\right)^d = \prod_{j=1}^n \frac{l_{j-1}^d}{l_j^d r_j} = \left(\prod_{j=1}^n f_j\right)^{-1}$$

and (26) follows.

So we can turn to the proof of statement (ii) in the proposition, without loss of generality $x < y$. First, we assume in addition that

(27) there is a $k \geq n$ with $G_k^1 \cap (x, y) = \emptyset$ but $\operatorname{card}(G_{k+1}^1 \cap [x, y]) \geq 2$.

Let $x' = \min(G_{k+1}^1 \cap [x, y])$, $y' = \max(G_{k+1}^1 \cap [x, y])$, thus $y' = x' + q_0 l_{k+1}$, $q_0 \geq 1$. We also notice that there are $a, b \in G_k^1$ with $b - a = l_k$ and $[x, y] \subset (a, b)$. Since the construction of h implies that

$$\operatorname{osc}(h, [s_1, s_2]) \leq v_m \quad \text{if } s_1, s_2 \in G_m^1 \text{ and } s_2 - s_1 = l_m\,,$$

we infer

$$|h(x) - h(y)| \leq \min(v_k, (2+q_0)v_{k+1}) \leq \min(v_k, 3q_0 v_{k+1})$$

$$\leq \min\left(v_k, 3\frac{|x-y|}{l_{k+1}}v_{k+1}\right)\,.$$

This implies

$$\frac{|h(x)-h(y)|}{\sqrt{y-x}} \leq \min\left(\frac{v_k}{\sqrt{|x-y|}}, \frac{3\sqrt{|x-y|}}{l_{k+1}}v_{k+1}\right) \leq \sqrt{3\frac{v_k v_{k+1}}{l_{k+1}}}$$

$$\leq \sqrt{3\frac{v_{k+1}}{l_{k+1}}} \leq \prod_{m=1}^k \left(\frac{v_{m+1}}{v_{m-1}}\frac{l_m}{l_{m+1}}\right)^{1/2}\sqrt{3\frac{v_0\, v_1}{l_1}}$$

$$\leq c_1 \prod_{m=1}^k \left(\frac{2^{-(m+1)^2+(m-1)^2}2^{4m+2}(2r_{m+1}+1)}{(2^{2m+1}-1)}\right)^{1/2}$$

$$\leq c_1 \prod_{m=1}^k \left(\frac{(2^{-4m})2^{4m+2}(2^{2m-p_{m+1}+1}+1)}{2^{2m+1}-1}\right)^{1/2}$$

$$\leq c_2 \prod_{m=1}^k \left(2^{2-p_{m+1}}\frac{1+2^{-1-2m+p_{m+1}}}{1-2^{-2m-1}}\right)^{1/2}$$

$$\leq c_3 2^{-\sum_{m=1}^k (p_{m+1}-2)/2} \leq c_3 2^{-\sum_{m=1}^n (p_m-2)/2}\,,$$

because our assumption in statement (ii) implies for c_4 sufficienly large and $m > c_4$ the estimate $1+2^{-1-2m+p_{m+1}} < 1+2^{-m/c_4}$, we see that $c_3 = c_{\{p_n\}} < \infty$.

To finish, we drop our additional assumption (27) and choose the maximal k such that $(x, y) \cap G^1_k = \emptyset$. If k satisfies (27) anyhow, then we are done. Otherwise, we pick $z \in G^1_{k+1} \cap (x, y)$ and notice that necessarily $\{z\} = G^1_{k+1} \cap (x, y)$ as else $l_{k+1} < |x - y|$ would imply $k \geq n$ and (27) would hold true. We consider now the interval $[x, z]$ and it is easily checked that the maximal k' with $(x, z) \cap G^1_{k'} = \emptyset$ satisfies (27) and therefore gives the desired inequality for x, z. Since the same argument works for z, y and as $\sqrt{z - x} + \sqrt{y - z} \leq \sqrt{2}\sqrt{y - x}$, our proof is finished. □

PROOF OF LEMMA 3.5. We choose the sequence $\{p_n\}^{\infty}_{n=1}$ by the simple rule $p_n = 4$ for all $n \geq 3$ and consider on $[0, 1]$ the function $h = h_{\{p_n\}}$ as obtained in Construction 3.6. We note that $h(0) = 0$, $h(1) = 1$ and extend the function to all of $\mathbb{R}$ by requiring $h(x + k) = h(x) + k$ for all $k \in \mathbb{Z}$ and $x \in [0, 1]$. Now it is easy, that in order to verify the statements (i) and (ii) we can restrict to the cases when $t, x, y \in [0, 1]$.

