Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. IV (2005), 79-128

Some relations among volume, intrinsic perimeter and one-dimensional restrictions of BV functions in Carnot groups

FRANCESCOPAOLO MONTEFALCONE

Abstract. Let $\mathbb{G}$ be a k-step Carnot group. The first aim of this paper is to show an interplay between volume and $\mathbb{G}$-perimeter, using one-dimensional horizontal slicing. What we prove is a kind of Fubini theorem for $\mathbb{G}$-regular submanifolds of codimension one. We then give some applications of this result: slicing of $BV_{\mathbb{G}}$ functions, integral geometric formulae for volume and $\mathbb{G}$-perimeter and, making use of a suitable notion of convexity, called $\mathbb{G}$-*convexity*, we state a Cauchy type formula for $\mathbb{G}$-convex sets. Finally, in the last section we prove a sub-Riemannian Santaló formula showing some related applications. In particular we find two lower bounds for the first eigenvalue of the Dirichlet problem for the Carnot sub-Laplacian $\Delta_{\mathbb{G}}$ on smooth domains.

Mathematics Subject Classification (2000): 49Q15 (primary); 46E35, 22E60 (secondary).

1. Introduction

In the last few years many efforts have been produced to develop a Geometric Measure Theory in very general metric spaces along the lines originally suggested in Federer's book [31]. In many respects, deep contributions to this program have been carried out, with different approaches, by De Giorgi [27, 28, 29], Gromov [45, 46], Preiss & Tisěr [67], David & Semmes [24], Cheeger [17], Ambrosio & Kirchheim [3, 4], and Montgomery [61], just to mention some examples. Moreover, progresses in these areas is somehow connected with the contemporary development of a theory of Sobolev spaces in abstract metric settings that culminated in the paper [48].

Geometries associated with a family of vector fields and Carnot Carathéodory spaces are the main models of these investigations. On this subject there is a wide literature and we shall refer the reader to [10, 14, 22, 34, 35, 37, 42, 51,

Partially supported by University of Bologna, Italy, founds for selected research topics.
Pervenuto alla Redazione il 12 febbraio 2004 e in forma definitiva il 26 novembre 2004.

58, 60, 61, 64, 66, 71, 74, 75]. Of course, this list is far from being complete but illustrates fairly well some of the directions followed by the present research. The closeness of Analysis and Geometry is here particularly stressed by the fact that, initially, these questions arose in the field of hypoelliptic differential equations; in this respect, we mention the remarkable paper of Rothschild and Stein [68]. Also we have to emphasize the special importance of the related studies on nilpotent Lie groups; as references we would cite the works of Folland and Stein [32, 33, 70] and Goodman [44] as regards the analytical aspects, and, for instance, those of Pansu [65, 66] and Korányi & Reimann [52] to better appreciate several geometrical features; see [49, 61] and [62] for useful comments.

The geometric setting of this paper is that of *Carnot groups*, also known in literature as non Abelian vector spaces or sub-Riemannian groups. They constitute a wide class of non trivial examples of Carnot Carathéodory spaces; see [10, 61]. Roughly speaking, a Carnot group $\mathbb{G}$ is a nilpotent and stratified Lie group endowed with a one-parameter family of dilations adapted to the stratification. They play a crucial role in the theory of Carnot Carathéodory geometries since a deep theorem of Mitchell states that the tangent cone (in the sense of Gromov) of a Carnot Carathéodory space is a suitable Carnot group; see [60] and [61] for clarifying discussions.

Since Carnot groups are also homogenous groups, according to the definition of [33], harmonic analysis and P.D.E.'s on them have become a rich and extensive subject of investigations.

Many classical tools of Calculus of Variations have been generalized to this context and, in particular, the theory of functions of bounded variation and that related of Caccioppoli sets.

A motivation for a great deal of new researches has been a De Giorgi type rectifiability result in the Heisenberg group (i.e. the most simple non Abelian Carnot group) due to Franchi, Serapioni and Serra Cassano; see [39] and [40, 41] for further generalizations. For a survey of results on these topics of Geometric Measure Theory and for more detailed bibliographic references, we shall refer the reader to [1, 5, 23, 40, 41, 49, 56, 61, 62, 63, 76].

In this paper we are mainly concerned with some elementary questions about measures on Carnot groups and our starting point is a Fubini type theorem for codimension one $\mathbb{G}$-regular submanifolds. We refer the reader to Theorem 2.2 for a precise statement.

The proof of Theorem 2.2 follows mainly by our Proposition 2.1, using some non-trivial approximation results. We would also remark that the main problem to get these formulae is that of a good choice of projection maps. Here we use, for a great number of integral formulae, the projections along the integral curves of a generating family of vector fields of $\mathbb{G}$, called *horizontal projections*; see Section 2.1 for more detailed comments.

These theorems enable us to consider one-dimensional slicing of functions and we apply this procedure to state a characterization of the space $BV_{\mathbb{G}}$ of functions of bounded variation on $\mathbb{G}$. Here the key point is that to link the total

$\mathbb{G}$-variation of a function with the variation of its one-dimensional restrictions; see Definition 1.13 and Theorem 3.7. We remark that a similar characterization was proved in [76] for Sobolev spaces in Carnot groups.

Secondly, we deduce some integral geometric formulae and one in particular for the intrinsic perimeter measure, in Proposition 3.13. Afterwards, we introduce a notion of convexity, called $\mathbb{G}$-*convexity*, analogous to that recently given in [23] and in [55]. We then prove that $\mathbb{G}$-convex sets verify an integral Cauchy type formula for the $\mathbb{G}$-perimeter and a related inequality, showing that, in a sense, this kind of convex set minimize the intrinsic $\mathbb{G}$-perimeter.

The last section is devoted to prove the validity of a Santaló type formula and some of its applications. We stress that our Theorem 4.4 generalizes to arbitrary Carnot groups a result already proved in Pansu's thesis [65]. This formula is strictly connected with the introduction of a measure on the unit horizontal bundle of $\mathbb{G}$ and with its invariance under a suitable restriction of the Riemannian geodesic flow. We refer to Section 4 for a detailed introduction.

We then apply Theorem 4.4 to establish a geometric inequality linking volume, $\mathbb{G}$-perimeter and diameter of smooth bounded domains.

Finally, as an application to Analysis in Carnot groups, we perform explicit computations to find two lower bounds for the first eigenvalue of the Dirichlet problem for the Carnot sub-Laplacian $\Delta_{\mathbb{G}}$ on smooth domains. This will be done quite easily, as we will see, by adapting some arguments of Riemannian geometry.

The following two subsections are devoted to introduce, in a self-contained way, definitions, results and preliminary tools necessary for the sequel.

ACKNOWLEDGEMENTS. I would like to thank my thesis advisor, Prof. B. Franchi, for the generous help he has given me throughout the preparation of this paper and Prof. L. Ambrosio for many clarifying comments about some topics here developed. Finally, I would also express my gratitude to Prof. E. Lanconelli for his constant support.

1.1. Carnot groups

Below we will introduce the geometric background for Carnot groups for which we refer the reader to [10, 19, 33, 50, 52, 45, 61, 65, 70, 73].

Let $\mathbb{G}$ be a connected, simply connected, nilpotent Lie group, with group law denoted by $\cdot$. It is well-known that any $x \in \mathbb{G}$ defines smooth maps $l_x, r_x : \mathbb{G} \to \mathbb{G}$, called *left translation* and *right translation*, respectively, by $l_x(y) := x \cdot y,\ r_x(y) := y \cdot x$. Left translations plays a key role in the theory of Lie groups since the *Lie algebra* $\mathfrak{g}$ associated with $\mathbb{G}$ is defined as the set of all *left invariant* vector fields. Explicitly, if $x \in \mathbb{G}$, then X is left invariant if

$$(X\psi)(l_x(y)) = X(\psi \circ l_x)(y) = \frac{d}{dt}\Big|_{t=0}\psi(x \cdot \mathrm{Exp}(tX))\ \forall\, \psi \in \mathbf{C}^{\infty}(\mathbb{G})\ \forall\, y \in \mathbb{G}\,. \quad (1)$$

The set of all left invariant vector fields X on $\mathbb{G}$ is a vector space and it becomes a Lie algebra since the bracket of left invariant vector fields is still a left invariant vector field. This algebra is canonically isomorphic to $T_e\mathbb{G}$, the tangent space at the identity e of $\mathbb{G}$, via the identification of any vector field X with its value X_e at e, where the isomorphism is given by the differential of the left translation at e, i.e. $d\,l_x : T_e\mathbb{G} \longrightarrow T_x\mathbb{G}$ for $x \in \mathbb{G}$. Let us consider now the following Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = x \in \mathbb{G} \end{cases}$$

where $X \in \mathfrak{g}$. From the elementary theory of O.D.E.'s we know that there exists a unique smooth solution which is defined on all of $\mathbb{R}$ since every left invariant vector field on a Lie group is complete. As usual we set

$$\gamma(t) = \exp[tX](x) \quad \forall\, t \in \mathbb{R}\,. \tag{2}$$

The integral curve of X is a *one-parameter subgroup* of $\mathbb{G}$ and one can think of it as the image in $\mathbb{G}$ of the one-dimensional vector subspace of $\mathfrak{g}$ generated by X. We may define the *exponential map* setting $\mathrm{Exp} : \mathfrak{g} \longmapsto \mathbb{G}$, $\mathrm{Exp}(X) := \exp[X](1)$. Since $\mathbb{G}$ is a simply connected Lie group we have that Exp is a global analytic diffeomorphism between $\mathfrak{g}$ and $\mathbb{G}$; see [19, 50, 73]. Hence the inverse of Exp is defined globally and we denote this map by Log. Notice that any solution γ of the above Cauchy problem is the right translation by $\mathrm{Exp}(tX)$ of $x \in \mathbb{G}$, i.e.

$$\gamma(t) = \exp[tX](x) = x \cdot \mathrm{Exp}(tX) = r_{\mathrm{Exp}(tX)}(x) \quad \forall\, t \in \mathbb{R}\,.$$

We have assumed that $\mathbb{G}$ is nilpotent and this means that its Lie algebra $\mathfrak{g}$ is nilpotent. To explain this condition first we define by induction $\mathfrak{g}_1 := \mathfrak{g}$ and $\mathfrak{g}_i := [\mathfrak{g}_1, \mathfrak{g}_{i-1}]$ for $i > 1$, where $[\mathfrak{g}_1, \mathfrak{g}_{i-1}]$ is the set of the Lie brackets $[X, Y]$ for $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_{i-1}$. We say that $\mathfrak{g}$ is *nilpotent of k-step* if $\mathfrak{g}_k \neq \{0\}$ and $\mathfrak{g}_{k+1} = \{0\}$. A connected, simply connected, nilpotent Lie group $\mathbb{G}$ is a *Carnot group of step k* if its Lie algebra $\mathfrak{g}$ admits a *step k stratification*, i.e. there exist linear subspaces $V_1, \ldots, V_k$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus \ldots \oplus V_k, \quad [V_1, V_{i-1}] = V_i \ \text{ for } i = 2, \ldots, k \ \text{ and } \ V_{k+1} = \{0\}\,. \tag{3}$$

Hereafter, we will assume that the underlying manifold of $\mathbb{G}$ is $\mathbb{R}^n$, for some $n \in \mathbb{N}$. Since $\mathfrak{g}$ nilpotent it follows that it is finite-dimensional as a vector space and that $\dim \mathfrak{g} = n$. We then choose as a vector basis of $\mathfrak{g}$ the standard one of $\mathbb{R}^n$, say $\mathrm{e}_1, \ldots, \mathrm{e}_n$. By means of the exponential map any $x \in \mathbb{G}$ can be written in a unique way as $x = \mathrm{Exp}(x_1\mathrm{e}_1 + \ldots + x_n\mathrm{e}_n)$. Therefore, using *exponential coordinates*, x is identified with the n-tuple $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbb{G}$ is identified with $\mathbb{R}^n$ equipped with the group law $\cdot$. The group law is completely determined by the Campbell-Hausdorff formula which states

$$\mathrm{Exp}(X) \cdot \mathrm{Exp}(Y) = \mathrm{Exp}(X * Y) \qquad \forall\, X,\ Y \in \mathfrak{g}\,,$$

where $X * Y$ is given by the following identity

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + R(X, Y), \quad (4)$$

where $R(X, Y)$ denotes a formal series of commutators which becomes a finite sum in our case, since $\mathbb{G}$ is a connected and simply connected nilpotent Lie group (see [19, 73]). Notice also that the group law turns out to be polynomial in the coordinates of $\mathbb{R}^n$. Moreover the Campbell-Hausdorff formula implies that the identity e of $\mathbb{G}$ is $0 \in \mathbb{R}^n$ and that $x^{-1} = -x$ for $x \in \mathbb{G}$.

We have seen that V_1, the first layer of $\mathfrak{g}$, generates the whole algebra by iterated Lie brackets. We set $m_i := \dim V_i$ for $i = 1, \ldots, k$ and $h_i := m_1 + \ldots + m_i$ where $h_0 := 0$ and $h_k := n$. The standard basis $\mathrm{e}_1, \ldots, \mathrm{e}_n$ of $\mathbb{R}^n$ can be adapted to the stratification of $\mathfrak{g}$ assuming that

$$\mathrm{e}_{h_{j-1}+1}, \ldots, \mathrm{e}_{h_j} \quad \text{is a basis of } V_j \text{ for each } j = 1, \ldots, k. \quad (5)$$

Therefore we may define a smooth global frame of left invariant vector fields for $\mathbb{G}$ related to the fixed basis of $\mathfrak{g}$ by setting

$$X_j(x) := d\, l_x \mathrm{e}_j \quad \text{for } x \in \mathbb{G} \text{ and } j = 1, \ldots, n, \quad (6)$$

or, equivalently, $X_j(0) = \mathrm{e}_j$ for $j = 1, \ldots, n$.

Any Carnot group $\mathbb{G}$ is endowed with a family of group automorphisms $\delta_\lambda : \mathbb{G} \longrightarrow \mathbb{G}$, the so-called *intrinsic dilations*, defined by

$$\delta_\lambda(x_1, \ldots, x_n) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n) \quad \text{for } x \in \mathbb{G} \text{ whenever } \lambda > 0, \quad (7)$$

where $\alpha_i \in \mathbb{N}$ is called *homogeneity* of the variable x_i and it is defined as $\alpha_j := i$ whenever $h_{i-1} + 1 \le j \le h_i$. Hence $1 = \alpha_1 = \ldots = \alpha_{m_1} < \alpha_{m_1+1} = 2 \le \ldots \le \alpha_n = k$; see [10, 33].

According to [33, 70], $\mathbb{G}$ is a *homogeneous group* with respect to the family of dilations δ_λ and hence we will denote by $Q := \sum_{i=1}^{k} i \dim V_i$ its *homogeneous dimension*.

The smooth subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by $X_1, \ldots, X_{m_1}$ is called the *horizontal bundle* $H\mathbb{G}$ on $\mathbb{G}$ and we refer to $X_1, \ldots, X_{m_1}$ as *generating vector fields* of the group. The fibers of this bundle are given by $H_z\mathbb{G} = \mathrm{span}\{X_1(z), \ldots, X_{m_1}(z)\}$, for $z \in \mathbb{G}$, and the horizontal bundle $H\mathbb{G}$ is the disjoint union of the *horizontal fibers*, i.e.

$$H\mathbb{G} := \bigsqcup_{z \in \mathbb{G}} H_z\mathbb{G}.$$

From now on we shall write any element of $H\mathbb{G}$ as an ordered pair $(z; Z)$ with $z \in \mathbb{G}$ and $Z \in H_z\mathbb{G}$. The projection map $\pi_{|H\mathbb{G}}$ on $\mathbb{G}$ is the restriction to $H\mathbb{G}$ of the projection map $\pi : T\mathbb{G} \longmapsto \mathbb{G}$, i.e. $\pi_{|H\mathbb{G}} : H\mathbb{G} \longmapsto \mathbb{G}$, $\pi_{|H\mathbb{G}}(z; Z) = z$ for $(z; Z) \in H\mathbb{G}$. We may define a *sub-Riemannian structure* on $\mathbb{G}$ endowing each fiber of $H\mathbb{G}$ with an inner product $\langle\cdot,\cdot\rangle_{H\mathbb{G}}$ and with a norm $|\cdot|_{H\mathbb{G}}$ that makes the basis $X_1,\dots,X_{m_1}$ an orthonormal basis. More precisely, if $(z;Z),(z;Z_1),(z;Z_2)\in H\mathbb{G}$ we shall write

$$\langle Z_1, Z_2\rangle_{H_z\mathbb{G}} = \sum_{j=1}^{m_1} a_j\, b_j, \quad |Z|_{H_z\mathbb{G}} = \sqrt{\langle Z, Z\rangle_{H_z\mathbb{G}}}\,,$$

where $Z_1(0)=\sum_{j=1}^{m_1} a_j \mathrm{e}_j,\ Z_2(0)=\sum_{j=1}^{m_1} b_j \mathrm{e}_j$.

Notation. Throughout this paper, unless mentioned otherwise, the notation $\langle\cdot,\cdot\rangle$ will be used to denote the euclidean inner product in $\mathbb{R}^k$ $(k=1,\ m_1 \text{ or } n)$ where k is given by the context and similarly for $|\cdot|$.

The sections of $H\mathbb{G}$ are called horizontal sections; the elements of $H\mathbb{G}$ are called horizontal vectors while those of $T\mathbb{G}$ that are not horizontal are called vertical vectors. Each horizontal section is identified with its canonical coordinates with respect to the moving frame $X_1,\dots,X_{m_1}$, so that a horizontal section ψ is regarded as a function $\psi=(\psi_1,\dots,\psi_{m_1}):\mathbb{R}^n\longrightarrow\mathbb{R}^{m_1}$. If $\mathcal{D}\subset\mathbb{G}$ we shall denote by $H\mathcal{D}$ the restriction to $\mathcal{D}$ of the structure of horizontal bundle, i.e.

$$H\mathcal{D} := \left\{(z;Z)\in H\mathbb{G} : z=\pi_{|H\mathbb{G}}(Z)\in\mathcal{D}\right\}.$$

Some of the next topics require us to introduce the notion of *unit horizontal bundle* on $\mathbb{G}$. To this end let us set $\mathring{H}\mathbb{G} := H\mathbb{G}\setminus\{0_{H\mathbb{G}}\}$, where $0_{H\mathbb{G}}$ is the zero section of $H\mathbb{G}$. Denoting by $UH\mathbb{G}$ the quotient of $\mathring{H}\mathbb{G}$ by the positive dilations we obtain a bundle structure on $\mathbb{G}$, called *unit horizontal bundle* on $\mathbb{G}$, whose projection map $\pi_{|UH\mathbb{G}}: UH\mathbb{G}\longmapsto\mathbb{G}$ is given by $\pi_{|UH\mathbb{G}}(z;Z)=z$ for $(z;Z)\in H\mathbb{G}$. Notice that each fiber $UH_z\mathbb{G}$ of $\pi_{|UH\mathbb{G}}$ can be identified with the unit sphere $\mathbb{S}^{m_1-1}$ of $\mathbb{R}^{m_1}$. Roughly speaking, $UH_z\mathbb{G}$ is the subset of $H_z\mathbb{G}$ of all unit vectors with respect to the norm on the fiber $|\cdot|_{H_z\mathbb{G}}$.

We shall now introduce the *Carnot-Carathéodory distance* and some related topics which can be found in [6, 10, 45, 46, 61, 62, 66].

We call a curve $\gamma:[0,1]\longrightarrow\mathbb{G}$ a *horizontal curve* if $\dot\gamma(t)\in H\mathbb{G}$ for all $t\in[0,1]$.