Concerning (i), each $d < 1/2$ obviously satisfies the assumption of Proposition 3.7 (i) and so $\mathcal{H}^d(h^{-1}(t)) > 0$ for all $t \in [0, 1]$, $d < 1/2$ which just says $\dim_{\mathrm{H}}(h^{-1}(t)) \geq 1/2$.

Considering (ii), we have $\sum^n_3 (p_k - 2)/2 = n - 2$ and we see from (23) that $1 > l_n/l_{n-1} \geq 2^{2n-2}/2^{4n-2}4r_n \geq 2^{-4n}$ and thus $l_n \geq 2^{-2n(n+1)}$. Hence, if $r \in [l_{n+1}, l_n)$ then $\log(1/r) \leq \log(2)2(n + 2)^2$. Now, given the $m \geq 1$ we infer from Proposition 3.7 (ii) that for $|x - y| \leq r \in [l_{n+1}, l_n)$

$$|h(x) - h(y)| \leq c\sqrt{r}2^{-n} \leq \sqrt{r}\frac{1}{n(2\log(2)(n + 2)^2)^m} \leq \frac{1}{n}\frac{\sqrt{r}}{(\log(1/r))^m}$$

provided n is sufficiently large. This finishes the proof of the lemma. □

REMARK 3.8. The construction presented above looks quite complicated, but as a compensation it does not only allow to control the modulus of continuity of the $\frac{1}{2}$-Hölder-functions involved up to the order of logarithmic terms but it also gives examples with optimal level set dimension for any Hölder exponent between $\frac{1}{2}$ and 1. Indeed, choosing p_k to be the integer part of ck with fixed $c \in (0, 2)$ we easily calculate from Proposition 3.7 that the resulting h has an Hölder exponent $\frac{2}{4-c}$ and all level sets of dimension at least $\frac{2-c}{4-c}$. On the other hand, if $f : [0, 1] \to [0, 1]$ is α-Hölder, then it is lipschitz on the $\frac{1}{\alpha}$-dimensional space $[0, 1]$ with the metric $\varrho(x, y) = \sqrt{[\alpha]}|x - y|$. Hence by the Eilenberg-Fubini type result given in 2.10.25 of [22] we see that almost all level sets of f are of dimension at most $\alpha^{-1} - 1$ metric ϱ and at most $1 - \alpha$ dimensional in the euclidean distance.

However, even our fine tuned construction could not answer the following natural question.

Can one find α-Hölder functions such that all level sets $f^{-1}(t)$ (or at least for all t from a set of positive measure) are of *positive* $1 - \alpha$-dimensional Hausdorff *measure*?

In the main application of our construction the $\frac{1}{2}$-dimensional measure is obviously zero since we have a "better" modulus of continuity then $\sqrt{|x-y|}$. But we were not able to improve the construction for the case of general $\frac{1}{2}$-Hölder functions nor to give a (presumably more likely) proof that level sets are always zero.

PROOF OF THEOREM 3.1. Let g be a function as in Lemma 3.4, so we have

$$|g(z) - g(w) - \langle \nabla g(w)\,,\, z-w\rangle| \;\le\; K'\,|z-w|^2(1+|\log(|z-w|)|)^K \qquad \forall z,\, w \in Q\,. \tag{28}$$

We can, of course, moreover require that $\|g\|_\infty \le 1$. Next we choose the function h from Lemma 3.5 and set $F^* := A_g \times [-1, 2]$. Now we can define the function $f^* : F^* \to \mathbb{R}$ by

$$f^*([z,t]) := h(t - g(z)) \quad \text{if } [z,t] \in F^*$$

and the section $k^* : F^* \to \mathrm{H}\mathbb{H}^1 \equiv \mathbb{R}^2$ as

$$k^*([z,t]) := (0,0) \quad \text{if } [z,t] \in F^*\,.$$

We claim now that the hypotheses of Theorem 2.25 are satisfied for $F = F^*$, $f = f^*$ and $k = k^*$. As the continuity of f^* and k^* is straightforward, we only have to show that

$$\lim_{\delta\to 0} \rho_F(\delta) = 0 \tag{29}$$

where ρ_F is the function defined in (17) just without localization to compact subsets. For this purpose consider any $P = [z,t]$, $P' = [w,s]$ in F^* and observe first that

$$\begin{aligned} |g(w) - g(z) + 2\Im m(w\bar z)| &= |g(w) - g(z) - 2\Im m(-w\bar z)| \\ &= |g(w) - g(z) - \Im m(-2\bar z(w-z))| \\ &= |g(w) - g(z) - \langle \nabla g(z), w - z\rangle_{\mathbb{R}^2}|, \text{ as } z \in A_g\,, \\ &\le K'\,|z-w|^2(1+|\log(|z-w|)|)^K\,. \end{aligned}$$

Moreover, $(P')^{-1}\cdot P = [z-w, t-s-2\Im m(w\bar z))]$ and hence our definition of norm as given in (3) and (4) ensures that

$$|g(w) - g(z) + 2\Im m(w\bar z)|,\, |t-s-2\Im m(w\bar z)| \le K' d(P,P')^2(1+|\log(d(P,Q))|)^K\,.$$

This shows that also

$$|(t-g(z)) - (s - g(w))| \le 2K' d(P,P')^2(1+|\log(d(P,P'))|)^K\,,$$

and now (21) implies that

$$|f^*(P) - f^*(P')| = |h(t-g(z)) - h(s-g(w))|$$
$$\leq \tilde{K}\frac{d(P,P')(1+|\log(d(P,P'))|)^{(K/2)}}{|\log(d(P,P')(1+|\log(d(P,P'))|)^{(K/2)})|^{(K+1)}}$$

and therefore

$$\frac{|f^*(P) - f^*(P')|}{d(P,P')} \to 0 \text{ if } P, P' \in F \text{ and } d(P,P') \to 0,$$

thus (29) follows.

Therefore applying Theorem 2.25 we can extend $f^* : F^* \to \mathbb{R}$ to a function $\tilde{f}^* : \mathbb{H}^1 \to \mathbb{R}$, $\tilde{f}^* \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^1)$ such that

$$\nabla_{\mathbb{H}} \tilde{f}_{|F^*} \equiv 0. \tag{30}$$

Define now $f : \mathbb{H}^1 \to \mathbb{R}$ as

$$f(x,y,t) := \tilde{f}^*(x,y,t) - x.$$

Then by construction and (30), since

$$|\nabla_{\mathbb{H}} f|_P = |(-1,0)|_{\mathbb{R}^2} = 1 \quad \forall P \in F^*$$

there is an open set $\Omega \supset F^*$ such that

$$|\nabla_{\mathbb{H}} f|_P \neq 0 \quad \forall P \in \Omega.$$

Let

$$S := \Omega \cap \{f = 0\},$$

then S is an $\mathbb{H}$-regular surface. Let us prove (18).

Observe that

$$S \supset A := \bigcup_{(x,y)\in A_g} \Big(\{(x,y)\} \times ((h^{-1}(x) + g((x,y)) \cap [-1,2]))\Big). \tag{31}$$

By the coarea inequality (see [22] 2.10.27)

$$\begin{aligned}
&\int_{A_g} \mathcal{H}^{(1-\varepsilon)/2}(h^{-1}(x) \cap [0,1])\, dxdy \\
&\leq \int_{A_g} \mathcal{H}^{(1-\varepsilon)/2}((h^{-1}(x) + g(x,y)) \cap [-1,2])\, dxdy \\
&\leq c_\varepsilon \mathcal{H}^{(5-\varepsilon)/2}(A).
\end{aligned} \tag{32}$$

Denote by $A_{g,y} := \{x \in [0,1] : (x,y) \in A_g\}$, the horizontal slice of A_g at height y. Since $\mathcal{L}^2(A_g) > 0$, there exists a suitable measurable set $I \subset [0,1]$ of positive measure so that $\mathcal{L}^1(A_{g,y}) > 0$ for $y \in I$. Since $\varepsilon > 0$ it follows from statement (i) of Corollary 3.5

$$\begin{aligned}
&\int_{A_g} \mathcal{H}^{(1-\varepsilon)/2}(h^{-1}(x) \cap [0,1])\, dxdy \\
&\geq \int_0^1 dy \int_{A_{g,y}} \mathcal{H}^{(1-\varepsilon)/2}(h^{-1}(x) \cap [0,1])\, dx > 0.
\end{aligned} \tag{33}$$