Definition 1.1. For every $x,y\in\mathbb{G}$ the *cc-distance* $d_c(x,y)$ is defined by

$$d_c(x,y) := \inf\int_0^1 |\dot\gamma(t)|_{H\mathbb{G}}\,dt$$

where the infimum is taken over all horizontal curves such that $\gamma(0)=x$ and $\gamma(1)=y$.

By Chow's theorem the set of horizontal curves joining two different points is not empty since the rank of the Lie algebra generated by $X_1, \dots, X_{m_1}$ is n and hence d_c is a left-invariant metric on $\mathbb{G}$. Note that d_c induces the same topology as the euclidean one on $\mathbb{R}^n$. We will denote by $\mathcal{U}_c(x, r)$ and $\mathcal{B}_c(x, r)$, respectively, the open and closed balls of center x and radius r with respect to d_c. Moreover d_c is well behaved with respect to left translation and dilations, indeed we have

$$\begin{gathered} d_c(z \cdot x, z \cdot y) = d_c(x, y), \quad d_c(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_c(x, y) \\ \forall\, x, y, z \in \mathbb{G} \qquad \forall\, \lambda \in \mathbb{R}\,. \end{gathered} \tag{8}$$

From now on, if $s \geq 0$, then $\mathcal{H}^s$ will denote the s-dimensional Hausdorff measure obtained from the euclidean distance in $\mathbb{R}^n$, while $\mathcal{H}_c^s$ will denote the s-dimensional spherical Hausdorff measure obtained from the cc distance d_c, using Carathédory's construction. We remark that a theorem of J. Mitchell states that the Hausdorff dimension of a Carnot group with respect to the cc-distance d_c equals its homogeneous dimension Q; see [60].

Later on we will use the following result.

Proposition 1.2. *Let $\gamma : [0, 1] \longrightarrow \mathbb{G}$ be an absolutely continuous horizontal curve. Then there exists the* **metric derivative** *$|\dot\gamma|$ of γ for $\mathcal{L}^1-$a.e. $t \in [0, 1]$ and we have*

$$|\dot\gamma|(t) := \lim_{\epsilon \to 0} \frac{d_c(\gamma(t+\epsilon), \gamma(t))}{|\epsilon|} = |\dot\gamma(t)|_{H\mathbb{G}} \quad \text{for } \mathcal{L}^1 - \text{a.e. } t \in [0, 1]\,.$$

Moreover, if $\mathrm{Var}(\gamma)$ *denotes the total variation of γ (with respect to the cc-distance d_c), then*

$$\mathrm{Var}(\gamma) = \int_0^1 |\dot\gamma(t)|_{H\mathbb{G}}\, dt \geq \mathcal{H}_c^1(\gamma([0, 1]))$$

and the equality holds if and only if γ is injective.

For a proof of this statement see Theorem 4.4.1 of [6] and Theorem 1.3.5 of [62]. Note that if γ is an integral curve of a fixed horizontal left invariant vector field $X \in H\mathbb{G}$ then $\dot\gamma(t)$ is constant being the vector of coordinates of X in V_1, and for all $\mathcal{K} \subset \gamma$ compact we get

$$\mathcal{H}_c^1(\mathcal{K}) = \int_{\gamma^{-1}(\mathcal{K})} |\dot\gamma(t)|_{H\mathbb{G}}\, dt = |X| \cdot \mathcal{L}^1(\gamma^{-1}(\mathcal{K}))\,. \tag{9}$$

We now summarize some features of differential forms on Lie groups; for more details the reader is referred to [50]. We say that a differential form ω on $\mathbb{G}$ is *left invariant* if $l_x^*\omega = \omega$ for all $x \in \mathbb{G}$, where the map l_x^* ($l_x^* : T_y^*\mathbb{G} \longmapsto T_{l_x(y)}^*\mathbb{G}$ for $y \in \mathbb{G}$) denotes the pullback by the left translation l_x. Analogously we define *right invariant* differential forms and we call *bi-invariant* a differential form that is both left and right invariant. A smooth *global coframe* for $\mathbb{G}$, i.e. a basis $\omega_1, \dots, \omega_n$ for $T^*\mathbb{G}$, is defined by the condition $\omega_i(X_j) = \delta_{ij}$ (for

$i, j = 1, \ldots, n$), where $X_1, \ldots, X_n$ is the chosen global frame for $\mathbb{G}$ and δ_{ij} denotes the Kronecker delta. A left invariant *volume form* on $\mathbb{G}$ is given by

$$\Theta := \omega_1 \wedge \ldots \wedge \omega_n \,. \tag{10}$$

With the previous assumptions on $\mathbb{G}$ we easily get that Θ turns out to be a bi-invariant n-form that is called the *Haar volume form* on $\mathbb{G}$. Since by integration of Θ we obtain the standard n-dimensional Lebesgue measure $\mathcal{L}^n$, this one is the Haar measure on $\mathbb{G}$.

Remark 1.3. We shall emphasize that, since also $\mathcal{H}_c^Q$, the Q-dimensional spherical Hausdorff measure of $\mathbb{G}$, is a Haar measure of $\mathbb{G}$ and since, up to scale, there is only one Haar measure on locally compact Lie groups, we must have

$$\mathcal{L}^n \llcorner \mathcal{B} = k_Q \cdot \mathcal{H}_c^Q \llcorner \mathcal{B} \qquad \forall\ \mathcal{B} \in \mathcal{B}or(\mathbb{G})\,, \tag{11}$$

where k_Q is an absolute constant and $\mathcal{B}or(\mathbb{G})$ denotes the family of Borel subsets of $\mathbb{G}$.

The next proposition will be used in the last section of this paper and introduces the so-called *Maurer-Cartan equations*, [50].

Proposition 1.4. *Let $\omega_1, \ldots, \omega_n$ be the global coframe for $\mathbb{G}$ determined by $\omega_i(X_j) = \delta_{ij}$ (for $i, j = 1, \ldots, n$), where $X_1, \ldots, X_n$ is the global frame for $\mathbb{G}$. Then*

$$d\omega_i = -\frac{1}{2} \sum_{j,h=1}^{n} c_{jh}^i \omega_j \wedge \omega_h \tag{12}$$

where c_{jh}^i are the **structural constants** *given by $[X_j, X_h] := \sum_{i=1}^n c_{jh}^i X_i$.*

Remark 1.5. The stratification of $\mathfrak{g}$ implies that if $X_j \in V_r$ and $X_h \in V_s$ then $[X_j, X_h] \in V_{r+s}$. Therefore

$$c_{jh}^i \neq 0 \Longrightarrow h_{r+s-1} < i < h_{r+s+1} \quad \forall\ i,\ j,\ h = 1, \ldots, n\,.$$

In particular $c_{ji}^i = c_{ih}^i = 0 \quad \forall\, i,\ j,\ h = 1, \ldots, n$. Moreover, let i be such that $h_{l-1} < i < h_{l+1}$. Then $c_{jh}^i \neq 0$ only if, for any j, h such that $h_{r-1} < j < h_{r+1}$ and $h_{s-1} < h < h_{s+1}$, we have that $l = r + s$. This means that we may rewrite the summation in (12) as follows

$$d\omega_i = -\frac{1}{2} \sum_{1 \leq j, h \leq h_{l-1}} c_{jh}^i \omega_j \wedge \omega_h \quad \text{whenever } h_{l-1} < i < h_{l+1}\,.$$

In what follows we collect several properties concerning group operation and canonical vector (resp. co-vector) fields of $\mathbb{G}$.

Proposition 1.6. *The group law has the form*

$$x \cdot y = \mathcal{P}(x, y) = x + y + \mathcal{Q}(x, y) \quad \forall\, x, y \in \mathbb{G}\,, \tag{13}$$

where $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_n) : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ *and* $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_n) : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ *are polynomial functions and, more precisely, for each* $i = 1, \dots, n$*, we have that* $\mathcal{P}_i$ *and* $\mathcal{Q}_i$ *are homogeneous polynomials of degree* α_i *with respect to the dilations of* $\mathbb{G}$*, i.e.*

$$\mathcal{P}_i(\delta_\lambda x, \delta_\lambda y) = \lambda^{\alpha_i} \mathcal{P}_i(x, y), \qquad \mathcal{Q}_i(\delta_\lambda x, \delta_\lambda y) = \lambda^{\alpha_i} \mathcal{Q}_i(x, y) \quad \forall\, x, y \in \mathbb{G}\,. \tag{14}$$

Moreover, for every $x, y \in \mathbb{G}$ *we have that*

(i) $\mathcal{Q}_1(x, y) = \dots = \mathcal{Q}_{m_1}(x, y) = 0$;
(ii) $\mathcal{Q}_j(x, 0) = \mathcal{Q}_j(0, y) = 0$ *and* $\mathcal{Q}_j(x, x) = \mathcal{Q}_j(x, -x) = 0$ *for* $m_1 < j \le n$;
(iii) $\mathcal{Q}_j(x, y) = \mathcal{Q}_j(x_1, \dots, x_{h_{i-1}}, y_1, \dots, y_{h_{i-1}})$ *if* $1 < i \le k$ *and* $j \le h_i$;
(iv) $\mathcal{Q}_j(x, y)$ *is a sum of terms each of which contains a factor* $(x_i y_l - x_l y_i)$ *for some* $1 \le i, l < j$*, whenever* $j > m_1$*.*

Proof. For the first part see [70], Chapter 12, Section 5, while the last statement follows by using Campbell-Hausdorff formula; see [63] for a detailed proof. □

Proposition 1.7. *Each left invariant vector fields of the moving frame for* $\mathbb{G}$ *have polynomial coefficients and it can be written as follows*

$$\begin{gathered} X_j(x) = \frac{\partial}{\partial x_j} + \sum_{i > h_l}^{n} a_{i,j}(x) \frac{\partial}{\partial x_i} \qquad \text{for } j = 1, \dots, n \text{ and } j \le h_l\,, \\ a_{i,j}(x) := \frac{\partial}{\partial y_j}\Big|_{y=0} \mathcal{Q}_i(x, y)\,. \end{gathered} \tag{15}$$

Thus if $j \le h_l$ *we have* $a_{i,j}(x) = a_{i,j}(x_1, \dots, x_{h_{l-1}})$, $a_{i,j}(0) = 0$ *and* $a_{i,j}(\delta_\lambda(x)) = \lambda^{\alpha_i - \alpha_j} a_{i,j}(x)$*. Moreover each* X_j $(j = 1, \dots, n)$ *turns out to be homogeneous of degree* α_j *with respect to positive dilations, i.e.*

$$X_j(\psi \circ \delta_\lambda)(x) = \lambda^{\alpha_j} X_j(\psi)((\delta_\lambda(x))) \quad \forall\, \psi \in \mathbf{C}^\infty(\mathbb{G}) \ \forall\, x \in \mathbb{G}\ \forall\, \lambda > 0\,;$$

see [33, 40, 63].

Finally, we recall some basic results about calculus in Carnot groups. We say that a map $L : \mathbb{G} \longrightarrow \mathbb{R}$ is $\mathbb{G}$-*linear* if is a group homomorphism of $(\mathbb{G}, \cdot)$ onto $(\mathbb{R}, +)$ and if it is positively homogeneous of degree 1 with respect to the positive dilations of $\mathbb{G}$, i.e. $L(\delta_\lambda x) = \lambda L(x)$ for every $\lambda > 0$ and $x \in \mathbb{G}$. The $\mathbb{R}$-linear set of $\mathbb{G}$-*linear* real valued functionals is indicated as $\mathcal{L}_{\mathbb{G}}$ and it is endowed with the norm $\|L\|_{\mathcal{L}_{\mathbb{G}}} := \sup\{|L(x)| : d_c(x, 0) \le 1\}$. For a fixed left invariant frame $X_1, \dots, X_n$ on $\mathbb{G}$, every $\mathbb{G}$-linear map can be represented as follows, [40].

Proposition 1.8. *A map $L : \mathbb{G} \longrightarrow \mathbb{R}$ is $\mathbb{G}$-linear only if there exists $a = (a_1, \dots, a_{m_1}) \in \mathbb{R}^{m_1}$ such that, whenever $v = (v_1, \dots, v_n) \in \mathbb{G}$, one has $L(v) = \sum_{j=1}^{m_1} a_j v_j$.*

Definition 1.9. Let $\Omega \subseteq \mathbb{G}$ be open and $x_0 \in \Omega$. We say that $f : \Omega \longrightarrow \mathbb{R}$ is *Pansu-differentiable* at x_0 if there exists a $\mathbb{G}$-linear map L such that

$$\lim_{\lambda \to 0^+} \frac{f(l_{x_0}(\delta_\lambda v)) - f(x_0)}{\lambda} = L(v)$$

uniformly with respect to v belonging to a compact set in $\mathbb{G}$. In particular, L is unique and we shall write $d_{\mathbb{G}} f(x_0)(v) := L(v)$.

This definition depends only on $\mathbb{G}$ and not on the particular choice of the canonical generating vector fields. If $\Omega \subseteq \mathbb{G}$ is open we denote by $\mathbf{C}^1_{\mathbb{G}}(\Omega)$ the set of all continuous real functions in Ω such that the map $d_{\mathbb{G}} f : \Omega \longrightarrow \mathcal{L}_{\mathbb{G}}$ is continuous in Ω and by $\mathbf{C}^1_{\mathbb{G}}(\Omega, H\mathbb{G})$ the set of all sections ψ of $H\mathbb{G}$ whose canonical coordinates ψ_j belongs to $\mathbf{C}^1_{\mathbb{G}}(\Omega)$ $(j = 1, \dots, m_1)$. We remark that $\mathbf{C}^1(\Omega) \subsetneqq \mathbf{C}^1_{\mathbb{G}}(\Omega)$, i.e. in general the inclusion is strict. We say that f is differentiable along X_j $(j = 1, \dots, m_1)$ at x_0 if the map $\lambda \mapsto f(l_{x_0}(\delta_\lambda \mathrm{e}_j))$ is differentiable at $\lambda = 0$ where e_j is the j-th vector of the standard basis of $\mathbb{R}^n$.

If we have fixed a generating family of left invariant vector fields, say $X_1, \dots, X_{m_1}$, then for any function $f : \mathbb{G} \longrightarrow \mathbb{R}$ for which the partial derivatives $X_j f$ $(j = 1, \dots, m_1)$ are defined, we will denote by $\nabla_{\mathbb{G}} f$ the horizontal section defined by

$$\nabla_{\mathbb{G}} f := \sum_{j=1}^{m_1} (X_j f) X_j \,,$$

so that $\nabla_{\mathbb{G}} f = (X_1 f, \dots, X_{m_1} f)$. Moreover, if $\psi = (\psi_1, \dots, \psi_{m_1})$ is a horizontal section such that $X_j \psi_j$ exists for every $j = 1, \dots, m_1$, we will denote by $\mathrm{div}_{\mathbb{G}} \psi$ the real valued function

$$\mathrm{div}_{\mathbb{G}}\, \psi := \sum_{j=1}^{m_1} X_j \psi_j \,.$$

We also remark that the following integration by parts formula holds; see [9, 49].

Proposition 1.10. *Let $\Omega \subset \mathbb{G}$ be a domain; let $f \in \mathbf{C}^\infty_0(\Omega)$ and $\psi \in \mathbf{C}^\infty_0(\Omega, H\mathbb{G})$. Then*

$$\int_\Omega f \,\mathrm{div}_{\mathbb{G}}\, \psi \, d\mathcal{L}^n = - \int_\Omega \langle \nabla_{\mathbb{G}} f, \psi \rangle_{H\mathbb{G}} \, d\mathcal{L}^n \,.$$

Remark 1.11. We stress that the notion of *horizontal gradient* $\nabla_{\mathbb{G}}$ depends only on the choice of the horizontal frame $X_1, \dots, X_{m_1}$ and therefore it is uniquely determined by the sub-Riemannian metric chosen. On the other hand, the notion of *horizontal divergence* $\mathrm{div}_{\mathbb{G}}$ turns out to be independent of the sub-Riemannian metric and it can be computed using the previous formula for the fixed basis; see [40, 41].

For a fixed $x_0 \in \mathbb{G}$ we set $\Pi_{x_0}(v) := \sum_{j=1}^{m_1} v_j\, X_j(x_0)$ for $v = (v_1, \dots, v_n) \in \mathbb{G}$. Notice that the map $v \longrightarrow \Pi_{x_0}(v)$ is a smooth section of $H\mathbb{G}$. The next proposition can be found in [63].

Proposition 1.12. *If f is Pansu-differentiable at x_0 then f is differentiable along X_j at x_0 for $j = 1, \dots, m_1$, and*

$$d_{\mathbb{G}} f(x_0)(v) = \langle \nabla_{\mathbb{G}} f, \Pi_{x_0}(v)\rangle_{H_{x_0}\mathbb{G}} \quad \textit{for any } v \in \mathbb{G}\,. \tag{16}$$

Finally, we shall introduce a notation that will be useful in some mean integral formulae. Fixing $x_0 \in \mathbb{G}$ and $X \in H\mathbb{G}$, we set

$$\mathcal{I}_{x_0}(X) := l_{x_0}(\mathrm{Exp}(X_0^{\perp})) = l_{x_0}\left(\left\{v \in \mathbb{G} : \langle \Pi_0(v), X\rangle_{H_0\mathbb{G}} = 0\right\}\right), \tag{17}$$

where $X_0^{\perp}$ denotes the orthogonal complement of $X(0)$ in $\mathfrak{g}$. Explicitly if $X(0) = \sum_{j=1}^{m_1} a_j \mathrm{e}_j$,

$$\mathcal{I}_{x_0}(X) = \left\{x \in \mathbb{G} : \sum_{j=1}^{m_1} \left(x_j - (x_0)_j\right) a_j = 0\right\}.$$

We call $\mathcal{I}_{x_0}(X)$ the *vertical hyperplane* through x_0 and orthogonal to X and we denote by $\mathcal{V}_{x_0}$ the family of all vertical hyperplanes through x_0, i.e. $\mathcal{V}_{x_0} := \left\{\mathcal{I}_{x_0}(X) : X \in H_{x_0}\mathbb{G}\right\}$.

1.2. $BV_{\mathbb{G}}$ and $\mathbb{G}$-Caccioppoli sets

For the classical theory of *BV* functions and Caccioppoli sets we shall refer the reader to [2, 30, 77], while many generalizations to metric spaces as Carnot-Carathéodory ones or Carnot groups can be found in [1, 3, 4, 14, 37, 38, 39, 40, 42, 60, 62, 63]. We shall make now a quick overview of main definitions and properties that will be used in the sequel.

Definition 1.13. If $\Omega \subseteq \mathbb{G}$ is open and $f \in L^1(\Omega)$, then f has *bounded $\mathbb{G}$-variation* in Ω if

$$|\nabla_{\mathbb{G}} f|(\Omega) := \sup\left\{\int_{\Omega} f\, \mathrm{div}_{\mathbb{G}}\psi\, d\mathcal{L}^n : \psi \in \mathbf{C}_0^1(\Omega, H\mathbb{G}),\ |\psi| \le 1\right\} < \infty\,, \tag{18}$$

where $|\nabla_{\mathbb{G}} f|(\Omega)$ is called $\mathbb{G}$-*variation* of f in Ω. We denote by $BV_{\mathbb{G}}(\Omega)$ the vector space of functions of bounded $\mathbb{G}$-variation in Ω and by $BV_{\mathbb{G},\mathrm{loc}}(\Omega)$ the set of functions belonging to $BV_{\mathbb{G}}(U)$ for each open set $U \Subset \Omega$.