Thus (31), (32) and (33) yield (18). □

4. – Hölder parameterization of $\mathbb{H}$-regular surfaces

In this section we will prove that each $\mathbb{H}$-regular surface $S \subset \mathbb{H}^1$ can be locally parameterized by means a Hölder continuous map of order $\frac{1}{2}$ and this parameterization is in some sense sharp. More precisely

THEOREM 4.1. *Let $S \subset \mathbb{H}^1$ be an $\mathbb{H}$-regular surface then for each $P_0 \in S$ there exist constants $\delta,\ r_0,\ L > 0$, an open neigborhood $\mathcal{U}$ of P_0 and a 1-to-1 map $\Phi : I := [-\delta, \delta] \times [-\delta^2, \delta^2] \to \mathbb{H}^1$ such that, if $\alpha = \frac{1}{2}$,*

(i) $$d(\Phi(u), \Phi(v)) \leq L\,|u - v|^{\alpha} \quad \forall u, v \in I\,;$$

(ii) $$\Phi(I) = S \cap \bar{\mathcal{U}}\,.$$

Moreover the $\mathbb{H}$-regular surface $S = \{(x, y, t) : x = 0\}$ cannot be locally parameterized by means any Hölder continuos map of order $\frac{1}{2} < \alpha \leq 1$ (i.e. by any map Φ satisfying (i) *with $\frac{1}{2} < \alpha \leq 1$).*

REMARK 4.2. In particular $\mathbb{H}$-regular surfaces cannot be seen as image through Lipschitz maps of a subspace of $(\mathbb{R}^2, |\cdot|^{2/3})$.

REMARK 4.3. Let $S \subset \mathbb{H}^1$ be a $\mathbb{H}$-regular surface. An interesting open question is wether there locally exists a map $\Phi : \Omega \subset \mathbb{R}^2 \to S$ such that $\Phi \in W^{1,4}(\Omega : \mathbb{H}^1)$. Here $W^{1,p}(\Omega : \mathbb{H}^1)$ denotes the Sobolev class of metric-space valued functions between $(\Omega, |\cdot|)$ and $(\mathbb{H}^1, d)$ studied in [33]. The exponent $p = 4$ should be natural according to the Sobolev embedding $W^{1,p}(\Omega : \mathbb{H}^1) \subset C^{0,1-\frac{2}{p}}_{\mathrm{loc}}(\Omega; \mathbb{H}^1)$ for $p > 2 = \dim\,\Omega$ proved in [33].

LEMMA 4.4. *Let $P,\ v = (v_1, v_2, v_3) \in \mathbb{H}^1$ and denote by $\gamma_{P,v} : [0, 1] \to \mathbb{H}^1$ the curve*

$$\gamma_{P,v}(s) = P \cdot (s\,v_1, s\,v_2, v_3)\,.$$

Then

(i) $\gamma_{P,v}$ *is a horizontal curve*;

(ii) *if* $P, Q \in U\left(P_0, \frac{r_0}{4}\right)$ *and* $v := P^{-1} \cdot Q$ *then* $\gamma_{P,v}([0, 1]) \subset U(P_0, r_0)$;

(iii) *for every* $g \in \mathbf{C}^1_{\mathbb{H}}(U(P_0, 2r_0))$, *and* $P, Q \in U\left(P_0, \frac{r_0}{4}\right)$ *with* $v := P^{-1} \cdot Q$, *there is some* $\bar{s} \in [0, 1]$ *such that*

$$g(\gamma_{P,v}(1)) - g(\gamma_{P,v}(0)) = v_1\,Xg(\gamma_{P,v}(\bar{s})) + v_2\,Yg(\gamma_{P,v}(\bar{s}))\,.$$

PROOF OF LEMMA 4.4. Observe that for every $s \in [0, 1]$

$$\begin{aligned} \dot{\gamma}_{P,v}(s) &= (v_1, v_2, 2\,(v_1 P_2 - v_2 P_1)) \\ &= v_1\,X(\gamma_{P,v}(s)) + v_2\,Y(\gamma_{P,v}(s)) \in \mathrm{H}\mathbb{H}^1_{\gamma_{P,v}(s)} \end{aligned} \tag{34}$$

then the statement (i) follows at once. Statement (ii) easily follows by means a simple calculations too. Thus let us prove (iii). First suppose that $g \in C^1(U(P_0, r_0))$ and put $G(s) := g(\gamma_{P,v}(s))$ if $0 \le s \le 1$. Now observe that by (34)

$$
\begin{aligned}
G'(s) &= \langle \nabla g(\gamma_{P,v}(s)), v_1\, X(\gamma_{P,v}(s)) + v_2\, Y(\gamma_{P,v}(s)) \rangle_{\mathbb{R}^3} \\
&= v_1\, Xg(\gamma_{P,v}(s)) + v_2\, Yg(\gamma_{P,v}(s))\,.
\end{aligned} \tag{35}
$$

On the other hand there exists $\bar{s} \in [0, 1]$ such that

$$G(1) - G(0) = G'(\bar{s})$$

and then by (35) the thesis follows in the case when $g \in C^1(U(P_0, r_0))$. In the general case we can approximate $g \in \mathbf{C}^1_{\mathbb{H}}(U(P_0, 2\, r_0))$ by a family of functions $g_\varepsilon \in C^1(U(P_0, r_0)$ such that

$$g_\varepsilon \to g, \quad Xg_\varepsilon \to Xg,\ Yg_\varepsilon \to Yg \text{ uniformly in } B(P_0, r_0) \tag{36}$$

(see, for instance, [26], step 1 of proof of Theorem 6.5). Then we can apply previous step to g_ε and then there exists $\bar{s_\varepsilon} \in (0, 1)$ such that

$$g_\varepsilon(\gamma_{P,v}(1)) - g_\varepsilon(\gamma_{P,v}(0)) = v_1\, Xg_\varepsilon(\gamma_{P,v}(\bar{s_\varepsilon})) + v_2\, Yg_\varepsilon(\gamma_{P,v}(\bar{s_\varepsilon}))\,. \tag{37}$$

On the other hand we can suppose that $\bar{s_\varepsilon} \to \bar{s} \in [0, 1]$ and so (36) and (37) yield the thesis. □

PROOF OF THEOREM 4.1. Without loss of generality we can assume that $P_0 = 0$ and

$$S \cap \Omega = \{f = 0\} \cap \Omega$$

with $\Omega = U(0, \frac{r_0}{4})$, $f \in \mathbf{C}^1_{\mathbb{H}}(U(0, 2\, r_0))$, $f(0) = 0$, $Xf > 0$ on $U(0, 2\, r_0)$. Then applying Theorem 2.23 and using the same notations there exists a 1-to-1, onto and continuous parameterization $\Phi : I := I_\delta \to \mathcal{U} \cap S$ of the type (16) with $\phi : I \to J$.

Let us prove that there exists a positive constant L_1 such that

$$|\phi(u) - \phi(u')| \le L_1\, |u - u'|^{1/2} \quad \forall\, u, u' \in I\,. \tag{38}$$

Let $u = (\eta, \tau)$, $u' = (\eta', \tau') \in I$, $P = \Phi(\eta, \tau)$, $Q = \Phi(\eta', \tau') \in \mathcal{U} \cap S$. Applying Lemma 4.4 with $g = f$ and

$$
\begin{aligned}
v = (v_1, v_2, v_3) &= P^{-1} \cdot Q \\
&= \big(\phi(\eta', \tau') - \phi(\eta, \tau), \eta' - \eta, \tau' - \tau + 2(\phi(\eta', \tau') + \phi(\eta, \tau))(\eta' - \eta)\big)
\end{aligned}
$$

$$
\begin{aligned}
0 = f(Q) - f(P) &= (f(\gamma_{P,v}(1)) - f(\gamma_{P,v}(0)) + (f(\gamma_{P,v}(0)) - f(P)) \\
&= Xf(\gamma_{P,v}(\bar{s}))\, v_1 + Yf(\gamma_{P,v}(\bar{s}))\, v_2 + (f(P \cdot (0, 0, v_3)) - f(P))
\end{aligned}
$$

and then

$$
(39)\qquad |v_1| \le \left| \frac{Yf(\gamma_{P,v}(\bar{s}))}{Xf(\gamma_{P,v}(\bar{s}))} \right| |v_2| + \frac{1}{|Xf(\gamma_{P,v}(\bar{s}))|} \left| f(P \cdot (0,0,v_3)) - f(P) \right| .
$$