Theorem 1.14. (Structure of $BV_{\mathbb{G}}$ functions.) *If $f \in BV_{\mathbb{G}}(\Omega)$ then $|\nabla_{\mathbb{G}} f|$ is a Radon measure in Ω and there exists a $|\nabla_{\mathbb{G}} f|-$measurable horizontal section $\sigma_f : \Omega \to H\mathbb{G}$ such that $|\sigma_f| = 1$ for $|\nabla_{\mathbb{G}} f|-$a.e. $x \in \Omega$ and*

$$\int_{\Omega} f\, \mathrm{div}_{\mathbb{G}}\, \psi\, d\mathcal{L}^n = \int_{\Omega} \langle \psi, \sigma_f\rangle_{H\mathbb{G}}\, d\,|\nabla_{\mathbb{G}} f| \quad \forall\ \psi \in \mathbf{C}_0^1(\Omega, H\mathbb{G})\,. \tag{19}$$

Moreover $\nabla_{\mathbb{G}}$ *can be extended as a vector valued measure for functions in* $BV_{\mathbb{G}}$ *setting*

$$\nabla_{\mathbb{G}} f := -\sigma_f \llcorner |\nabla_{\mathbb{G}} f| = \Big(-(\sigma_f)_1 \llcorner |\nabla_{\mathbb{G}} f|, \dots, -(\sigma_f)_{m_1} \llcorner |\nabla_{\mathbb{G}} f| \Big), \tag{20}$$

where $(\sigma_f)_j$ $(j = 1, \dots, m_1)$ *is the* j*-th component of* σ_f *with respect to the horizontal frame.*

The following two theorems hold in the general context of Carnot-Carathéodory geometries associated with vector fields; see [37, 42].

Theorem 1.15. (Lower semicontinuity.) *Let* $f, f_k \in L^1(\Omega)$*,* $k \in \mathbb{N}$*, be such that* $f_k \to f$ *in* $L^1(\Omega)$*; then*

$$|\nabla_{\mathbb{G}} f|(\Omega) \le \liminf_{k\to\infty} |\nabla_{\mathbb{G}} f_k|(\Omega). \tag{21}$$

Theorem 1.16. (Compactness.) $BV_{\mathbb{G},\mathrm{loc}}(\mathbb{G})$ *is compactly embedded in* $L^p_{\mathrm{loc}}(\mathbb{G})$ *for* $1 \le p < \frac{Q}{Q-1}$*, where* Q *denotes the homogeneous dimension of* $\mathbb{G}$*.*

Definition 1.17. Let $\Omega \subseteq \mathbb{G}$ be open; then a measurable set $E \subset \mathbb{G}$ has finite $\mathbb{G}$-perimeter in Ω, or is a $\mathbb{G}$-*Caccioppoli set* in Ω, if its characteristic function $\mathbf{1}_E$ belongs to $BV_{\mathbb{G},\mathrm{loc}}(\Omega)$. In this case we call $\mathbb{G}$-*perimeter* of E in Ω the (Radon) measure given by

$$|\partial E|_{\mathbb{G}} := |\nabla_{\mathbb{G}} \mathbf{1}_E| \tag{22}$$

and we call *generalized inward* $\mathbb{G}$-*normal* along ∂E in Ω the vector valued measure

$$\nu_E := -\sigma_{\mathbf{1}_E}. \tag{23}$$

We stress that the notion of $\mathbb{G}$-perimeter depends only on the sub-Riemannian metric chosen (see also Remark 1.11).

Remark 1.18. The $\mathbb{G}$-perimeter measure is invariant under group translations, i.e.

$$|\partial E|_{\mathbb{G}}(\mathcal{B}) = |\partial(l_x E)|_{\mathbb{G}}(l_x \mathcal{B}) \ \ \forall\, x \in \mathbb{G}\ \forall\, \mathcal{B} \in \mathcal{B}or(\mathbb{G}); \tag{24}$$

indeed $\mathrm{div}_{\mathbb{G}}$ is invariant under group translations and the Jacobian determinant of l_x is equal to 1. Moreover the $\mathbb{G}$-perimeter is $(Q-1)$-homogeneous with respect to the intrinsic dilations, i.e.

$$|\partial(\delta_\lambda E)|_{\mathbb{G}}(\delta_\lambda \mathcal{B}) = \lambda^{Q-1} |\partial E|_{\mathbb{G}}(\mathcal{B}) \ \ \forall\, \mathcal{B} \in \mathcal{B}or(\mathbb{G}). \tag{25}$$

This fact can be easily proved by changing variable in formula (18).

Proposition 1.19. [14]. *If E is a $\mathbb{G}$-Caccioppoli set in Ω with $\mathbf{C}^1$ smooth boundary, then*

$$|\partial E|_{\mathbb{G}}(\Omega) = \int_{\partial E\cap\Omega} \sqrt{\langle X_1, \mathrm{n}\rangle^2 + \ldots + \langle X_{m_1}, \mathrm{n}\rangle^2}\, d\,\mathcal{H}^{n-1}\,, \tag{26}$$

where n *is the euclidean unit inward normal along ∂E. In this case we have*

$$\nu_E(x) = \frac{\Big(\langle X_1(x), \mathrm{n}(x)\rangle, \ldots, \langle X_{m_1}(x), \mathrm{n}(x)\rangle\Big)}{\sqrt{\langle X_1(x), \mathrm{n}(x)\rangle^2 + \ldots + \langle X_{m_1}(x), \mathrm{n}(x)\rangle^2}} \quad \forall x \in \partial E \cap \Omega\,.$$

We would point out that the regularization technique of convolution with mollifiers enables us to obtain several approximation results for both Sobolev and BV functions in Carnot groups as well as in more general contexts; see, for instance, [37, 42]. To this end we introduce a family of spherically symmetric mollifiers J_ϵ ($\epsilon > 0$) by $J_\epsilon(x) := \epsilon^{-n} J(\epsilon^{-1}x)$, where $J \in \mathbf{C}_0^\infty(\mathbb{R}^n), J \geq 0$, $\mathrm{spt}\,(J) \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\int_{\mathbb{G}} J\, d\mathcal{L}^n = 1$.

Lemma 1.20. *Let $\Omega \subseteq \mathbb{G}$ be open and $f \in BV_{\mathbb{G}}(\Omega)$. If $\widetilde{\Omega} \Subset \Omega$ is open and $|\nabla_{\mathbb{G}} f|(\partial\widetilde{\Omega}) = 0$, then*

$$\lim_{\epsilon\to 0} |\nabla_{\mathbb{G}}(J_\epsilon * f)|(\widetilde{\Omega}) = |\nabla_{\mathbb{G}} f|(\widetilde{\Omega})\,. \tag{27}$$

Theorem 1.21. (Density for $BV_{\mathbb{G}}$ functions.) *Let $f \in BV_{\mathbb{G}}(\Omega)$; then there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathbf{C}^\infty(\Omega) \cap BV_{\mathbb{G}}(\Omega)$ such that*

$$\lim_{j\to\infty} \|f_j - f\|_{L^1(\Omega)} = 0 \quad \textit{and} \quad \lim_{j\to\infty} |\nabla_{\mathbb{G}} f_j|(\Omega) = |\nabla_{\mathbb{G}} f|(\Omega)\,. \tag{28}$$

The following coarea formula for $BV_{\mathbb{G}}$ functions is a key tool to understand the interplay between $BV_{\mathbb{G}}$ functions and $\mathbb{G}$-Caccioppoli sets; for a proof see [42, 37, 63].

Theorem 1.22. *Let $f \in BV_{\mathbb{G}}(\Omega)$ and set $E_t := \{x \in \Omega : f(x) > t\}$ for $t \in \mathbb{R}$. Then*

(i) *E_t has finite $\mathbb{G}$-perimeter in Ω for $\mathcal{L}^1$−a.e. $t \in \mathbb{R}$;*
(ii) *$|\nabla_{\mathbb{G}} f|(\Omega) = \int_{-\infty}^{+\infty} |\partial E_t|(\Omega)\, dt$.*
(iii) *Conversely, if $f \in L^1(\Omega)$ and $\int_{-\infty}^{+\infty} |\partial E_t|(\Omega)\, dt < \infty$, then $f \in BV_{\mathbb{G}}(\Omega)$ and* (ii) *holds.*

In $\mathbb{R}^n$ a $\mathbf{C}^1$ smooth hypersurface can be viewed as the zero set of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with non-vanishing gradient. In Carnot groups it is possible to follow the same approach to define the so-called $\mathbb{G}$-*regular hypersurfaces*, [39, 40, 41].

Definition 1.23. We say that $\mathcal{S} \subset \mathbb{G}$ is a $\mathbb{G}$-*regular hypersurface* if for every $x \in \mathcal{S}$ there exist a neighborhood $\mathcal{U}$ of x and a function $f \in \mathbf{C}^1_{\mathbb{G}}(\mathcal{U})$ such that

(i) $\mathcal{S} \cap \mathcal{U} = \{y \in \mathcal{U} : f(y) = 0\}$;
(ii) $\nabla_{\mathbb{G}} f(y) \neq 0 \qquad \forall\ y \in \mathcal{U}$.

The following *Implicit Function Theorem* has been recently proved in [40].

Theorem 1.24. (Implicit Function Theorem.) *Let $\Omega \subseteq \mathbb{G}$ be open such that $0 \in \Omega$; let $f \in \mathbf{C}^1_{\mathbb{G}}(\Omega)$ be such that $f(0) = 0$ and $X_1 f(0) > 0$. Put $E := \{x \in \Omega : f(x) < 0\}$, $\mathcal{S} := \{x \in \Omega : f(x) = 0\}$ and for $h, \delta > 0$ set $J_h := [-h, h]$ and $I_\delta := \{\xi = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1} : |\xi_j| \le \delta,\ j = 2, \dots, n\}$. If $\xi \in \mathbb{R}^{n-1}$ and $t \in J_h$ we denote by $\gamma^1_{(0,\xi)}(t)$ the integral curve of the horizontal left invariant vector field $X_1 \in H\mathbb{G}$ at the time t issued from $(0, \xi) \in \{(0, \eta) \in \mathbb{G} : \eta \in \mathbb{R}^{n-1}\}$, that is*

$$\gamma^1_{(0,\xi)}(t) = \exp[tX_1](0, \xi)\,.$$

Then there exist $\delta, h > 0$ such that $\mathbb{R} \times \mathbb{R}^{n-1} \ni (t, \xi) \longmapsto \gamma^1_{(0,\xi)}(t)$ is a diffeomorphism of a neighborhood of $J_h \times I_\delta$ onto an open subset of $\mathbb{R}^n$ and denoting by $\mathcal{U} \Subset \Omega$ the image of $\mathrm{Int}\{J_h \times I_\delta\}$ *under this mapping the following statements hold*:

(i) *E has finite $\mathbb{G}$-perimeter in $\mathcal{U}$*;
(ii) $\partial E \cap \Omega = \mathcal{S} \cap \mathcal{U}$;
(iii) *if ν_E is the generalized inner unit normal of E then*

$$\nu_E(x) = -\frac{\nabla_{\mathbb{G}} f(x)}{|\nabla_{\mathbb{G}} f(x)|_{H\mathbb{G}}} \quad \forall\, x \in \mathcal{S} \cap \Omega\,,$$
$$|\nu_E|_{H\mathbb{G}} = 1 \qquad \textit{for } |\partial E|_{\mathbb{G}} - \textit{a.e. } x \in \mathcal{U}\,.$$

Moreover there exists a unique continuous function $\phi = \phi(\xi) : I_\delta \longrightarrow J_h$ such that, setting $\Phi(\xi) = \gamma^1_{(0,\xi)}(\phi(\xi))$ for $\xi \in I_\delta$, we have

(iv) $\mathcal{S} \cap \mathcal{U} = \{x \in \mathcal{U} : x = \Phi(\xi),\ \xi \in I_\delta\}$;
(v) *the $\mathbb{G}$-perimeter has the following integral representation*

$$|\partial E|_{\mathbb{G}}(\mathcal{U}) = \int_{I_\delta} \frac{\sqrt{\sum_{j=1}^{m} |X_j f(\Phi(\xi))|^2}}{X_1 f(\Phi(\xi))}\, d\xi\,.$$

We conclude this subsection with the definition of *partial perimeter* along a horizontal direction, while in the next Lemma 1.26 we explicitly characterize it.

Definition 1.25. Let Ω be open and let $X \in H\mathbb{G}$. Let E be a Lebesgue measurable subset of $\mathbb{G}$ such that $\mathcal{L}^n(E \cap \Omega) < \infty$. Then we say that E has finite X-perimeter in Ω if

$$|\partial_X E|_{\mathbb{G}}(\Omega) := \sup\left\{\int_\Omega \mathbf{1}_E\, X\varphi\, d\mathcal{L}^n : \varphi \in \mathbf{C}^1_0(\Omega),\ |\varphi| \le 1\right\} < \infty \qquad (29)$$

and we call this quantity the *X-perimeter* of E in Ω; see also [37].

We will see in Section 3.1 that this notion agrees with that more general of X-variation of a L^1 function; see, for instance, Definition 3.1 and Remark 3.2 below.

Lemma 1.26. *Let Ω be open and let $X \in H\mathbb{G}$. If E is a $\mathbb{G}$-Caccioppoli set in Ω, then*

$$|\partial_X E|_{\mathbb{G}}(\Omega) = \int_{\Omega} |\langle X, \nu_E\rangle_{H\mathbb{G}}|\, d\,|\partial E|_{\mathbb{G}}\,.$$

Proof. Firstly, putting $\Phi := \varphi X \in H\mathbb{G}$, where $\varphi \in \mathbf{C}^1_0(\Omega)$, $|\varphi| \le 1$, we get

$$\begin{aligned}\int_{\Omega} \mathbf{1}_E X\,\varphi\, d\mathcal{L}^n &= \int_{\Omega} \mathbf{1}_E\, \langle \nabla_{\mathbb{G}}\varphi, X\rangle_{H\mathbb{G}}\, d\mathcal{L}^n = \int_{\Omega} \mathbf{1}_E\, \mathrm{div}_{\mathbb{G}}\,\Phi\, d\mathcal{L}^n\\ &= -\int_{\Omega} \langle \Phi, \nu_E\rangle_{H\mathbb{G}}\, d\,|\partial E|_{\mathbb{G}} = -\int_{\Omega} \varphi\,\langle X, \nu_E\rangle_{H\mathbb{G}}\, d\,|\partial E|_{\mathbb{G}}\,.\end{aligned}$$

Since for every $x \in \Omega$ we have $\varphi\,\langle X, \nu_E\rangle_{H\mathbb{G}} \le |\langle X, \nu_E\rangle_{H\mathbb{G}}|$, from Definition 1.25 it follows that

$$|\partial_X E|_{\mathbb{G}}(\Omega) \le \int_{\Omega} |\langle X, \nu_E\rangle_{H\mathbb{G}}|\, d\,|\partial E|_{\mathbb{G}}\,.$$

Now we shall prove the reverse inequality. So let $\epsilon > 0$ and set

$$\Omega_\epsilon := \left\{ x \in \Omega : |x| < \frac{1}{\epsilon},\ \mathrm{dist}(x, \partial\Omega) > \epsilon \right\},$$

$$\zeta_\epsilon := \frac{J_\epsilon * \left(\mathbf{1}_{\Omega_\epsilon} \mathrm{sign}(\langle X, \nu_E\rangle_{H\mathbb{G}})\right)}{\sqrt{\epsilon^2 + \left(J_\epsilon * (\mathbf{1}_{\Omega_\epsilon} \mathrm{sign}(\langle X, \nu_E\rangle_{H\mathbb{G}}))\right)^2}}\,,$$

where, as above, J_ϵ is a Friedrichs' mollifier. Using standard properties of mollifiers we get that $\zeta_\epsilon \in \mathbf{C}^\infty_0(\Omega)$, $|\zeta_\epsilon| < 1$, and $\zeta_\epsilon \longrightarrow \mathbf{1}_\Omega\, \mathrm{sign}(\langle X, \nu_E\rangle_{H\mathbb{G}})$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{G}$ as $\epsilon \to 0$. Finally, from Definition 1.25 together with previous computations and Fatou's lemma we get

$$\begin{aligned}|\partial_X E|_{\mathbb{G}}(\Omega) &\ge \liminf_{\epsilon\to 0} \int_{\Omega} \zeta_\epsilon\, \langle X, \nu_E\rangle_{H\mathbb{G}}\, d\,|\partial E|_{\mathbb{G}}\\ &\ge \int_{\Omega} \liminf_{\epsilon\to 0}\, \zeta_\epsilon\, \langle X, \nu_E\rangle_{H\mathbb{G}}\, d\,|\partial E|_{\mathbb{G}} = \int_{\Omega} |\langle X, \nu_E\rangle_{H\mathbb{G}}|\, d\,|\partial E|_{\mathbb{G}}\,. \qquad \square\end{aligned}$$

Remark 1.27. From Lemma 1.26 and from the regularity of the measures $|\partial E|_{\mathbb{G}}$ and $|\partial_X E|_{\mathbb{G}}$ one gets equality of measures, i.e.

$$|\partial_X E|_{\mathbb{G}} \llcorner \mathcal{B} = |\langle X, \nu_E\rangle_{H\mathbb{G}}| \cdot |\partial E|_{\mathbb{G}} \llcorner \mathcal{B} \quad \forall\, \mathcal{B} \in \mathcal{B}or(\mathbb{G})\,.$$

2. A Fubini type theorem in Carnot groups

2.1. Statement of results

Let $\mathcal{S} \subset \mathbb{G}$ be a $\mathbf{C}^1$ smooth hypersurface. By the usual Implicit Function Theorem, without loss of generality, we may assume that $\mathcal{S} = \partial E$, locally, where E is an open $\mathbb{G}$-Caccioppoli set. Furthermore, again arguing locally, let us assume that $X \in H\mathbb{G}$ is a horizontal left invariant vector field which is *transverse* to $\mathcal{S}$, i.e.

$$\langle X(y), \mathrm{n}(y)\rangle \neq 0 \qquad \forall\, y \in \mathcal{S}\,, \tag{30}$$

where n is the euclidean unit inward normal along $\mathcal{S}$. We explicitly notice that if $X \in H\mathbb{G}$ is a horizontal left invariant vector field and $\mathcal{S} \subset \mathbb{G}$ is a $\mathbf{C}^1$ smooth hypersurface we have that

$$\langle X, \nu_E\rangle_{H_y\mathbb{G}} \neq 0 \Longleftrightarrow \langle X(y), \mathrm{n}(y)\rangle \neq 0 \qquad \forall\, y \in \mathcal{S}\,.$$

Indeed by Proposition 1.19 the inward unit $\mathbb{G}$-normal along $\mathcal{S} = \partial E$ is given by

$$\nu_E(y) = \frac{\sum_{j=1}^{m_1} \langle X_j(y), \mathrm{n}(y)\rangle X_j(y)}{\sqrt{\sum_{j=1}^{m_1} \langle X_j(y), \mathrm{n}(y)\rangle^2}} \qquad \forall\, y \in \mathcal{S}$$

and if $X = \sum_{i=1}^{m_1} a_i X_i$ we get

$$\langle X(y), \nu_E(y)\rangle_{H_y\mathbb{G}} = \frac{\sum_{j=1}^{m_1} \langle X_j(y), \mathrm{n}(y)\rangle\, a_j}{\sqrt{\sum_{j=1}^{m_1} \langle X_j(y), \mathrm{n}(y)\rangle^2}} = \frac{\langle X(y), \mathrm{n}(y)\rangle}{\sqrt{\sum_{j=1}^{m_1} \langle X_j(y), \mathrm{n}(y)\rangle^2}}\,.$$