Notice now because $Xf,\ Yf \in C^0(U(0, 2r_0))$, $\phi : I \to \mathbb{R}$ is continuous, by Theorem 2.20 and (9)

$$
M_1 = \sup_{U(0,r_0)} \frac{|Yf|}{|Xf|} < \infty, \qquad M_2 = \sup_{U(0,r_0)} \frac{1}{|Xf|} < \infty
$$

$$
M_3 = \sup_{R \neq S \in U(0,r_0)} \frac{|f(R) - f(S)|}{|R - S|^{\frac{1}{2}}} < \infty, \quad M_4 = \sup_I |\phi| < \infty .
$$

From (39)

$$
|\phi(u') - \phi(u)| \le M_1 |\eta' - \eta| + M_2 M_3 \left(|\tau' - \tau| + 4 M_4 |\eta' - \eta| \right)^{\frac{1}{2}}
$$

and then (38) for a suitable constant $L_1 > 0$. Then simple calculations yield the estimate (i) for a suitable constant $L > 0$.

Let us prove now the remaining part of the theorem. Let us denote by $I \subset \mathbb{R}^2$ a general open set. Suppose, by contradiction, the existence of a mapping $\Phi = (\phi_1, \phi_2, \phi_3) : I \to \mathbb{H}^1$ satisfying (i) and (ii) with $\frac{1}{2} < \alpha \le 1$ and $S = \{(0, y, t) : y, t \in \mathbb{R}\}$. Then from (ii) $\phi_1 \equiv 0$ in I. On the other hand (i) yields by a simple calculations

$$
|\phi_3(v) - \phi_3(u) + 2(\phi_1(u)\phi_2(v) - \phi_1(v)\phi_2(u))| \le L^2 |u - v|^{2\alpha} \quad \forall u, v \in I
$$

whence $\phi_3 : I \to R$ would be constant and then a contradiction arises. $\square$

REFERENCES

[1] L. Ambrosio – B. Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math. Ann. **318** (2000), 527-555.

[2] L. Ambrosio – B. Kirchheim, *Currents in metric spaces*, Acta Math. **185** (2000), 1-80.

[3] L. Ambrosio – V. Magnani, *Weak differentiability of BV function on stratified groups*, Math. Z. **245** (2003), 123-153.

[4] Z. Balogh, *Size of characteristic sets and functions with prescribed gradient*, J. Reine Angew. Math. **564** (2003), 63-84.

[5] Z. Balogh – M. Rickly – F. Serra Cassano, *Comparison of Hausdorff measures with respect to the Euclidean and the Heisenberg metric*, Publ. Mat. **47** (2003), 237-259.

[6] Z. Balogh – H. Hofer-Isenegger – J. T. Tyson, *Lifts of Lipischitz maps and horizontal fractals in the Heisenberg group*, Preprint (2003).

[7] A. BELLAÏCHE, "The tangent space in subriemannian geometry", In: "Subriemannian Geometry", Progress in Mathematics **144**, A. Bellaiche – J. Risler (eds.), Birkhauser Verlag, Basel, 1996.

[8] G. BELLETTINI – M. PAOLINI – S. VENTURINI, *Some results in surface measure in Calculus of Variations*, Ann. Mat. Pura Appl. (4) **170** (1996), 329-357.

[9] M. BIROLI – U. MOSCO, *Sobolev and isoperimetric inequalities for Dirichlet forms on homogeneous spaces*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **6** (1995), 37-44.

[10] L. CAPOGNA – D. DANIELLI – N. GAROFALO, *The geometric Sobolev embedding for vector fields and the isoperimetric inequality*, Comm. Anal. Geom. **12** (1994), 203-215.

[11] T. COULHON – L. SALOFF-COSTE, *Isopérimétrie pour les groupes et les variétés*, Rev. Mat. Iberoamericana **9** (1993), 293-314.

[12] M. CHLEBÍK, *Hausdorff lower s-densities and rectifiability of sets in n-space*, Preprint.