Condition (30) is equivalent to require that $X(y) \in H_y\mathbb{G} \setminus T_y\mathcal{S}$ for $y \in \mathcal{S}$. Consider now the following Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)) \\ \gamma(0) = y \in \mathcal{S}\,. \end{cases}$$

There exists a unique smooth solution of this problem which is defined on all of $\mathbb{R}$ and, throughout this section, we shall write $\gamma_y^X(t) = \exp[tX](y)$ for $t \in \mathbb{R}$ and $y \in \mathcal{S}$. If $X \in H\mathbb{G}$ is fixed, we shall remove the superindex just writing γ_y. Notice that $\gamma_y^X(t) = y \cdot \mathrm{Exp}(tX) = \mathcal{P}(y, \mathrm{Exp}(tX))$. Following [58], we call such a trajectory a *horizontal X-line,* or simply *horizontal line*. Now let us consider the family of horizontal X-lines starting from $\mathcal{S}$. We denote by $\mathcal{R}^X_{\mathcal{S}}$ the subset of $\mathbb{G}$ *reachable* from $\mathcal{S}$ by means of horizontal X-lines, i.e.

$$\mathcal{R}^X_{\mathcal{S}} := \Big\{ x \in \mathbb{G} : \exists\ y \in \mathcal{S},\ \exists\ t \in \mathbb{R} \text{ s.t. } x = \exp[tX](y) \Big\}.$$

From now on, we assume that $\mathcal{S}$ enjoys the following further property:

$$\gamma_y(\mathbb{R}) \cap \mathcal{S} = \{y\} \qquad \forall\ y \in \mathcal{S}. \tag{31}$$

Since X is transverse to $\mathcal{S}$, from the uniqueness of the solutions of the Cauchy problem and the hypothesis (31), it follows that any subset $\mathcal{D}$ of $\mathcal{R}^X_{\mathcal{S}}$ has a natural projection on $\mathcal{S}$ along the horizontal direction X. More precisely, we may define a mapping $pr^X_{\mathcal{S}} : \mathcal{D} \subseteq \mathcal{R}^X_{\mathcal{S}} \longmapsto \mathcal{S}$ as follows; for $x \in \mathcal{D}$ and $y \in \mathcal{S}$ we set $y = pr^X_{\mathcal{S}}(x)$ if, and only if, there exists $t \in \mathbb{R}$ such that $x = \exp[tX](y)$. Using this type of projection every subset $\mathcal{D}$ of $\mathcal{R}^X_{\mathcal{S}}$ can be foliated with one-dimensional leaves that are horizontal X-lines. In fact, setting $\mathcal{D}_y := \gamma_y(\mathbb{R}) \cap \mathcal{D}$, one has

$$\mathcal{D} = \bigsqcup_{y \in pr^X_{\mathcal{S}}(\mathcal{D})} \mathcal{D}_y \qquad \text{and} \qquad y_1 \neq y_2 \Longrightarrow \mathcal{D}_{y_1} \cap \mathcal{D}_{y_2} = \emptyset \quad \forall\ y_1,\ y_2 \in pr^X_{\mathcal{S}}(\mathcal{D}).$$

In some of our results we will often use vertical hyperplanes (see (17) of Section 1.1). It is important to note that every subset of $\mathbb{G}$ is reachable from any vertical hyperplane. Finally, we would emphasize that, although this projection turns out to be useful in the proof of many integral formulas, it is not Lipschitz with respect to the Carnot-Carathéodory distance d_c and so one cannot to assimilate it to an euclidean orthogonal projection; see for more details [52].

We may state our first result of this section.

Proposition 2.1. *Let $\mathcal{S} \subset \mathbb{G}$ be a $\mathbf{C}^1$ smooth hypersurface and let $X \in H\mathbb{G}$, $|X|_{H\mathbb{G}} = 1$, be a unit horizontal left invariant vector field which is transverse to $\mathcal{S}$, i.e.*

$$\langle X, \nu_E \rangle_{H_y\mathbb{G}} \neq 0 \qquad \forall\ y \in \mathcal{S}.$$

Let γ_y be the horizontal X-line starting from $y \in \mathcal{S}$, i.e.

$$\gamma_y : \mathbb{R} \longmapsto \mathbb{G}, \quad \gamma_y(t) = \exp[tX](y) \ \textit{for}\ \ y \in \mathcal{S}.$$

Moreover we assume that

$$\gamma_y(\mathbb{R}) \cap \mathcal{S} = \{y\} \qquad \forall\ y \in \mathcal{S}.$$

Let $\mathcal{D} \subseteq \mathcal{R}^X_{\mathcal{S}}$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from $\mathcal{S}$ by means of horizontal X-lines. Since locally $\mathcal{S} = \partial E$, for a suitable open set $E \subset \mathbb{G}$, without loss of generality we may assume that $\mathcal{S} = \partial E$ globally, where E has locally finite $\mathbb{G}$-perimeter. Then we have

(i) *$\mathcal{D}_y := \gamma_y(\mathbb{R}) \cap \mathcal{D}$ is $\mathcal{H}^1_c$–measurable for $|\partial E|_{\mathbb{G}}$–a.e. $y \in \mathcal{S}$;*
(ii) *the mapping $\mathcal{S} \ni y \longmapsto \mathcal{H}^1_c(\mathcal{D}_y)$ is $|\partial E|_{\mathbb{G}}$-measurable on $\mathcal{S}$ and*

$$\mathcal{L}^n(\mathcal{D}) = \int_{pr^X_{\mathcal{S}}(\mathcal{D})} \mathcal{H}^1_c(\mathcal{D}_y)\, |\langle X, \nu_E \rangle_{H_y\mathbb{G}}|\, d\, |\partial E|_{\mathbb{G}}(y) = \int_{pr^X_{\mathcal{S}}(\mathcal{D})} \mathcal{H}^1_c(\mathcal{D}_y)\, d\, |\partial_X E|_{\mathbb{G}}(y),$$

where $pr^X_{\mathcal{S}}(\mathcal{D}) \subseteq \mathcal{S}$ is the horizontal X-projection of $\mathcal{D}$ on $\mathcal{S}$.

This proposition may be generalized to $\mathbb{G}$-regular hypersurfaces and, more precisely, we can state our main theorem as follows.

Theorem 2.2. *Let $\mathcal{S} \subset \mathbb{G}$ be a $\mathbb{G}$-regular hypersurface. By Theorem* 1.24, *without loss of generality, we may assume that $\mathcal{S} = \partial E$ globally, where $E \subset \mathbb{G}$ is an open $\mathbb{G}$-Caccioppoli set with locally $\mathbf{C}^1_{\mathbb{G}}$ boundary. Let $X \in H\mathbb{G}$, $|X|_{H\mathbb{G}} = 1$, be a unit horizontal left invariant vector field which is transverse to $\mathcal{S}$. Let γ_y be the horizontal X-line starting from $y \in \mathcal{S}$ and let us suppose that $\gamma_y(\mathbb{R}) \cap \mathcal{S} = \{y\}$ for every $y \in \mathcal{S}$. Let $\mathcal{D} \subseteq \mathcal{R}^X_{\mathcal{S}}$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from $\mathcal{S}$. Then we have*

(i) *$\mathcal{D}_y := \gamma_y(\mathbb{R}) \cap \mathcal{D}$ is $\mathcal{H}^1_c$−measurable for $|\partial E|_{\mathbb{G}}$−a.e. $y \in \mathcal{S}$;*
(ii) *the mapping $\mathcal{S} \ni y \longmapsto \mathcal{H}^1_c(\mathcal{D}_y)$ is $|\partial E|_{\mathbb{G}}$-measurable on $\mathcal{S}$ and*

$$\mathcal{L}^n(\mathcal{D}) = \int_{pr^X_{\mathcal{S}}(\mathcal{D})} \mathcal{H}^1_c(\mathcal{D}_y)\, |\langle X, \nu_E\rangle_{H_y\mathbb{G}}|\, d\,|\partial E|_{\mathbb{G}}(y) = \int_{pr^X_{\mathcal{S}}(\mathcal{D})} \mathcal{H}^1_c(\mathcal{D}_y)\, d\,|\partial_X E|_{\mathbb{G}}(y)\,.$$

The proof of these results will be given in the next subsection. Nevertheless we state a first useful consequence.

Corollary 2.3. *Let $\mathcal{S} \subset \mathbb{G}$ be a $\mathbb{G}$-regular hypersurface and assume that $\mathcal{S} = \partial E$ globally, where $E \subset \mathbb{G}$ is a suitable open $\mathbb{G}$-Caccioppoli set. Let $X \in H\mathbb{G}$, $|X|_{H\mathbb{G}} = 1$, be a unit horizontal left invariant vector field which is transverse to $\mathcal{S}$ and denote by γ_y the horizontal X-line starting from $y \in \mathcal{S}$. We assume that $\gamma_y(\mathbb{R}) \cap \mathcal{S} = \{y\}$ for every $y \in \mathcal{S}$. Finally let $\mathcal{D} \subseteq \mathcal{R}^X_{\mathcal{S}}$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from $\mathcal{S}$ by means of X-lines. Then, for every function $\psi \in L^1(\mathcal{D})$ the following statements hold*

(i) *let $\psi_{|\mathcal{D}_y}$ denote the restriction of ψ to $\mathcal{D}_y := \gamma_y(\mathbb{R}) \cap \mathcal{D}$ and let us define the mapping*

$$\psi_y : \gamma_y^{-1}(\mathcal{D}_y) \subseteq \mathbb{R} \longmapsto \mathbb{R}, \qquad \psi_y(t) = (\psi \circ \gamma_y)(t)\,.$$

Then ψ_y is $\mathcal{L}^1$-measurable for $|\partial E|_{\mathbb{G}}$−a.e. $y \in \mathcal{S}$ or, equivalently, the restriction $\psi_{|\mathcal{D}_y}$ is $\mathcal{H}^1_c$-measurable for $|\partial E|_{\mathbb{G}}$−a.e. $y \in \mathcal{S}$;
(ii) *the mapping defined by*

$$\mathcal{S} \ni y \longmapsto \int_{\mathcal{D}_y} \psi\, d\mathcal{H}^1_c = \int_{\gamma_y^{-1}(\mathcal{D}_y)} \psi_y(t)\, dt$$

is $|\partial E|_{\mathbb{G}}$-measurable on $\mathcal{S}$ and the following formula holds

$$\begin{aligned}\int_{\mathcal{D}} \psi\, d\,\mathcal{L}^n &= \int_{pr^X_{\mathcal{S}}(\mathcal{D})} \left[\int_{\mathcal{D}_y} \psi\, d\mathcal{H}^1_c\right] d\,|\partial_X E|_{\mathbb{G}}(y)\\ &= \int_{pr^X_{\mathcal{S}}(\mathcal{D})} \left[\int_{\gamma_y^{-1}(\mathcal{D}_y)} \psi_y(t)\, dt\right] |\langle X, \nu_E\rangle_{H_y\mathbb{G}}|\, d\,|\partial E|_{\mathbb{G}}(y)\,.\end{aligned}$$

Proof. Having at our disposal Theorem 2.2, is enough to use a standard argument of measure theory to approximate the function ψ with a finite linear combination of characteristic functions, as for instance in Theorem 3.2.5 of [31]. □

Till now we have used only the intrinsic $\mathbb{G}$-perimeter as a measure for hypersurfaces in $\mathbb{G}$ but also different measures can be considered. In fact, the comparison of different surfaces measures is a one of the main problems of Geometric Measure Theory in Carnot groups and in general Carnot-Carathéodory spaces. In particular, an interesting problem for Carnot groups is that to compare the $\mathbb{G}$-perimeter with the $(Q-1)$-dimensional Hausdorff measure associated with either the cc-distance d_c or with some suitable homogeneous distance on $\mathbb{G}$, in the case of euclidean smooth hypersurfaces; see [7, 40, 41, 57]. In this regard, we have that the following result holds true for general Carnot groups; see [57].

Remark 2.4. Let $\mathcal{S}$ be a $\mathbf{C}^1$ smooth hypersurface and let us assume that $\mathcal{S}$ is locally the boundary of an open set E. Then

$$|\partial E|_{\mathbb{G}} \llcorner \mathcal{B} = k_{Q-1}(\nu_E)\, \mathcal{H}_c^{Q-1} \llcorner \big(\mathcal{S} \cap \mathcal{B}\big) \quad \forall\, \mathcal{B} \in \mathcal{B}or(\mathbb{G}) \tag{32}$$

where the measure $\mathcal{H}_c^{Q-1}$ is the spherical[(1)] $(Q-1)$-dimensional Hausdorff measure associated with the cc-distance d_c and k_{Q-1} is a function depending on ν_E, called *metric factor* (see Definition 2.17 in [57]).

Therefore, we may reformulate Proposition 2.1 by using Hausdorff measures with respect to the cc-distance d_c and, more precisely, we have the following

Corollary 2.5. *Let $\mathcal{S} \subset \mathbb{G}$ be a $\mathbf{C}^1$ smooth hypersurface and let $X \in H\mathbb{G}$, $|X|_{H\mathbb{G}} = 1$, be a unit horizontal left invariant vector field which is transverse to $\mathcal{S}$. Let γ_y be the horizontal X-line starting from $y \in \mathcal{S}$ and assume that $\gamma_y(\mathbb{R}) \cap \mathcal{S} = \{y\}$ for every $y \in \mathcal{S}$. Finally, let $\mathcal{D} \subseteq \mathcal{R}_{\mathcal{S}}^X$ be a $\mathcal{H}_c^Q$-measurable subset of $\mathbb{G}$ that is reachable from $\mathcal{S}$ by means of horizontal X-lines. Then*

(i) *$\mathcal{D}_y := \gamma_y(\mathbb{R}) \cap \mathcal{D}$ is $\mathcal{H}_c^1$–measurable for $\mathcal{H}_c^{Q-1}$–a.e. $y \in \mathcal{S}$;*
(ii) *the mapping $\mathcal{S} \ni y \longmapsto \mathcal{H}_c^1(\mathcal{D}_y)$ is $\mathcal{H}_c^{Q-1}$-measurable on $\mathcal{S}$ and*

$$\mathcal{H}_c^Q(\mathcal{D}) = \int_{pr_{\mathcal{S}}^X(\mathcal{D})} \mathcal{H}_c^1(\mathcal{D}_y)\, |\langle X, \nu_E\rangle_{H_y\mathbb{G}}|\, \frac{k_{Q-1}(\nu_E)}{k_Q}\, d\,\mathcal{H}_c^{Q-1}(y)$$

where k_Q is the constant defined in Remark 1.3. *Moreover*

$$\int_{\mathcal{D}} \psi\, d\,\mathcal{H}_c^Q = \int_{pr_{\mathcal{S}}^X(\mathcal{D})} \left[\int_{\mathcal{D}_y} \psi\, d\mathcal{H}_c^1\right] |\langle X, \nu_E\rangle_{H_y\mathbb{G}}|\, \frac{k_{Q-1}(\nu_E)}{k_Q}\, d\,\mathcal{H}_c^{Q-1}(y)\,.$$

(1)Notice that $\mathcal{H}_c^{Q-1}(\mathcal{S}) = \lim_{\delta\to 0^+} \mathcal{H}_{c,\delta}^{Q-1}(\mathcal{S})$ where, up to a constant multiple,

$$\mathcal{H}_{c,\delta}^{Q-1}(\mathcal{S}) = \inf\left\{\sum_i \Big(\mathrm{diam}_c(\mathcal{B}_i)\Big)^{Q-1} \;:\; \mathcal{S} \subset \bigcup_i \mathcal{B}_i;\ \mathrm{diam}_c(\mathcal{B}_i) < \delta\right\}$$

and the infimum is taken with respect to closed d_c-balls $\mathcal{B}_i$.

Proof. We have already observed in Remark 1.3 that Lebesgue measure $\mathcal{L}^n$ and Q-dimensional spherical Hausdorff measure $\mathcal{H}_c^Q$ coincide up to the constant k_Q. Thus, using Proposition 2.1, Corollary 2.3 and the identity of measures stated in (32) the thesis follows. □

2.2. Proofs of Proposition 2.1 and Theorem 2.2

This subsection is entirely devoted to prove Proposition 2.1 and Theorem 2.2. The proof of Proposition 2.1 relies mainly on Lemma 2.7 below and on the classical change of variables formula with some non trivial computations. The proof of Theorem 2.2 follows from Proposition 2.1 using an approximation argument inspired by a recent work of Franchi, Serapioni and Serra Cassano about an implicit function theorem in Carnot groups; see Theorem 1.24 or [40].

We begin by stating two technical lemmas. For the notation used in the sequel we refer the reader to Section 1.1. We just recall here that the group law $\cdot$ on $\mathbb{G}$ is also denoted by $\mathcal{P}(x, y) = x + y + \mathcal{Q}(x, y)$ for $x, y \in \mathbb{G}$, where $\mathcal{P}_j(x, y) = x_j + y_j$ for $1 \leq j \leq m_1 (= \dim V_1)$ and $\mathcal{P}_j(x, y) = x_j + y_j + \mathcal{Q}_j(x, y)$ for $j > m_1$.

Lemma 2.6. *If $X \in V_1$ and $j > m_1$, then*

$$\mathcal{Q}_j(y, \mathrm{Exp}((t_1+t_2)X)) = \mathcal{Q}_j(y, \mathrm{Exp}(t_1X)) + \mathcal{Q}_j(\mathcal{P}(y, \mathrm{Exp}(t_1X)), \mathrm{Exp}(t_2X)) \tag{33}$$

whenever $y \in \mathbb{G}$ and $t_1, t_2 \in \mathbb{R}$.

Proof. Firstly, by Proposition 1.6 we get that if $X \in V_1$

$$\mathcal{P}(\mathrm{Exp}(t_1X), \mathrm{Exp}(t_2X)) = \mathrm{Exp}(t_1X) + \mathrm{Exp}(t_2X) \quad \forall\, t_1, t_2 \in \mathbb{R}\,. \tag{34}$$

Now, starting from the associativity property of the group law and using (34), it follows that

$$\mathcal{P}(\mathcal{P}(y, \mathrm{Exp}(t_1X)), \mathrm{Exp}(t_2X)) = \mathcal{P}(y, \mathcal{P}(\mathrm{Exp}(t_1X), \mathrm{Exp}(t_2X)))$$

and so

$$\mathcal{P}_j(\mathcal{P}(y, \mathrm{Exp}(t_1X)), \mathrm{Exp}(t_2X)) = \mathcal{P}_j(y, \mathcal{P}(\mathrm{Exp}(t_1X), \mathrm{Exp}(t_2X)))\,. \tag{35}$$

Moreover the following identities hold

$$\begin{aligned} &\mathcal{P}_j(\mathcal{P}(y, \mathrm{Exp}(t_1X)), \mathrm{Exp}(t_2X)) \\ &= \mathcal{P}_j(y, \mathrm{Exp}(t_1X)) + \mathcal{Q}_j(\mathcal{P}(y, \mathrm{Exp}(t_1X)), \mathrm{Exp}(t_2X)) \\ &= y_j + \mathcal{Q}_j(y, \mathrm{Exp}(t_1X)) + \mathcal{Q}_j(\mathcal{P}(y, \mathrm{Exp}(t_1X)), \mathrm{Exp}(t_2X))\,; \end{aligned} \tag{36}$$

$$\begin{aligned} \mathcal{P}_j(y, \mathcal{P}(\mathrm{Exp}(t_1X), \mathrm{Exp}(t_2X))) &= y_j + \mathcal{Q}_j(y, \mathcal{P}(\mathrm{Exp}(t_1X), \mathrm{Exp}(t_2X))) \\ &= y_j + \mathcal{Q}_j(y, \mathrm{Exp}((t_1+t_2)X))\,. \end{aligned} \tag{37}$$

Thus the claim easily follows by substituting (36) and (37) in (35). □

Lemma 2.7. *If $X \in V_1$ we have that*

$$\frac{\partial}{\partial t}\mathcal{P}(y, \mathrm{Exp}(tX)) = \left[\frac{\partial}{\partial y}\mathcal{P}(y, \mathrm{Exp}(tX))\right] X(y) \quad \forall\, t \in \mathbb{R}\ \forall\, y \in \mathbb{G}\,. \tag{38}$$

Notation. In some of the following formulae we will write

$$\mathcal{J}_y\mathcal{P}(y, z) := \frac{\partial}{\partial y}\mathcal{P}(y, z) \qquad (y, z \in \mathbb{G})\,.$$

Proof. We shall prove this lemma by considering components. First, let $X = \sum_{j=1}^{m_1} a_j \mathrm{e}_j$ so that $\mathrm{Exp}(tX) = (ta_1, \dots, ta_{m_1}, 0, \dots, 0)$. If $1 \le j \le m_1$ we have that $\mathcal{P}_j(y, \mathrm{Exp}(tX)) = y_j + ta_j$ and since

$$\left\langle \left[\mathcal{J}_y\mathcal{P}(y, \mathrm{Exp}(tX))\right] X(y), \mathrm{e}_j \right\rangle = a_j\,,$$

in this case the thesis follows. Now if $j > m_1$, we have to show that

$$\frac{\partial}{\partial t}\mathcal{P}_j(y, \mathrm{Exp}(tX)) = \langle \nabla_y \mathcal{P}_j(y, \mathrm{Exp}(tX)), X(y)\rangle\,.$$

Since $(\mathrm{Exp}(tX))_j = 0$, we have that $\mathcal{P}_j(y, \mathrm{Exp}(tX)) = y_j + \mathcal{Q}_j(y, \mathrm{Exp}(tX))$. Moreover the following identities hold

$$\frac{\partial}{\partial t}\mathcal{P}_j(y, \mathrm{Exp}(tX)) = \frac{\partial}{\partial t}\mathcal{Q}_j(y, \mathrm{Exp}(tX))\,; \tag{39}$$

$$\langle \nabla_y \mathcal{P}_j(y, \mathrm{Exp}(tX)), X(y)\rangle = (X(y))_j + \langle \nabla_y \mathcal{Q}_j(y, \mathrm{Exp}(tX)), X(y)\rangle\,. \tag{40}$$

Therefore, by (39) and (40) we have to prove that

$$\begin{gathered}\frac{\partial}{\partial t}\mathcal{Q}_j(y, \mathrm{Exp}(tX)) = (X(y))_j + \langle \nabla_y \mathcal{Q}_j(y, \mathrm{Exp}(tX)), X(y)\rangle \\ \forall\, t \in \mathbb{R}\ \forall\, y \in \mathbb{G}\,.\end{gathered} \tag{41}$$

Now, by differentiating both sides of (33) of the previous Lemma 2.6 with respect to t_1 at the *time* $t_1 = 0$ and putting $t_2 = t$, we get that

$$\begin{aligned}
&\frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{Q}_j(y, \mathrm{Exp}((t_1 + t)X)) \\
&= \frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{Q}_j(y, \mathrm{Exp}(t_1X)) + \frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{Q}_j(\mathcal{P}(y, \mathrm{Exp}(t_1X)), \mathrm{Exp}(tX)) \\
&= \frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{P}_j(y, \mathrm{Exp}(t_1X)) \\
&\quad + \left\langle \nabla_y \mathcal{Q}_j(\mathcal{P}(y, 0), \mathrm{Exp}(tX)), \left[\frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{P}(y, \mathrm{Exp}(t_1X))\right]\right\rangle \\
&= (X(y))_j + \left\langle \nabla_y \mathcal{Q}_j(y, \mathrm{Exp}(tX)), X(y)\right\rangle
\end{aligned}$$

that is nothing but (41). $\square$

Proof of Proposition 2.1. Let $\mathcal{S}_\alpha$ be an open neighborhood of $pr^X_{\mathcal{S}}(\mathcal{D})$ on $\mathcal{S}$. Of course, without loss of generality, we may think of $\mathcal{S}_\alpha$ as globally parameterized through a smooth mapping Φ_α, where $\Phi_\alpha : \mathcal{U}_\alpha \subseteq \mathbb{R}^{n-1} \longmapsto \mathcal{S}_\alpha$ and $\Phi_\alpha \in \mathbf{C}^1(\mathcal{U}_\alpha, \mathbb{G})$. In the general case we shall use a partition of unity related to an atlas $\{(\mathcal{S}_\alpha, \Psi_\alpha)\}_{\alpha\in\mathcal{A}}$ of $\mathcal{S}$, where $\Psi_\alpha := \Phi_\alpha^{-1}$ for $\alpha \in \mathcal{A}$ and $(\mathcal{S}_\alpha, \Psi_\alpha)$ is a coordinate chart on $\mathcal{S}$. However, for sake of simplicity, we omit the index α from $\mathcal{U}_\alpha$, Φ_α and $\mathcal{S}_\alpha$ just writing $\mathcal{U}$, Φ and $\mathcal{S}$. Let us consider the map $\mathcal{S} \times \mathbb{R} \ni (y, t) \longmapsto \gamma_y(t) \in \mathbb{G}$ given by $\gamma_y(t) = \exp[tX](y)$. The last one enables us to carry out the parametrization of $\mathcal{D}$ we were looking for. Indeed, more precisely, starting from the parametrization of $\mathcal{S}$, we may put

$$\gamma_{\Phi(\xi)}(t) = \exp[tX](\Phi(\xi))$$

whenever $\xi \in \mathcal{U}$ and $t \in \mathbb{R}$. In the sequel, for simplicity, we shall drop the dependence on the variables and we denote this mapping just by γ_Φ. This one enjoys an important property that we summarize in the next lemma.

Lemma 2.8. *The Jacobian matrix of the mapping γ_Φ with respect to $(\xi, t) \in \mathcal{U} \times \mathbb{R}$ satisfies the following identity*

$$\left|\det\left[\mathcal{J}_{(\xi,t)}\gamma_\Phi\right]\right| = \left|\left\langle X, \nu_E\right\rangle_{H_\Phi\mathbb{G}}\right| \left(\sum_{j=1}^{m_1}\langle X_j(\Phi), \mathrm{n}(\Phi)\rangle^2\right)^{\frac{1}{2}} |\Phi_{\xi_1} \wedge \ldots \wedge \Phi_{\xi_{n-1}}|\,, \quad (42)$$

where we have set

$$\Phi_{\xi_h} := \frac{\partial\,\Phi}{\partial\,\xi_h} \qquad \textit{for } h = 1, \ldots, n-1\,.$$

Proof of Lemma 2.8. We have to compute the expression of the Jacobian matrix of γ_Φ, i.e.

$$\mathcal{J}_{(\xi,t)}\gamma_\Phi = \left[\frac{\partial\,\gamma_\Phi}{\partial\,\xi}, \frac{\partial\,\gamma_\Phi}{\partial\,t}\right] = \left[\frac{\partial\,\gamma_\Phi}{\partial\,\xi_1}, \ldots, \frac{\partial\,\gamma_\Phi}{\partial\,\xi_{n-1}}, \frac{\partial\,\gamma_\Phi}{\partial\,t}\right].$$

By definition we have that $\gamma_{\Phi(\xi)}(t) = \mathcal{P}(\Phi(\xi), \mathrm{Exp}(tX))$ and so we get

$$\frac{\partial\,\gamma_\Phi}{\partial\,\xi} = \left[\frac{\partial}{\partial\,y}\bigg|_{y=\Phi(\xi)} \mathcal{P}(y, \mathrm{Exp}(tX))\right]\frac{\partial\,\Phi}{\partial\,\xi}\,.$$

We have then

$$\mathcal{J}_{(\xi,t)}\gamma_\Phi = \left[\left[\frac{\partial}{\partial\,y}\bigg|_{y=\Phi(\xi)} \mathcal{P}(y, \mathrm{Exp}(tX))\right]\frac{\partial\,\Phi}{\partial\,\xi}\,,\ \frac{\partial}{\partial\,t}\mathcal{P}(\Phi(\xi), \mathrm{Exp}(tX))\right] \quad (43)$$

and, for sake of simplicity, in the following computations we will set

$$\mathbf{A} := \left[\frac{\partial}{\partial y}\bigg|_{y=\Phi(\xi)} \mathcal{P}(y, \operatorname{Exp}(tX)) \right] \qquad \text{and} \qquad \mathbf{b} := \frac{\partial}{\partial t}\mathcal{P}(\Phi(\xi), \operatorname{Exp}(tX))\,.$$

Then

$$\left| \det \left[\mathcal{J}_{(\xi,t)}\gamma_\Phi \right] \right| = \left| \det \left[\mathbf{A}\,\frac{\partial\,\Phi}{\partial\,\xi}\,,\ \mathbf{b} \right] \right| = \left| \det \left[\mathbf{A}\,\frac{\partial\,\Phi}{\partial\,\xi}\,,\ \mathbf{A}\mathbf{A}^{-1}\,\mathbf{b} \right] \right|.$$

Next, note that $|\det \mathbf{A}\,| = 1$. Indeed, in general, one has

$$\frac{\partial}{\partial\,y}\mathcal{P}(y,z) = \mathbf{I}_n + \frac{\partial}{\partial\,y}\mathcal{Q}(y,z)$$

whenever $y, z \in \mathbb{G}$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\frac{\partial}{\partial y}\mathcal{Q}$ is a $n \times n$ nilpotent matrix, because it is lower triangular with the entries in the main diagonal all equal to 1. Furthermore, by Lemma 2.7 we infer that

$$X(y) = \left[\frac{\partial}{\partial\,y}\mathcal{P}(y, \operatorname{Exp}(tX)) \right]^{-1} \frac{\partial}{\partial\,t}\mathcal{P}(y, \operatorname{Exp}(tX))$$

whenever $y \in \mathbb{G}$ and $t \in \mathbb{R}$ and so, in particular, we get that $\mathbf{A}^{-1}\,\mathbf{b} = X(\Phi(\xi))$. Therefore

$$\begin{aligned}
&\left| \det \left[\mathcal{J}_{(\xi,t)}\gamma_\Phi \right] \right| = |\det \mathbf{A}\,| \cdot \left| \det \left[\frac{\partial\,\Phi}{\partial\,\xi}\,,\ \mathbf{A}^{-1}\,\mathbf{b} \right] \right| = \left| \det \left[\frac{\partial\,\Phi}{\partial\,\xi}\,,\ X(\Phi(\xi)) \right] \right| \\
= &\left| \det \left[\frac{\partial\,\Phi}{\partial\,\xi_1}, \dots, \frac{\partial\,\Phi}{\partial\,\xi_{n-1}}\,,\ X(\Phi(\xi)) \right] \right| = \left| \left\langle \frac{\partial\,\Phi}{\partial\,\xi_1} \wedge \dots \wedge \frac{\partial\,\Phi}{\partial\,\xi_{n-1}}, X(\Phi(\xi)) \right\rangle \right| \\
= &\left| \left\langle \mathrm{n}(\Phi(\xi)), X(\Phi(\xi)) \right\rangle \right| \cdot |\Phi_{\xi_1} \wedge \dots \wedge \Phi_{\xi_{n-1}}|\,.
\end{aligned}$$

Here above we have used two standard properties of Linear Algebra and, more precisely, the following identity

$$\det\left[\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n-1}, \mathbf{b}\right] = \left\langle \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \dots \wedge \mathbf{a}_{n-1}, \mathbf{b} \right\rangle \qquad \forall\ \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n-1}, \mathbf{b} \in \mathbb{R}^n\,,$$

and the fact that

$$\det\left[\mathbf{A}\mathbf{b}_1, \mathbf{A}\mathbf{b}_2, \dots, \mathbf{A}\mathbf{b}_n\right] = \det \mathbf{A} \cdot \det\left[\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\right]$$

for any invertible $n \times n$ matrix $\mathbf{A}$. Notice also that in the last line, we have used the explicit expression of the euclidean unit inward normal along a parametric hypersurface. Now, keeping in mind that, whenever $\mathcal{S} = \partial E$ is smooth, we have

$$\nu_E(y) = \frac{\Big(\langle X_1(y), \mathrm{n}(y)\rangle, \ldots, \langle X_{m_1}(y), \mathrm{n}(y)\rangle\Big)}{\left(\displaystyle\sum_{j=1}^{m_1} \langle X_j(y), \mathrm{n}(y)\rangle^2\right)^{\frac{1}{2}}}$$

for every $y \in \mathcal{S}$, the thesis follows by observing that

$$\left|\Big\langle \mathrm{n}(y), X(y)\Big\rangle\right| = \left|\Big\langle \nu_E(y), X(y)\Big\rangle_{H_y\mathbb{G}}\right| \cdot \left(\sum_{j=1}^{m_1} \langle X_j(y), \mathrm{n}(y)\rangle^2\right)^{\frac{1}{2}}. \qquad \square$$

Starting from Lemma 2.8 we carry out the proof of Proposition 2.1 by means of a partition of unity $\{(\mathcal{W}_\alpha, \sigma_\alpha)\}_{\alpha\in\mathcal{A}}$ related to the atlas $\{(\mathcal{S}_\alpha, \Psi_\alpha)\}_{\alpha\in\mathcal{A}}$ for $\mathcal{S}$, where $\mathcal{W}_\alpha = \mathrm{spt}\{\sigma_\alpha\} \Subset \mathcal{S}_\alpha$. Indeed, by the classical change of variables formula we get that

$$\mathcal{L}^n(\mathcal{D}) = \sum_{\alpha\in\mathcal{A}} \int_{\Psi_\alpha(pr_{\mathcal{S}}^X(\mathcal{D})\cap\mathcal{S}_\alpha)} (\sigma_\alpha \circ \Phi_\alpha)(\xi) \left[\int_{\gamma^{-1}_{\Phi_\alpha(\xi)}(\mathcal{D}_{\Phi_\alpha(\xi)})} \left|\det\left[\mathcal{J}_{(\xi,t)}\gamma_{\Phi_\alpha(\xi)}(t)\right]\right| dt\right] d\xi, \tag{44}$$

where $\mathcal{D}_{\Phi_\alpha(\xi)} := \gamma_{\Phi_\alpha(\xi)}(\mathbb{R}) \cap \mathcal{D}$ and $\gamma^{-1}_{\Phi_\alpha(\xi)}(\mathcal{D}_{\Phi_\alpha(\xi)}) = \Big\{t \in \mathbb{R} : \gamma_{\Phi_\alpha(\xi)}(t) \cap \mathcal{D} \neq \emptyset\Big\}$. Therefore, by (42) we have

$$\begin{aligned}
\mathcal{L}^n(\mathcal{D}) = \sum_{\alpha\in\mathcal{A}} \int_{\Psi_\alpha(pr_{\mathcal{S}}^X(\mathcal{D})\cap\mathcal{S}_\alpha)} &(\sigma_\alpha \circ \Phi_\alpha)(\xi) \left[\int_{\gamma^{-1}_{\Phi_\alpha(\xi)}(\mathcal{D}_{\Phi_\alpha(\xi)})} \left|\Big\langle X, \nu_E\Big\rangle_{H_{\Phi_\alpha(\xi)}\mathbb{G}}\right|\right.\\
&\left.\cdot \left(\sum_{j=1}^{m} \langle X_j(\Phi_\alpha(\xi)), \mathrm{n}(\Phi_\alpha(\xi))\rangle^2\right)^{\frac{1}{2}} \cdot \left|(\Phi_\alpha)_{\xi_1} \wedge \ldots \wedge (\Phi_\alpha)_{\xi_{n-1}}\right| dt\right] d\xi\\
= \int_{pr_{\mathcal{S}}^X(\mathcal{D})} &\left[\int_{\mathbb{R}} \mathbf{1}_{\mathcal{D}_y}(t)\, dt\right] \left|\langle X, \nu_E\rangle_{H_y\mathbb{G}}\right| \left(\sum_{j=1}^{m} \langle X_j(y), \mathrm{n}(y)\rangle^2\right)^{\frac{1}{2}} d\mathcal{H}^{n-1}(y)\\
= \int_{pr_{\mathcal{S}}^X(\mathcal{D})} &\mathcal{H}^1_c(\mathcal{D}_y)\, |\langle X, \nu_E\rangle_{H_y\mathbb{G}}|\, d\,|\partial E|_{\mathbb{G}}(y) = \int_{pr_{\mathcal{S}}^X(\mathcal{D})} \mathcal{H}^1_c(\mathcal{D}_y)\; d\,|\partial_X E|_{\mathbb{G}}(y),
\end{aligned}$$

where we have used Proposition 1.2, Proposition 1.19 and Remark 1.27. $\square$

Before the beginning of the proof of Theorem 2.2 we recall the basic statements of Implicit Function Theorem 1.24. We assume, by hypothesis, that $\mathcal{S}$ is a $\mathbb{G}$-regular hypersurface and so for every $\widetilde{x} \in \mathcal{S}$ there exist an open neighborhood $\mathcal{U}$ of $\widetilde{x}$ and a real valued function $f \in \mathbf{C}^1_{\mathbb{G}}(\mathcal{U})$ such that $\mathcal{S} \cap \mathcal{U} = \{x \in \mathcal{U} : f(x) = 0\}$ and $\nabla_{\mathbb{G}} f(x) \neq 0$ for all $x \in \mathcal{U}$. Thus $\mathcal{S}$ is locally the boundary of

$E = \{x \in \mathcal{U} : f(x) < 0\}$ and, without loss of generality, we can assume that $X_1 f(x) > 0$ for $x \in \mathcal{U}$. Let now h, $\delta > 0$ and set

$$J_h := [-h, h], \quad I_\delta := \{\xi = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1} : |\xi_j| \le \delta,\ j = 2, \dots, n\}.$$

If $\xi \in \mathbb{R}^{n-1}$ and $t \in J_h$ we denote by $\gamma^1_{(0,\xi)}(t)$ the integral curve of the left invariant horizontal vector field $X_1 \in H\mathbb{G}$ at the time t issued from $(0, \xi) \in \{(0, \eta) \in \mathbb{G} : \eta \in \mathbb{R}^{n-1}\}$. Then Theorem 1.24 states that there exist δ, $h > 0$ such that the mapping

$$\mathbb{R} \times \mathbb{R}^{n-1} \ni (t, \xi) \longmapsto \gamma^1_{(0,\xi)}(t)$$

is a diffeomorphism of a neighborhood of $J_h \times I_\delta$ onto an open subset of $\mathbb{G}$. In what follows we denote by $\mathcal{U}$ the image of $\operatorname{Int}\{J_h \times I_\delta\}$ through this mapping. The set E has finite $\mathbb{G}$-perimeter in $\mathcal{U}$ and if ν_E is the generalized inward unit normal of E we have

$$\nu_E(x) = -\frac{\nabla_{\mathbb{G}} f(x)}{|\nabla_{\mathbb{G}} f(x)|_{H_x\mathbb{G}}} \quad \forall x \in \mathcal{S} \cap \mathcal{U}.$$

Furthermore, there exists a unique continuous function $\phi = \phi(\xi) : I_\delta \longrightarrow J_h$ such that, setting $\Phi(\xi) = \gamma^1_{(0,\xi)}(\phi(\xi))$ for $\xi \in I_\delta$, we have $\mathcal{S} \cap \mathcal{U} = \{x \in \mathcal{U} : x = \Phi(\xi),\ \xi \in I_\delta\}$ and the $\mathbb{G}$-perimeter has the following integral representation

$$|\partial E|_{\mathbb{G}}(\mathcal{U}) = \int_{I_\delta} \frac{\sqrt{\sum_{j=1}^{m} |X_j f(\Phi(\xi))|^2}}{X_1 f(\Phi(\xi))}\, d\xi\,. \tag{45}$$