[13] D. DANIELLI – N. GAROFALO – D. M. NHIEU, *Traces inequalities for Carnot–Caratheodory spaces and applications*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **27** (1998), 195-252.

[14] G. DAVID – S. SEMMES, "Fractured Fractals and Broken Dreams. Self-Similar Geometry through Metric and Measure", Oxford University Press, 1997.

[15] G. DAVID – T. TORO, *Reifenberg flat metric spaces, snowballs, and embeddings*, Math. Ann. **315** (1999), 641-710.

[16] E. DE GIORGI, *Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni*, Ann.Mat.Pura Appl. (4) **36** (1954), 191-213.

[17] E. DE GIORGI, *Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni*, Ricerche Mat. **4** (1955), 95-113.

[18] E. DE GIORGI – F. COLOMBINI – L. C. PICCININI, "Frontiere orientate di misura minima e questioni collegate", Scuola Normale Superiore, Pisa, 1972.

[19] E. DE GIORGI, *Problema di Plateau generale e funzionali geodetici*, Atti Sem. Mat. Fis. Univ. Modena **43** (1995), 285-292.

[20] E. DE GIORGI, *Un progetto di teoria unitaria delle correnti, forme differenziali, varietà ambientate in spazi metrici, funzioni a variazione limitata*, Manuscript, (1995).

[21] E. DE GIORGI, *Un progetto di teoria delle correnti, forme differenziali e varietà non orientate in spazi metrici*, In: "Variational Methods, Non Linear Analysys and Differential Equations in Honour of J. P. Cecconi", (Genova 1993), M. Chicco et al. (eds.), ECIG, Genova, 67-71.

[22] H. FEDERER, "Geometric Measure Theory", Springer, 1969.

[23] B. FRANCHI – S. GALLOT – R. L. WHEEDEN, *Sobolev and isoperimetric inequalities for degenerate metrics*, Math. Ann. **300** (1994), 557-571.

[24] B. FRANCHI – R. SERAPIONI – F. SERRA CASSANO, *Meyers-Serrin Type Theorems and Relaxation of Variational Integrals Depending Vector Fields*, Houston J. Math. **22** (1996), 859-889.

[25] B. FRANCHI – R. SERAPIONI – F. SERRA CASSANO, *Sur les ensembles des périmètre fini dans le groupe de Heisenberg*, C.R. Acad. Sci. Paris Ser. I Math. **329** (1999), 183-188.

[26] B. FRANCHI – R. SERAPIONI – F. SERRA CASSANO, *Rectifiability and perimeter in the Heisenberg group*, Math. Ann. **321** (2001), 479-531.

[27] B. FRANCHI – R. SERAPIONI – F. SERRA CASSANO, *Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups*, Comm. Anal. Geom. **11** (2003) 909-944.

[28] B. FRANCHI – R. SERAPIONI – F. SERRA CASSANO, *Rectifiability and perimeter in step 2 groups*, Proceedings of Equadiff10, 2001, Math. Bohem. **127** (2002), 219-228.

[29] B. FRANCHI – R. SERAPIONI – F. SERRA CASSANO, *On the structure of finite perimeter sets in step 2 Carnot groups*, J. Geom. Anal. **13** (2003), 421-466.

[30] N. GAROFALO – D. M. NHIEU, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), 1081-1144.

[31] M. GROMOV, "Carnot-Carathéodory spaces seen from within", In: "Subriemannian Geometry", Progress in Mathematics **144**, A. Bellaiche and J. Risler (eds.), Birkhauser Verlag, Basel, 1996.

[32] P. HAJŁASZ – P. KOSKELA, "Sobolev met Poincare", Mem. AMS **145**, 2000.

[33] J. HEINONEN – P. KOSKELA – N. SHANMUGANLINGAM – J. TYSON, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*, J. Anal. Math. **85** (2001), 87-139.

[34] B. KIRCHHEIM, *Rectifiable metric spaces: local structure and regularity of the Hausdorff measure*, Proc. Amer. Math. Soc. **121** (1994), 113-123.

[35] B. KIRCHHEIM – V. MAGNANI, *A counterexample to the metric differentiability*, Proc. Edinburgh Math. Soc. **46** (2003), 221-227.