Let now J_ϵ be a Friedrichs' mollifier; putting $f_\epsilon = f * J_\epsilon$ by the continuity of f we have that $f_\epsilon \longrightarrow f$ as $\epsilon \to 0$ uniformly in $\mathcal{U}$ and analogously $(X_j f) * J_\epsilon \longrightarrow X_j f$ as $\epsilon \to 0$ uniformly in $\mathcal{U}$ (for $j = 1, \dots, m$). Arguing as in [38], p. 90, we obtain

$$X_j f_\epsilon = (X_j f) * J_\epsilon - ((X_j f) * J_\epsilon - X_j f_\epsilon) \quad \text{for } j = 1, \dots, m$$

and also

$$(X_j f) * J_\epsilon - X_j f_\epsilon \longrightarrow 0$$

uniformly in $\mathcal{U}$ as $\epsilon \to 0$. We note that starting from the regularization of f by the usual Implicit Function Theorem we get the existence of a smooth function $\phi_\epsilon : I_\delta \longrightarrow J_h$ such that $\phi_\epsilon \longrightarrow \phi$ as $\epsilon \to 0$ uniformly in I_δ. Thus we may construct a family $\{\mathcal{S}_\epsilon\}_{\epsilon>0}$ of smooth hypersurfaces which uniformly converges in $\mathcal{U}$ to $\mathcal{S} \cap \mathcal{U}$ as $\epsilon \to 0$. Moreover every hypersurface $\mathcal{S}_\epsilon$ is the boundary of a smooth open set E_ϵ which also converges in $\mathcal{U}$ to $E \cap \mathcal{U}$ as $\epsilon \to 0$. An explicit parametrization of $\mathcal{S}_\epsilon$ is given by the mapping $\Phi_\epsilon : I_\delta \longrightarrow \mathbb{G}$,

$\Phi_\epsilon(\xi) := \gamma^1_{(0,\xi)}(\phi_\epsilon)$ for $\xi \in I_\delta$. Finally, we have that $\Phi_\epsilon \longrightarrow \Phi$ uniformly for $\xi \in I_\delta$ as $\epsilon \to 0$. To see this, notice that

$$|\Phi_\epsilon(\xi) - \Phi(\xi)| = |\gamma^1_{(0,\xi)}(\phi_\epsilon(\xi)) - \gamma^1_{(0,\xi)}(\phi(\xi))| \leq \left| \int_{\phi_\epsilon(\xi)}^{\phi(\xi)} |X_1(\exp[tX_1](0,\xi))|\, dt \right|$$

and that $d_c(\exp[tX_1](0,\xi),(0,\xi)) \leq |t| \leq h$. Therefore, if $\mathcal{K}$ is a compact subset of I_δ

$$\exp[tX_1]((0,\xi)) \in \mathcal{K}_h := \{z \in \mathbb{G} : d_c(z, \{0 \times \mathcal{K}\}) \leq h\},$$

and, keeping in mind that $\phi_\epsilon \longrightarrow \phi$ as $\epsilon \to 0$ uniformly in I_δ, the proof of the previous assertion follows by observing that

$$|\Phi_\epsilon(\xi) - \Phi(\xi)| \leq |\phi_\epsilon(\xi) - \phi(\xi)| \cdot \max_{z \in \mathcal{K}_h} |X_1(z)|.$$

Proof of Theorem 2.2. This proof is divided into several claims. In what follows we will use the notation introduced in Theorem 1.24. From now on we assume that the hypersurface $\mathcal{S}$ is parameterized by a unique map Φ as above and, more precisely, we may suppose that there exists $\delta > 0$ such that $\mathcal{S}$ is the image of $\Phi : \mathrm{Int}\{I_\delta\} \longmapsto \mathbb{G}$, where $\Phi(\xi) = \gamma^1_{(0,\xi)}(\phi(\xi))$ and $I_\delta = \{\xi \in \mathbb{R}^{n-1} : |\xi|_\infty \leq \delta\}$. So we have

$$\mathcal{S} = \{y \in \mathbb{G} : y = \Phi(\xi),\ \xi \in I_\delta\} = \{y \in \mathbb{G} : f(y) = 0\}$$

where $f \in \mathbf{C}^1_{\mathbb{G}}(\mathbb{G})$ is an *implicit function* which defines $\mathcal{S}$ and such that $X_1 f > 0$ near $\mathcal{S}$.

Claim 1. Let $\alpha \in L^\infty(\mathbb{G}) \cap \mathbf{C}^\infty(\mathbb{G})$ be such that $\alpha \geq 0$. Then we have

$$\int_{\mathbb{G}} \alpha\, d\,\mathcal{L}^n = \int_{\mathcal{S}} \left[\int_{\mathbb{R}} (\alpha \circ \gamma_y)(t)\, dt \right] d\,|\partial_X E|_{\mathbb{G}}(y).$$

Proof of Claim 1. More explicitly, we note that the right-hand side is equal to

$$\int_{\mathcal{S}} \left[\int_{\mathbb{R}} (\alpha \circ \gamma_y)(t)\, |\langle X, \nu_E \rangle_{H_y\mathbb{G}}|\, dt \right] d\,|\partial E|_{\mathbb{G}}(y).$$

To prove this claim, we first set

$$I := \int_{\mathcal{S}} \left[\int_{\mathbb{R}} (\alpha \circ \gamma_y)(t)\, dt \right] d\,|\partial_X E|_{\mathbb{G}}(y).$$

By (iii) and (v) of Theorem 1.24 we get that

$$I = \int_{I_\delta} \int_{\mathbb{R}} \frac{|\langle X(\Phi(\xi)), \nabla_{\mathbb{G}} f(\Phi(\xi))\rangle_{H_{\Phi(\xi)}\mathbb{G}}|\,(\alpha \circ \gamma_{\Phi(\xi)})(t)}{X_1 f(\Phi(\xi))}\, dt\, d\xi\,.$$

Now we shall prove that

$$I = \lim_{\epsilon \to 0} \int_{I_\delta} \int_{\mathbb{R}} \frac{|\langle X(\Phi_\epsilon(\xi)), \nabla_{\mathbb{G}} f_\epsilon(\Phi_\epsilon(\xi))\rangle_{H_{\Phi_\epsilon(\xi)}\mathbb{G}}|\,(\alpha \circ \gamma_{\Phi_\epsilon(\xi)})(t)}{X_1 f(\Phi_\epsilon(\xi))}\, dt\, d\xi\,. \qquad (46)$$

Indeed if (46) holds, Claim 1 will hold by observing that

$$I = \lim_{\epsilon \to 0} \int_{\mathcal{S}_\epsilon} \int_{\mathbb{R}} (\alpha \circ \gamma_y)(t)\, |\langle X, \nu_{E_\epsilon}\rangle_{H_y\mathbb{G}}|\, dt\, d\,|\partial E_\epsilon|_{\mathbb{G}}(y)$$

and that Corollary 2.3 implies that

$$\int_{\mathbb{G}} \alpha\, d\,\mathcal{L}^n = \int_{\mathcal{S}_\epsilon} \int_{\mathbb{R}} (\alpha \circ \gamma_y)(t)\, |\langle X, \nu_{E_\epsilon}\rangle_{H_y\mathbb{G}}|\, dt\, d\,|\partial E_\epsilon|_{\mathbb{G}}(y)\,.$$

To prove (46), we first notice that, as we have seen above, $\Phi(\xi) \longrightarrow \Phi_\epsilon(\xi)$ uniformly in I_δ as $\epsilon \to 0$ and so, keeping in mind that $\nabla_{\mathbb{G}} f_\epsilon \longrightarrow \nabla_{\mathbb{G}} f$ uniformly on compact sets, we get

$$\nabla_{\mathbb{G}} f_\epsilon(\Phi_\epsilon(\xi)) \longrightarrow \nabla_{\mathbb{G}} f(\Phi(\xi)) \qquad (47)$$

as $\epsilon \to 0$ for $\xi \in I_\delta$. Thus, by (47) and by the continuous dependence of the Cauchy problem on the initial data, the integrand in (46) tends to the integrand of I. On the other hand $\Phi_\epsilon(\xi)$ lies in a fixed compact neighborhood of $\Phi(I_\delta)$ so that, by Weierstrass theorem and our assumptions on α, the integrand in (46) is bounded by a constant for $(\xi, t) \in I_\delta \times \mathbb{R}$ and (46) follows by Dominate Convergence Theorem. □

Claim 2. Let $Q \subset \mathcal{R}^X_{\mathcal{S}}$ be a compact, rectangular n-box. Then

$$\mathcal{L}^n(Q) \geq \int_{\mathcal{S}} \mathcal{H}^1_c(\gamma_y(\mathbb{R}) \cap Q)\, d\,|\partial_X E|_{\mathbb{G}}(y)\,.$$

Proof of Claim 2. Let us choose a sequence of functions $\{\alpha_h\}_{h\in\mathbb{N}}$ such that

$$\lim_{h\to\infty} \alpha_h(x) = \mathbf{1}_Q(x) \quad \forall\, x \in \mathbb{G}\,.$$

For $y \in \mathcal{S}$ we set $\gamma_y^{-1}(Q) := \Big\{t \in \mathbb{R} : \gamma_y(t) \in Q\Big\}$. So we have

$$\alpha_h(\gamma_y(t)) \longrightarrow \mathbf{1}_{\gamma_y^{-1}(Q)}(t) \quad \forall\, (y, t) \in \mathcal{S} \times \mathbb{R}$$

as $h \to \infty$. Therefore, the proof follows by observing that

$$\begin{aligned}
\int_{\mathcal{S}} \mathcal{H}^1_c(\gamma_y(\mathbb{R}) \cap Q)\, d\, |\partial_X E|_{\mathbb{G}}(y) &= \int_{\mathcal{S}} \int_{\mathbb{R}} \mathbf{1}_{\gamma_y^{-1}(Q)}(t)\, d\, t\, d\, |\partial_X E|_{\mathbb{G}}(y) \\
&= \int_{\mathcal{S}} \int_{\mathbb{R}} \lim_{h\to\infty} \alpha_h(\gamma_y(t))\, d\, t\, d\, |\partial_X E|_{\mathbb{G}}(y) \\
&\leq \liminf_{h\to\infty} \int_{\mathcal{S}} \int_{\mathbb{R}} \alpha_h(\gamma_y(t))\, d\, t\, d\, |\partial_X E|_{\mathbb{G}}(y) \\
&= \lim_{h\to\infty} \int_{\mathbb{G}} \alpha_h(x)\, d\, \mathcal{L}^n(x) = \mathcal{L}^n(Q)\,. \qquad \square
\end{aligned}$$

Claim 3. Let $F \subset \mathcal{R}^X_{\mathcal{S}}$ be a measurable subset of $\mathbb{G}$ such that $\mathcal{L}^n(F) = 0$. Setting

$$\mathcal{S}_0 := \left\{ y \in \mathcal{S} : \mathcal{H}^1_c(\gamma^1_y(\mathbb{R}) \cap F) > 0 \right\},$$

we have that $|\partial E|_{\mathbb{G}}(\mathcal{S}_0) = 0$.

Proof of Claim 3. Let $\epsilon > 0$ and $\{Q_j\}_{j\in\mathbb{N}}$ be a countable family of compact, rectangular, n-boxes such that

$$F \subseteq \bigcup_{j=1}^{\infty} Q_j, \qquad \sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) < \epsilon\,.$$

We have then

$$\begin{aligned}
\int_{\mathcal{S}} \mathcal{H}^1_c(\gamma_y(\mathbb{R}) \cap F)\, d\, |\partial_X E|_{\mathbb{G}}(y) &\leq \int_{\mathcal{S}} \sum_{j=1}^{\infty} \mathcal{H}^1_c(\gamma_y(\mathbb{R}) \cap Q_j)\, d\, |\partial_X E|_{\mathbb{G}}(y) \\
&= \int_{\mathcal{S}} \lim_{k\to\infty} \sum_{j=1}^{k} \mathcal{H}^1_c(\gamma_y(\mathbb{R}) \cap Q_j)\, d\, |\partial_X E|_{\mathbb{G}}(y) \\
&\leq \lim_{k\to\infty} \sum_{j=1}^{k} \int_{\mathcal{S}} \mathcal{H}^1_c(\gamma_y(\mathbb{R}) \cap Q_j)\, d\, |\partial_X E|_{\mathbb{G}}(y) \\
&\leq \sum_{j=1}^{\infty} \int_{\mathcal{S}} \mathcal{H}^1_c(\gamma_y(\mathbb{R}) \cap Q_j)\, d\, |\partial_X E|_{\mathbb{G}}(y) \\
&\leq \sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) < \epsilon\,.
\end{aligned}$$

Therefore

$$\int_{\mathcal{S}} \mathcal{H}^1_c(\gamma_y(\mathbb{R}) \cap F) \cdot |\langle X, \nu_E \rangle_{H_y\mathbb{G}}|\, d\, |\partial E|_{\mathbb{G}}(y) = 0$$

and since $\langle X, \nu_E\rangle_{H_y\mathbb{G}} \neq 0$ for any $y \in \mathcal{S}$, the proof of Claim 3 follows by observing that

$$\mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap F) = 0 \quad \text{for } |\partial E|_{\mathbb{G}} - \text{a. e. } y \in \mathcal{S}. \qquad \square$$

At this point we can achieve the proof of Theorem 2.2 in the following way. Let J_ϵ be a Friedrichs' mollifier and put $\alpha_\epsilon := \mathbf{1}_{\mathcal{D}} * J_\epsilon$. Since $\alpha_\epsilon \in L^\infty(\mathbb{G}) \cap \mathbf{C}^\infty(\mathbb{G})$ and $\alpha_\epsilon \longrightarrow \mathbf{1}_{\mathcal{D}}$ in L^1_{loc}, up to a subsequence, we may assume that

$$\lim_{\epsilon\to 0} \alpha_\epsilon = \mathbf{1}_{\mathcal{D}} \quad \text{for } \mathcal{L}^n - \text{a.e. } x \in \mathbb{G}\,.$$

Now setting

$$F := \mathcal{D}\setminus\left\{x \in \mathcal{D} : \lim_{\epsilon\to 0} \alpha_h(x) = \mathbf{1}_{\mathcal{D}}(x)\right\} \text{ and } \mathcal{S}_0 := \left\{y \in S : \mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap F) > 0\right\}$$

by Claim 3 we get that $|\partial E|_{\mathbb{G}}(\mathcal{S}_0) = 0$. Moreover by Claim 1 we have

$$\int_{\mathbb{G}} \alpha_\epsilon \, d\,\mathcal{L}^n = \int_{\mathcal{S}} \int_{\mathbb{R}} (\alpha_\epsilon \circ \gamma_y)(t)\, dt\, d\,|\partial_X E|_{\mathbb{G}}(y) \tag{48}$$

$$= \int_{\mathcal{S}\setminus\mathcal{S}_0} \int_{\mathbb{R}} (\alpha_\epsilon \circ \gamma_y)(t)\, dt\, d\,|\partial_X E|_{\mathbb{G}}(y)\,. \tag{49}$$

Therefore $\alpha_\epsilon(\gamma_y(t)) \longrightarrow \mathbf{1}_{\gamma_y^{-1}(Q)}(t)$ for $\mathcal{L}^1$ – a.e. $t \in \mathbb{R}$ and $|\partial E|_{\mathbb{G}}$ – a.e. $y \in \mathcal{S}$ as $\epsilon \to 0$. Thus the thesis follows by letting ϵ to 0 in (48). $\square$

3. Sections of $BV_{\mathbb{G}}$ functions and $\mathbb{G}$-Perimeter

3.1. One-dimensional restrictions of $BV_{\mathbb{G}}$ functions

We now introduce the concept of variation along a horizontal direction of a locally summable function in a Carnot group $\mathbb{G}$ and we summarize its main properties. Afterwards, we define the notion of X-variation along a horizontal line and we consider the space of functions of bounded variation along a fixed horizontal line. Then, in Theorem 3.7, we establish a link between the notion of variation of a function along a horizontal direction and that of variation of the restrictions of such a function to a family of horizontal lines. Finally, we generalize to Carnot groups a well-known characterization of the usual space *BV* by means of one-dimensional restrictions of its elements. These topics in the classical setting can be found in [2], or in [30], while many other results about function of bounded variation in Carnot-Carathéodory spaces can be found in [1, 5, 14, 37, 38, 42, 63, 62, 76].

Definition 3.1. Let $\Omega \subseteq \mathbb{G}$ be open and let $X \in H\mathbb{G}$ be a horizontal left invariant vector field. We say that $f \in L^1(\Omega)$ has bounded X-variation in Ω if

$$|Xf|(\Omega) = \sup\left\{ \int_\Omega f X\varphi \, d\mathcal{L}^n : \varphi \in \mathbf{C}^1_0(\Omega),\ |\varphi| \leq 1 \right\} < \infty\,;$$

we refer to the quantity $|Xf|(\Omega)$ as the X-*variation* of f in Ω and we denote by $BV_X(\Omega)$ the vector space of bounded X-variation functions in Ω.

In the next remark we summarize, without proof, some well-known properties of the variation.

Remark 3.2. Let $\Omega \subseteq \mathbb{G}$ be open and let $X \in H\mathbb{G}$. Then the following items hold:

(i) let $f, f_k \in L^1(\Omega)$ for $k \in \mathbb{N}$ be such that $f_k \longrightarrow f$ in $L^1(\Omega)$ as $k \to \infty$. Then
$$|Xf|(\Omega) \leq \liminf_{k \longrightarrow \infty} |Xf_k|(\Omega)\,;$$

(ii) if $f \in BV_X(\Omega)$ then $|Xf|$ is a Radon measure in Ω and
$$\int_\Omega f\, X\varphi \, d\,\mathcal{L}^n = -\int_\Omega \varphi \, d\,|Xf| \quad \forall\ \varphi \in \mathbf{C}^\infty_0(\Omega)\,;$$

(iii) $|Xf|(\Omega) = \int_\Omega |Xf|\, d\mathcal{L}^n \qquad \forall\ f \in \mathbf{C}^1(\Omega)\,;$

(iv) if $f \in BV_X(\Omega)$ then there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathbf{C}^\infty(\Omega) \cap BV_X(\Omega)$ such that
$$\lim_{j\to\infty} \|f_j - f\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \lim_{j\to\infty} |Xf_j|(\Omega) = |Xf|(\Omega)\,.$$

From now on, let Ω denote an open subset of $\mathbb{G}$ and let $f : \Omega \longrightarrow \mathbb{R}$. Moreover let us fix a horizontal direction $X \in H\mathbb{G}$ and let us denote by $\gamma : \mathbb{R} \longrightarrow \Omega$ the corresponding horizontal X-line. Proposition 1.2 implies that for all compact set $\mathcal{K} \subset \gamma$ one has
$$\mathcal{H}^1_c(\mathcal{K}) = \int_{\gamma^{-1}(\mathcal{K})} |X|_{H\mathbb{G}}\, dt\,.$$

Therefore, if $f \circ \gamma \in L^1(\gamma^{-1}(\mathcal{K}))$, putting $|X|_{H\mathbb{G}} = 1$, we get that the integral of f along the horizontal X-line γ is given by

$$\int_{\mathcal{K}} f\, d\mathcal{H}^1_c = \int_{\gamma^{-1}(\mathcal{K})} (f \circ \gamma)(t)\, dt \tag{50}$$

for every compact $\mathcal{K} \subset \gamma$. In the sequel, if$\mathcal{U} \subset \gamma$ is an open subset of γ, we shall denote by $L^1(\mathcal{U}, d\mathcal{H}^1_c \llcorner \gamma)$ the space of all $\mathcal{H}^1_c$-summable functions defined on $\mathcal{U}$.

Proposition 3.3. *Let $X \in H\mathbb{G}$, $|X|_{H\mathbb{G}} = 1$, and let γ be a horizontal X-line starting from $x \in \mathbb{G}$, i.e. $\gamma(t) = \exp[tX](x)$for $t \in \mathbb{R}$. If $\mathcal{U}$ is an open subset of γ and*

$f \in L^1(\mathcal{U},\, d\mathcal{H}^1_c \llcorner \gamma)$ the following two statements are equivalent:

(i) $f \circ \gamma \in BV(\gamma^{-1}(\mathcal{U}))$;

(ii) $|D(f \circ \gamma)|(\gamma^{-1}(\mathcal{U})) = \sup \left\{ \int_\gamma f\, d\psi,\ \psi \in \mathbf{C}^1_0(\mathcal{U}), |\psi| \le 1 \right\} < \infty\,.$

Moreover, setting

$$\mathrm{var}^1_X[f](\mathcal{U}) :$$
$$= \sup \left\{ \int_{\mathcal{U}} f\, X\varphi\, d\mathcal{H}^1_c : \varphi \in \mathbf{C}^1_0(\mathcal{B}),\ |\varphi| \le 1,\ \mathcal{B} \subset \mathbb{G},\ \mathcal{B} \text{ open s.t. } \gamma \cap \mathcal{B} = \mathcal{U} \right\},$$

we get that

$$\mathrm{var}^1_X[f](\mathcal{U}) = |D(f \circ \gamma)|(\gamma^{-1}(\mathcal{U}))\,.$$

Remark 3.4. Here we have used the usual definition (see [2, 30]) of total variation for real functions of one variable. We remind that, whenever $h : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, $h \in L^1(I)$, the total variation $|Dh|(I)$ of h in I is given by

$$|Dh|(I) := \sup \left\{ \int_I h\, \frac{d\phi}{dt}\, dt : \phi \in \mathbf{C}^1_0(I),\ |\phi| \le 1| \right\}\,.$$

$BV(I)$ will denote the space of functions belonging to $L^1(I)$ and of finite total variation in I.

Proof of Proposition 3.1. Since

$$\int_\gamma f\, d\psi = \int_{\mathbb{R}} (f \circ \gamma) \frac{d}{dt}(\psi \circ \gamma)\, dt$$

it follows that (i) is equivalent to (ii) because if $\psi \in \mathbf{C}^1_0(\mathcal{U})$, $|\psi| \le 1$, we may put

$$\phi = (\phi \circ \gamma^{-1}) \circ \gamma = \psi \circ \gamma\,,$$

where $\phi \in \mathbf{C}^1_0(\mathbb{R})$, $\mathrm{spt}(\phi) \subset \gamma^{-1}(\mathcal{U})$, $|\phi| \le 1$. To prove the last statement we notice that, for any $\psi \in \mathbf{C}^1_0(\mathcal{U})$, $|\psi| \le 1$, we may find $\varphi \in \mathbf{C}^1_0(\mathbb{R}^n)$ such that $\psi = \varphi_{|\gamma}$, $\mathrm{spt}(\varphi) \cap \gamma = \mathrm{spt}(\psi)$ and $|\varphi| \le 1$. Thus the following chain of equalities holds:

$$\begin{aligned}
&\sup \left\{ \int_\gamma f\, d\psi : \psi \in \mathbf{C}^1_0(\mathcal{U}), |\psi| \le 1 \right\} \\
&= \sup \left\{ \int_\gamma f\, d\varphi : \varphi \in \mathbf{C}^1_0(\mathbb{R}^n),\ \mathrm{spt}(\varphi) \cap \gamma \subset \mathcal{U},\ |\varphi| \le 1 \right\} \\
&= \sup \left\{ \int_{\mathbb{R}} (f \circ \gamma) \frac{d}{dt}(\varphi \circ \gamma)\, dt : \varphi \in \mathbf{C}^1_0(\mathbb{R}^n),\ \mathrm{spt}(\varphi) \cap \gamma \subset \mathcal{U},\ |\varphi| \le 1 \right\} \\
&= \sup \left\{ \int_{\mathbb{R}} (f \circ \gamma) \left\langle \dot\gamma(t), \nabla\varphi(\gamma(t)) \right\rangle dt : \varphi \in \mathbf{C}^1_0(\mathbb{R}^n),\ \mathrm{spt}(\varphi) \cap \gamma \subset \mathcal{U}, |\varphi| \le 1 \right\}.
\end{aligned} \tag{51}$$

So we may find an open set $\mathcal{B} \subset \mathbb{G}$ such that $\gamma \cap \mathcal{B} = \mathcal{U}$ and we get that (51) is equal to

$$\sup \left\{ \int_{\mathcal{U}} f\, X\varphi\, d\mathcal{H}^1_c : \varphi \in \mathbf{C}^1_0(\Omega),\ |\varphi| \leq 1 \right\}. \qquad \square$$

Definition 3.5. Let $X \in H\mathbb{G}$, $|X|_{H\mathbb{G}} = 1$, and let γ be a horizontal X-line. If $\mathcal{U}$ is an open subset of γ and $f \in L^1(\mathcal{U}, d\mathcal{H}^1_c \llcorner \gamma)$ we call $\mathrm{var}^1_X[f](\mathcal{U})$ the *X-variation of f along γ* and we define $BV^1_X(\mathcal{U})$ as the space of functions of finite X-variation in $\mathcal{U} \subset \gamma$.

Proposition 3.6. *Let $X \in H\mathbb{G}$, $|X|_{H\mathbb{G}} = 1$; let γ be a horizontal X-line. Then for every $\mathcal{H}^1_c$-measurable set $\mathcal{E} \subset \gamma$ one has*

$$\mathrm{var}^1_X[\mathbf{1}_{\mathcal{E}}](\gamma) = |D\mathbf{1}_{\gamma^{-1}(\mathcal{E})}|(\mathbb{R}) = \mathrm{var}^1_X[\mathbf{1}_{l_y\mathcal{E}}](l_y\gamma) \quad \forall\ y \in \mathbb{G} \tag{52}$$

where $\gamma^{-1}(\mathcal{E}) = \Big\{t \in \mathbb{R} : \gamma(t) \in \mathcal{E}\Big\}$; moreover

$$\mathrm{var}^1_X[\mathbf{1}_{\mathcal{E}}](\gamma) \geq 2 \tag{53}$$

and equality holds if and only if $\gamma^{-1}(\mathcal{E})$ is a bounded interval of $\mathbb{R}$.

Proof. Equalities (52) follow from Definition 3.5. Moreover, using the first identity of (52) we get that $\mathrm{var}^1_X[\mathbf{1}_{\mathcal{E}}](\gamma)$ is equal to the euclidean one-dimensional perimeter of $\gamma^{-1}(\mathcal{E})$ in $\mathbb{R}$. Thus, using the *one-dimensional isoperimetric inequality* of [72], p. 103, Section 3.6, we get (53). $\square$

It seems interesting to find some results that reduce the study of $BV_{\mathbb{G}}$ functions to that one of their one-dimensional restrictions. Indeed, this is a very useful approach in classical Calculus of Variations; see [2, 43]. Here below we state a theorem modeled on an analogous euclidean result for which we refer the reader to [2] and [30]. A similar theorem has been proved in [76] for Sobolev functions in Carnot groups and in [18] in the case of vertical planes in Heisenberg type groups.

Theorem 3.7. *Let $\mathcal{S} \subset \mathbb{G}$ be a $\mathbb{G}$-regular hypersurface and assume that $\mathcal{S} = \partial E$ globally, where $E \subset \mathbb{G}$ is a suitable open $\mathbb{G}$-Caccioppoli set. Let $X \in H\mathbb{G}$, $|X|_{H\mathbb{G}} = 1$, be a unit horizontal left invariant vector field which is transverse to $\mathcal{S}$ and denote by γ_y the horizontal X-line starting from $y \in \mathcal{S}$. We assume that $\gamma_y(\mathbb{R}) \cap \mathcal{S} = \{y\}$ for every $y \in \mathcal{S}$. Finally let $\Omega \subseteq \mathcal{R}^X_{\mathcal{S}}$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from $\mathcal{S}$ by means of X-lines. Then*

$$|Xf|(\Omega) = \int_{pr^X_{\mathcal{S}}(\Omega)} \mathrm{var}^1_X[f_{\gamma_y}](\Omega_y)\, d\,|\partial_X E|_{\mathbb{G}}(y) \tag{54}$$

where $f_{\gamma_y} := f \circ \gamma_y$ and $\Omega_y := \gamma_y \cap \Omega$.

Proof. Using (ii) of Corollary 2.3 we get

$$\begin{aligned}\int_\Omega f X\varphi \, d\mathcal{L}^n &= \int_{pr_{\mathcal{S}}^X(\Omega)} \int_{\gamma_y^{-1}(\Omega_y)} (f \circ \gamma_y) \frac{d}{dt}(\varphi \circ \gamma_y) \, dt \, d\,|\partial_X E|_{\mathbb{G}}(y) \\ &\le \int_{pr_{\mathcal{S}}^X(\Omega)} \operatorname{var}_X^1[f_{\gamma_y}](\Omega_y) \, d\,|\partial_X E|_{\mathbb{G}}(y)\,,\end{aligned}$$

whenever $\varphi \in \mathbf{C}_0^1(\Omega)$. In a similar way we obtain the equality if $f \in \mathbf{C}^1(\Omega)$. Now let us set

$$\Omega^h := \left\{ x \in \Omega : |x| < \frac{1}{h},\ \operatorname{dist}(x, \partial\Omega) > h \right\}$$

and choose $h > 0$ such that $|Xf|(\partial\Omega^h) = 0$. Notice that this can be done for $\mathcal{L}^1-$a.e. $h > 0$, as for instance in [2], Example 1.63. Therefore, using Lemma 1.20, we get that

$$\begin{aligned}&\lim_{\epsilon\to 0} \int_{\Omega^h} |(f * J_\epsilon) - f| \, d\mathcal{L}^n \\ &= \lim_{\epsilon\to 0} \int_{pr_{\mathcal{S}}^X(\Omega^h)} \|(f * J_\epsilon)_y - f_y\|_{L^1(\gamma_y{}^{-1}((\Omega^h)_y))} \, d\,|\partial_X E|_{\mathbb{G}}(y) = 0\,,\end{aligned}$$

and so we may choose a sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} \int_{\gamma_y{}^{-1}((\Omega^h)_y)} |(f * J_{\epsilon_j})_y - f_y| \, dt = 0 \qquad \text{for } |\partial_X E|_{\mathbb{G}} - \text{a.e. } y \in pr_{\mathcal{S}}^X(\Omega^h)\,.$$

By the lower semicontinuity of the X-variation (see (i) of Remark 3.2) we get

$$\begin{aligned}&\int_{pr_{\mathcal{S}}^X(\Omega^h)} \operatorname{var}_X^1[f_{\gamma_y}]((\Omega^h)_y) \, d\,|\partial_X E|_{\mathbb{G}}(y) \\ &\le \int_{pr_{\mathcal{S}}^X(\Omega^h)} \liminf_{j\to\infty} \operatorname{var}_X^1[(f * J_{\epsilon_j})_{\gamma_y}]((\Omega^h)_y) \, d\,|\partial_X E|_{\mathbb{G}}(y) \\ &\le \liminf_{j\to\infty} \int_{pr_{\mathcal{S}}^X(\Omega^h)} \operatorname{var}_X^1[(f * J_{\epsilon_j})_y]((\Omega^h)_y) \, d\,|\partial_X E|_{\mathbb{G}}(y) \\ &= \lim_{j\to\infty} |X(f * J_{\epsilon_j})|(\Omega^h) \\ &= |Xf|(\Omega^h) \le |Xf|(\Omega)\end{aligned}$$

and the claim follows by letting $h \to 0$. $\square$

We would notice that for any $j = 1, \dots, m_1$, the following inequalities hold

$$|X_j f|(\Omega) \leq |\nabla_{\mathbb{G}} f|(\Omega) \leq \sum_{j=1}^{m_1} |X_j f|(\Omega)\,,$$

whenever $f \in BV_{\mathbb{G}}(\Omega)$, where $X_1, \dots, X_{m_1}$ are the canonical generating vector fields of the group. This easily follows from Definition 1.13 and Definition 3.1 and, using Theorem 3.7, it allows to state the following

Corollary 3.8. (Characterization of $BV_{\mathbb{G}}$ by sections.) *Let $X_1, \dots, X_{m_1}$ be the generating vector fields of the global frame for $\mathbb{G}$ and let $j = 1, \dots, m_1$. Let $\mathcal{S}_j \subset \mathbb{G}$ be a $\mathbb{G}$-regular hypersurface such that $\mathcal{S}_j = \partial E_j$ globally, where $E_j \subset \mathbb{G}$ is a suitable open $\mathbb{G}$-Caccioppoli set, and suppose that X_j is transverse to $\mathcal{S}_j$. Denoting by γ_y^j the horizontal X_j-line starting from $y \in \mathcal{S}_j$, we assume that $\gamma_y^j(\mathbb{R}) \cap \mathcal{S}_j = \{y\}$ for every $y \in \mathcal{S}_j$. Finally let $\Omega \subseteq \mathcal{R}^{X_j}_{\mathcal{S}_j}$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from each $\mathcal{S}_j$ by means of X_j-lines. Then, we have that $f \in BV_{\mathbb{G}}(\Omega)$ if and only if $f_{\gamma_y^j} \in BV^1_{X_j}(\Omega_y^{X_j})$ for $|\partial E_{X_j}|_{\mathbb{G}}$−a.e. $y \in pr^{X_j}_{\mathcal{S}_j}(\Omega)$ and*

$$\int_{pr^{X_j}_{\mathcal{S}_j}(\Omega)} \mathrm{var}^1_{X_j}\left[f_{\gamma_y^j}\right]\left(\Omega_y^{X_j}\right) d\,|\partial_{X_j} E|_{\mathbb{G}}(y) < \infty \qquad \forall\ j = 1, \dots, m_1\,.$$

Remark 3.9. Denoting by $\mathcal{I}_0(X_j)$ the vertical hyperplane through $0 \in \mathbb{G}$ and orthogonal to X_j (see (17) of Section 1.1), we may assume that $\mathcal{S}_j = \mathcal{I}_0(X_j)$ for every $j = 1, \dots, m_1$, and for such hypersurfaces the hypotheses of Corollary 3.8 are automatically verified since each subset of $\mathbb{G}$ is reachable from a given vertical hyperplane. More precisely, if $\Omega \subseteq \mathbb{G}$ and $j = 1, \dots, m_1$, we have that Ω can be foliated with a family of horizontal X_j-lines starting from $\mathcal{I}_0(X_j)$ and hence the above characterization of $BV_{\mathbb{G}}(\Omega)$ can be reformulated by means of vertical hyperplanes.

3.2. Integral geometric measures, $\mathbb{G}$-normal sets and $\mathbb{G}$-convexity

In this subsection we give some applications of the previous results. To this end, we introduce a measure μ_0 on $UH\mathbb{G}$ (i.e. the unit horizontal bundle over $\mathbb{G}$) that we need to state some integral geometric formulae for volume and $\mathbb{G}$-perimeter. Afterwards, we give a definition of $\mathbb{G}$-*normality* with respect to a vertical hyperplane that generalizes the euclidean one (see [26, 72]). Then we formulate an intrinsic definition of convexity, called $\mathbb{G}$-*convexity* (see Definition 3.15 below), that seems to be natural from a geometric point of view. Indeed, using this notion we prove a *Cauchy-type formula* and a related inequality which says that, in a sense, among all sets containing a fixed $\mathbb{G}$-convex set, this one minimizes the $\mathbb{G}$-perimeter. See Theorem 3.19 and Corollary 3.20 below, and also [16] and [69]for the classical statements. We would emphasize that equivalent definitions of convexity in Carnot groups has been introduced recently in [23] and in [55]; see also [8] and [47] for some further developments.

We first consider the volume form on $UH\mathbb{G}$ given by $\Theta \wedge \sigma_{m_1-1}$, where $\Theta = \omega_1 \wedge \ldots \wedge \omega_n$ is the bi-invariant volume form on $\mathbb{G}$ and σ_{m_1-1} is the canonical volume form on the unit sphere $\mathbb{S}^{m_1-1}$ of $\mathbb{R}^{m_1}$ that is identified with the generic fiber of $UH\mathbb{G}$. More explicitly, we note that if $(x; X) \in UH\mathbb{G}$ $(X(0) = (a_1, \ldots, a_{m_1}, 0, \ldots, 0))$, then

$$\sigma_{m_1-1}(X) = \sum_{i=1}^{m_1-1} (-1)^{i+1} a_i \, da_1 \wedge \ldots \wedge \widehat{da_i} \wedge \ldots \wedge da_{m_1}$$

and

$$\begin{aligned} &(\Theta \wedge \sigma_{m_1-1})(x; X)(X_1, \ldots, X_n; Y_1, \ldots, Y_{m_1}) \\ &= \Theta(x)(X_1, \ldots, X_n) \cdot \sigma_{m_1-1}(X)(Y_1, \ldots, Y_{m_1}) \quad \forall\, X_1, \ldots, X_n \in T_x\mathbb{G} \\ &\qquad \forall\, Y_1, \ldots, Y_{m_1} \in UH_x\mathbb{G}\,. \end{aligned}$$

Definition 3.10. We denote by μ_0 the measure on $UH\mathbb{G}$ obtained by integration of $\Theta \wedge \sigma_{m_1-1}$ and by μ_{0x} the measure on $UH_x\mathbb{G}$ (i.e. the fiber at x) obtained by integration of σ_{m_1-1}. Thus, for every function $f \in L^1(UH\mathbb{G})$ we have

$$\int_{UH\mathbb{G}} f(x; X)\, d\,\mu_0(x; X) = \int_{\mathbb{G}} d\,\mathcal{L}^n(x) \int_{UH_x\mathbb{G}} f(x; X)\, d\,\mu_{0x}(X)\,. \tag{55}$$

We remind that if $\mathcal{D}$ is a subset of $\mathbb{G}$, then $UH\mathcal{D}$ denotes the restriction of the bundle structure $UH\mathbb{G}$ to $\mathcal{D}$, i.e.

$$UH\mathcal{D} := \left\{ X \in UH\mathbb{G} : \pi_{|UH\mathbb{G}}(X) \in \mathcal{D} \right\}.$$

Furthermore, if $x_0 \in \mathbb{G}$ and $X \in UH\mathbb{G}$, then $\mathcal{I}_{x_0}(X)$ denotes the vertical hyperplane through x_0 and orthogonal to X and $\mathcal{V}_{x_0}$ denotes the family of all vertical hyperplanes through x_0. Finally, γ_y^X is the horizontal X-line starting from $y \in \mathcal{I}_{x_0}(X)$, i.e. $\gamma_y^X(t) = \exp[tX](y)$ for $t \in \mathbb{R}$, and if $\mathcal{D} \subset \mathbb{G}$ we set $\mathcal{D}_y^X := \gamma_y^X(\mathbb{R}) \cap \mathcal{D}$. Note that, if $X(0) = \sum_{j=1}^{m_1} a_j \mathrm{e}_j$, then $\mathcal{I}_{x_0}(X)$ is the boundary of the half-space

$$\mathcal{I}_{x_0}^-(X) = \left\{ x \in \mathbb{G} : \sum_{j=1}^{m_1} \left(x_j - (x_0)_j \right) a_j \le 0 \right\}$$

and we get that

$$\nu_{\mathcal{I}_{x_0}^-(X)}(y) = (a_1, \ldots, a_{m_1}) \qquad \forall\, y \in \mathcal{I}_{x_0}(X)\,.$$

Hence, the $\mathbb{G}$-perimeter of $\mathcal{I}_{x_0}^-(X)$ is the usual $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on the vertical hyperplane $\mathcal{I}_{x_0}(X)$ and, using Proposition 2.1, we may state the following

Corollary 3.11. *Let $\mathcal{D}$ be a Lebesgue measurable subset of $\mathbb{G}$ and fix $x_0 \in \mathbb{G}$. Then*

$$\mu_0(UH\mathcal{D}) = \int_{UH_{x_0}\mathbb{G}} d\,\mu_{0x_0}(X) \int_{pr^X_{\mathcal{I}_{x_0}(X)}(\mathcal{D})} \mathcal{H}^1_c(\mathcal{D}^X_y)\, d\,\mathcal{H}^{n-1}(y)$$

or, equivalently,

$$\mathcal{L}^n(\mathcal{D}) = \frac{1}{O_{m_1-1}} \int_{UH_{x_0}\mathbb{G}} d\,\mu_{0x_0}(X) \int_{pr^X_{\mathcal{I}_{x_0}(X)}(\mathcal{D})} \mathcal{H}^1_c(\mathcal{D}^X_y)\, d\,\mathcal{H}^{n-1}(y)\,,$$

where O_{m_1-1} denotes the (m_1-1)-dimensional surface measure of the sphere $\mathbb{S}^{m_1-1}$ of $\mathbb{R}^{m_1}$.