[36] A. KORÁNYI – H. M. REIMANN, *Foundation for the Theory of Quasiconformal Mappings on the Heisenberg Group*, Adv. Math. **111** (1995), 1-87.

[37] A. LORENT, *Rectifiability of measures with locally uniform cube density*, Proc. London Math. Soc. (3) **86** (2003), 153-249.

[38] V. MAGNANI, *Differentiability and Area formula on stratified Lie groups*, Houston J. Math. **27** (2001), 297-323.

[39] V. MAGNANI, *Characteristic points, rectifiability and perimeter measure on stratified groups*, Preprint (2003).

[40] M. A. MARTIN – P. MATTILA, *Hausorff measures, Hölder continuous maps and self-similar fractals*, Math. Proc. Cambridge Philos. Soc. **114** (1993), 37-42.

[41] M. A. MARTIN – P. MATTILA, *On The Parametrization of Self-Similar And Other Fractal Sets*, Trans. Amer. Math. Soc. **128** (2000), 2641-2648.

[42] P. MATTILA, "Geometry of sets and measures in Euclidean spaces", Cambridge U.P., 1995.

[43] J. MITCHELL, *On Carnot-Carathèodory metrics*, J. Differential Geom. **21** (1985), 35-45.

[44] R. MONTI – F. SERRA CASSANO, *Surface measures in Carnot-Carathéodory spaces*, Calc. Var. Partial Differential Equations **13** (2001), 339-376.

[45] A. NAGEL – E. M. STEIN – S. WAINGER, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math. **155** (1985), 103-147.

[46] P. PANSU, *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un*, Ann. of Math. **129** (1989), 1-60.

[47] P. PANSU, *Une inégalité isopérimétrique sur le groupe de Heisenberg*, C.R. Acad. Sci. Paris **295** I (1982), 127-130.

[48] P. PANSU, *Geometrie du Group d'Heisenberg*, These pour le titre de Docteur 3ème cycle, Universite Paris VII, (1982).

[49] S. D. PAULS, *A notion of rectifiability modelled on Carnot groups*, Indiana Univ. Math. J. **53** (2004), 49-81.

[50] D. Preiss – J. Tišer, *On Besicovitch* $1/2$*-problem*, J. London Math. Soc. **45** (1992), 279-287.

[51] E. Reifenberg, *Solution of the Plateau problem for m-dimensional surfaces of varying topological type*, Acta Math. **104** (1960), 198-223.

[52] S. Semmes, *Chord- arc surfaces with small constant I*, Adv. Math. **85** (1991), 198-223.

[53] S. Semmes, *Chord- arc surfaces with small constant II*, Adv. Math. **88** (1991), 170-189.

[54] S. Semmes, *On the non existence of bilipschitz parameterization and geometric problems about* A_∞ *weights*, Rev. Mat. Iberoamericana **12** (1996), 337-410.

[55] S. Semmes, *Good metric spaces without good parameterization*, Rev. Mat. Iberoamericana **12** (1996), 187-275.

[56] L. Simon, "Lectures on Geometric Measure Theory", Proc. Centre for Math. Anal., Australian Nat. Univ. **3**, 1983.

[57] E. M. Stein, "Harmonic Analysis", Princeton University Press, 1993.

[58] R. Strichartz, *Self-similarity on nilpotent Lie groups*, Contemp. Math. **140** (1992), 123-157.

[59] T. Toro, *Geometric conditions and existence of bi-lipschitz parameterizations*, Duke Math. J. **77** (1995), 193-227.

[60] N. Th. Varopoulos, *Analysis on Lie Groups*, J. Funct. Anal. **76** (1988), 346-410.

[61] N. Th. Varopoulos – L. Saloff-Coste – T. Coulhon, "Analysis and Geometry on Groups", Cambridge University Press, Cambridge, 1992.

[62] S. K. Vodop'yanov, $\mathcal{P}$*-differentiability on Carnot groups in different topologies and related topics*, Proc. on Analysis and Geometry, 603-670, Sobolev Institute Press, Novosibirsk, 2000.

Mathematical Institute
University of Oxford
24-29 St Giles'
Oxford, OX1 3LB, UK
kirchhei@maths.ox.ac.uk

Dipartimento di Matematica
Università di Trento
Via Sommarive, 14
38050 Povo (Trento), Italia
cassano@science.unitn.it