Proof. From Proposition 2.1 we have that

$$\mathcal{L}^n(\mathcal{D}) = \int_{pr^X_{\mathcal{I}_{x_0}(X)}(\mathcal{D})} \mathcal{H}^1_c(\mathcal{D}^X_y)\, d\,\mathcal{H}^{n-1}(y) \quad \forall\, X \in UH\mathbb{G}\,,$$

so we get the claim by integrating both sides of the last identity over $X \in UH_{x_0}\mathbb{G}$. □

Corollary 3.12. *Let $\Omega \subseteq \mathbb{G}$ be open and let $X \in UH\mathbb{G}$. If $\mathcal{D} \subset \mathbb{G}$ is a $\mathbb{G}$-Caccioppoli set, then*

$$|\partial_X \mathcal{D}|_{\mathbb{G}}(\Omega) = \int_{pr^X_{\mathcal{I}_{x_0}(X)}(\mathcal{D}\cap\Omega)} \mathrm{var}^1_X[\mathbf{1}_{\mathcal{D}^X_y}](\Omega^X_y)\, d\,\mathcal{H}^{n-1}(y)\,. \tag{56}$$

Proof. This follows using Lemma 1.26 and Theorem 3.7 and observing that, for the half-space $\mathcal{I}^-_{x_0}(X)$, we have

$$|\partial_X \mathcal{I}^-_{x_0}(X)|_{\mathbb{G}}(\mathcal{B}) = \mathcal{H}^{n-1}(\mathcal{B} \cap \mathcal{I}_{x_0}(X)) \quad \forall\, \mathcal{B} \in \mathcal{B}or(\mathbb{G})\,.$$ □

As an application of the last corollary we may establish the following

Proposition 3.13. (Integral geometric $\mathbb{G}$-perimeter.) *Let $\Omega \subseteq \mathbb{G}$ be open and fix $x_0 \in \mathbb{G}$. If $\mathcal{D} \subset \mathbb{G}$ is a $\mathbb{G}$-Caccioppoli set, we have*

$$|\partial\mathcal{D}|_{\mathbb{G}}(\Omega) = \frac{1}{2\kappa_{m_1-1}} \int_{UH_{x_0}\mathbb{G}} d\,\mu_{0x_0}(X) \int_{pr^X_{\mathcal{I}_{x_0}(X)}(\mathcal{D}\cap\Omega)} \mathrm{var}^1_X[\mathbf{1}_{\mathcal{D}^X_y}](\Omega^X_y)\, d\,\mathcal{H}^{n-1}(y)\,, \tag{57}$$

where κ_{m_1-1} denotes the m_1-1-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{m_1-1}$.

Proof. Starting from Corollary 3.12, we integrate both sides of (56) over $X \in UH_{x_0}\mathbb{G}$. Thus

$$\begin{aligned}
\int_{UH_{x_0}\mathbb{G}} d\,\mu 0_{x_0}(X) \int_{pr^X_{\mathcal{I}_{x_0}(X)}(\mathcal{D}\cap\Omega)} & \mathrm{var}^1_X[\mathbf{1}_{\mathcal{D}^X_y}](\Omega^X_y)\, d\,\mathcal{H}^{n-1}(y) \\
&= \int_{UH_{x_0}\mathbb{G}} |\partial\mathcal{D}|_{\mathbb{G}}(\Omega)\, d\,\mu 0_{x_0}(X) \\
&= \int_{UH_{x_0}\mathbb{G}} d\,\mu 0_{x_0}(X) \int_{\mathcal{D}\cap\Omega} |\langle X, \nu_{\mathcal{D}}\rangle_{H\mathbb{G}}|\, d\,|\partial\mathcal{D}|_{\mathbb{G}} \\
&= \int_{\mathcal{D}\cap\Omega} d\,|\partial\mathcal{D}|_{\mathbb{G}} \int_{UH_{x_0}\mathbb{G}} |\langle X, \nu_{\mathcal{D}}\rangle_{H\mathbb{G}}|\, d\,\sigma_{m_1-1}(X) \\
&= 2\kappa_{m_1-1}|\partial\mathcal{D}|_{\mathbb{G}}(\Omega)\,,
\end{aligned}$$

where we have used Fubini's theorem and spherical coordinates to compute the integrating of the last line. □

We now introduce the notion of $\mathbb{G}$-*normality* with respect to a fixed vertical hyperplane.

Definition 3.14. ($\mathbb{G}$-normality). If $x_0 \in \mathbb{G}$ and $X \in H\mathbb{G}$ is a horizontal direction, let $\mathcal{I}_{x_0}(X)$ denote the vertical hyperplane through x_0 and orthogonal to X. We say that $\mathcal{D} \subseteq \mathbb{G}$ is *pointwise* X*-normal* with respect to $\mathcal{I}_{x_0}(X)$ if for all $y \in \mathcal{I}_{x_0}(X)$ we have that $(\gamma^X_y)^{-1}(\gamma^X_y(\mathbb{R}) \cap \mathcal{D})$ is the empty set or a connected subset of $\mathbb{R}$ or, equivalently, if $\gamma^X_y(\mathbb{R}) \cap \mathcal{D}$ is either empty or a connected subset of $\gamma^X_y(\mathbb{R})$. Moreover we say that $\mathcal{D}$ is X-normal with respect to $\mathcal{I}_{x_0}(X)$ if $\mathcal{D}$ is $\mathcal{L}^1$-equivalent to a subset of $\mathbb{G}$ that is pointwise X-normal with respect to $\mathcal{I}_{x_0}(X)$.

Usually, we term this property *pointwise* $\mathbb{G}$*-normality* (resp. $\mathbb{G}$*-normality*) with respect to a vertical hyperplane. As already observed, for any point $x \in \mathbb{G}$ and for any horizontal direction $X \in H\mathbb{G}$ there exists a unique horizontal X-line passing from x. This implies that the notion of $\mathbb{G}$-normality is invariant under group translations, as left translations send a vertical hyperplane orthogonal to $X \in H\mathbb{G}$ into a vertical hyperplane which is still orthogonal to X. Let now $x_0 \in \mathbb{G}$ and consider the family $\mathcal{V}_{x_0}$ of vertical hyperplanes through x_0. The invariance under group translations of the notion of $\mathbb{G}$-normality allows to see that the following two conditions are equivalent:

(i) $\mathcal{D} \subseteq \mathbb{G}$ *is pointwise* $\mathbb{G}$*-normal with respect to any vertical hyperplane* $\mathcal{I}_{x_0}(X) \in \mathcal{V}_{x_0}$;
(ii) $\mathcal{D} \subseteq \mathbb{G}$ *is pointwise* $\mathbb{G}$*-normal with respect to any vertical hyperplane* $\mathcal{I}_z(Z)$ *where* $z \in \mathbb{G}$ *and* $Z \in H\mathbb{G}$.

We would emphasize that the notions introduced above generalize the euclidean ones because, if $\mathbb{G}$ is $(\mathbb{R}^n, +)$, they coincide, as it can be easily proved. Moreover, the analogy with the euclidean case suggests the following

Definition 3.15. ($\mathbb{G}$-convexity). We say that $\mathcal{D} \subseteq \mathbb{G}$ is $\mathbb{G}$-*convex* if for every $x \in \mathbb{G}$, whenever $X \in H\mathbb{G}$, we have that $(\gamma_x^X)^{-1}(\gamma_x^X(\mathbb{R}) \cap \mathcal{D})$ is the empty set or a connected subset of $\mathbb{R}$ or, equivalently, if $\gamma_x^X(\mathbb{R}) \cap \mathcal{D}$ is either empty or a connected subset of $\gamma_x^X(\mathbb{R})$.

Also in this case, if the Carnot group reduces to $(\mathbb{R}^n, +)$, the definitions coincide. Moreover, $\mathbb{G}$-convexity is invariant under group translations and it is stable under intersection, i.e. if $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathbb{G}$ are $\mathbb{G}$-convex sets, then also $\mathcal{D}_1 \cap \mathcal{D}_2$ is a $\mathbb{G}$-convex set.

We refer the reader to [23] and [55] for different, but in fact equivalent, definitions of convexity in Carnot groups. See also [8] for a detailed discussion on this topic.

Remark 3.16. We would point out that $\mathbb{G}$-convexity turns out to be equivalent to condition (ii) above, i.e.

- $\mathcal{D}$ is $\mathbb{G}$-convex if, and only if, $\mathcal{D}$ is pointwise $\mathbb{G}$-normal with respect any vertical hyperplane.

Of course, if $\mathcal{D}$ is just $\mathbb{G}$-normal with respect all of vertical hyperplanes of $\mathbb{G}$, then it is only $\mathcal{L}^1$-equivalent to a $\mathbb{G}$-convex set. To better explain the geometric meaning of $\mathbb{G}$-convexity we make use of horizontal m_1-planes. We remind that, if $z \in \mathbb{G}$, then $H_z\mathbb{G} := l_z(\mathrm{Exp}(V_1))$ denotes the horizontal m_1-plane through z, i.e. the set of all horizontal lines starting from z. Using just the previous definitions it is easy to see that $\mathcal{D}$ is $\mathbb{G}$-convex if and only if

- $\mathrm{Log}\left(l_{-z}(H_z\mathbb{G} \cap \mathcal{D})\right)$ is starshaped in V_1 with respect to $0 \in \mathfrak{g}$ for all $z \in \mathcal{D}$.

In particular, the following implication holds:

- if $\mathrm{Log}\left(l_{-z}(H_z\mathbb{G} \cap \mathcal{D})\right)$ is euclidean convex in V_1 for all $z \in \mathcal{D}$, then $\mathcal{D}$ is $\mathbb{G}$-convex.

Finally, if $z \in \mathrm{Exp}(V_k)$, where V_k is the center of the Lie algebra $\mathfrak{g}$, then the horizontal plane $H_z\mathbb{G}$ through z is an affine m_1-dimensional plane and we get that

- if $\mathcal{D}$ is $\mathbb{G}$-convex, then $H_z\mathbb{G} \cap \mathcal{D}$ is starshaped in $H_z\mathbb{G}$ with respect to z for all $z \in \mathrm{Exp}(V_k)$.

Remark 3.17. ($\mathbb{G}$-convexity in 2-step Carnot groups). If $\mathbb{G}$ is a 2-step Carnot group, then its horizontal lines are also euclidean lines. This is a straightforward consequence of the group law that is completely determined by Campbell-Hausdorff formula, as we have seen in Section 1.1. Thus, from the definition of $\mathbb{G}$-convexity, it follows that euclidean convex sets are $\mathbb{G}$-convex sets. In general the converse it is not true, as proved in the next example.

Example 3.18. (An $\mathbb{H}^1$-convex that is not euclidean convex). Let us consider the Heisenberg group $\mathbb{H}^1 = (\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}, \cdot)$, where $(z, t) \cdot (z', t') = (z + z', t + t' + 2\Im(z\bar{z}'))$. Then, the truncated cone of width $\alpha > 0$ given by

$$\mathcal{C}_\alpha = \left\{(z, t) \in \mathbb{C} \times \mathbb{R} : |z| \leq \alpha\,|t|,\ |z| \leq 1,\ \alpha\,|t| \leq 1\right\}$$

is an $\mathbb{H}^1$-convex set for any $\alpha \geq 2$ but it is not convex. This easily follows observing that the maximal slope of the horizontal lines having initial data in the cylinder $\{(z,t) \in \mathbb{H}^1 : |z| \leq 1\}$ is 2 so that any such line intercepts $\mathcal{C}_\alpha$ in a segment line.

This definition of $\mathbb{G}$-convexity can be used to generalize the Cauchy's formula for the area of euclidean convex sets. For the statement of this classical theorem see [13, 16, 69].

Theorem 3.19. (Cauchy type formula.) *Let $\mathcal{D}$ be a $\mathbb{G}$-convex subset of $\mathbb{G}$ and $x_0 \in \mathbb{G}$. Then*

$$|\partial\mathcal{D}|_{\mathbb{G}}(\mathbb{G}) = \frac{1}{\kappa_{m_1-1}} \int_{UH_{x_0}\mathbb{G}} \mathcal{H}^{n-1}\Big(pr^X_{\mathcal{I}_{x_0}(X)}(\mathcal{D})\Big)\, d\,\mu_{0x_0}(X) \tag{58}$$

where κ_{m_1-1} is the m_1-1-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{m_1-1}$.

Proof. Using Proposition 3.6 and Proposition 3.13 we get the thesis observing that, since $\mathcal{D}$ is $\mathbb{G}$-convex, then $\mathrm{var}^1_X[\mathbf{1}_{\mathcal{D}^X_y}](\gamma^X_y) = 2$ for $\mathcal{H}^{n-1}$–a.e. $y \in pr^X_{\mathcal{I}_{x_0}(X)}(\mathcal{D})$ for any $X \in UH_{x_0}\mathbb{G}$. □

The above theorem, analogously to the euclidean case, allows to see that, in a sense, $\mathbb{G}$-convex sets minimize the $\mathbb{G}$-perimeter. Indeed, as an immediate application, we have the following

Corollary 3.20. *If $\mathcal{D} \subset \mathbb{G}$ is a $\mathbb{G}$-convex set, then for any open set Ω containing $\mathcal{D}$ we have*

$$|\partial\mathcal{D}|_{\mathbb{G}}(\mathbb{G}) \leq |\partial\Omega|_{\mathbb{G}}(\mathbb{G})\,.$$

Proof. Fixing $x_0 \in \mathbb{G}$, the claim follows by the previous Theorem 3.19 by observing that, for every $X \in H\mathbb{G}$, one has $pr^X_{\mathcal{I}_{x_0}(X)}(\mathcal{D}) \subseteq pr^X_{\mathcal{I}_{x_0}(X)}(\Omega)$. □

4. A Santaló type formula and some related topics

From now on will be discussing the integration on the unit horizontal bundle $UH\mathbb{G}$ of a k-step Carnot group $\mathbb{G}$ endoweed with the measure μ_0 (see Definition 3.10). The results here exposed rely on the invariance of μ_0 under the action of the *horizontal flow*, i.e. the flow generated by restriction to $H\mathbb{G}$ of the Riemannian geodesic flow. In fact, the measure μ_0 generalizes to Carnot groups the classical notion of *Kinematic density* (see [11, 15]). More precisely, an integral formula is given in Theorem 4.4, which generalizes the well-known Santaló formula [69]. We emphasize that, in the case of the Heisenberg group $\mathbb{H}^1$, a Santalò-type formula was proved by Pansu, [65]. We then give some applications of Theorem 4.4. In particular, we find two lower bounds for the first eigenvalue of the Dirichlet problem for the Carnot sub-Laplacian $\Delta_{\mathbb{G}}$ on smooth domains (see Proposition 4.8 and Theorem 4.9).

In the tangent bundle $T\mathbb{G}$ we use coordinates given by $(x; X) = ((x_1, \dots, x_n); (a_1, \dots, a_n))$, where $(x_1, \dots, x_n)$ are the exponential coordinates of $x \in \mathbb{G}$ and $(a_1, \dots, a_n)$ are the coordinates of X in the Lie algebra $\mathfrak{g} \cong T_0\mathbb{G}$, i.e. $X(0) = \sum_{i=1}^n a_i \mathrm{e}_i$. We endowed $\mathfrak{g}$ with an inner product denoted by $\langle \cdot, \cdot \rangle$ that is the usual one in $\mathbb{R}^n$. This uniquely determines a left invariant metric on $\mathbb{G}$, also denoted by $\langle \cdot, \cdot \rangle$, that is defined by setting

$$\langle X, Y \rangle := \langle X(0), Y(0) \rangle \quad \forall\; X, Y \in \mathfrak{g}\,.$$

The *energy function* of a vector field $X \in \mathfrak{g}$ associated with $\langle \cdot, \cdot \rangle$ is given by

$$E(X) := \frac{1}{2}\langle X, X \rangle = \frac{1}{2}\sum_{i=1}^n a_i^2\,.$$

Moreover we denote by α the canonical 1-form on the cotangent bundle $T^*\mathbb{G}$ that is given, with our notations, by $\alpha := \sum_{i=1}^n a_i \omega_i$. Following Besse [11], we call *geodesic vector field* on $T\mathbb{G}$ the solution of the equation[(2)]

$$i(T)d\alpha = -dE\,. \tag{59}$$

We call *geodesic flow* the flow generated by T. We remark that T is given by

$$T = \sum_{i=1}^n a_i X_i - \frac{1}{2}\sum_{l,i,j,k=1}^n a_i\, a_l\, c^i_{jk}\left(\delta_{lj}\frac{\partial}{\partial a_k} - \delta_{lk}\frac{\partial}{\partial a_j}\right).$$

To prove this is enough to use equation (59) together with the definitions of α, E and T. The result follows by applying Proposition 1.4. Now we shall prove that the restriction of the canonical 1-form α to the unit horizontal bundle is invariant under the geodesic flow. This means that the Lie derivative of α by T is zero. Indeed, by Cartan's identity (see [54]), we get

$$\mathfrak{L}_T\, \alpha = i(T)\, d\alpha + d\, i(T)\, \alpha = -dE + 2\sum_{i=1}^n a_i\, da_i = dE\,.$$

Now, since we consider unit horizontal vectors, the thesis follows observing that $a_i = 0$ for any $i = m_1 + 1, \dots, n$, and that $\sum_{i=1}^{m_1} a_i^2 = 1$. Therefore, setting $\alpha_0 := \alpha_{|UH\mathbb{G}}$ to denote the restriction of α to the unit horizontal bundle $UH\mathbb{G}$, we have that α_0 is invariant under the restriction of T to the horizontal bundle. From now on we denote this vector by T_0, i.e. $T_0 = T_{|H\mathbb{G}}$, and we call *horizontal flow* the flow on $H\mathbb{G}$ generated by T_0.

Before the next theorem, which asserts a *Liouville type property* of the measure μ_0, we fix some notations. We set $\tau_1 := \omega_1 \wedge \dots \wedge \omega_{m_1}$, $\tau_2 := \omega_{m_1+1} \wedge \dots \wedge \omega_{m_2}$, $\tau_k := \omega_{m_{k-1}+1} \wedge \dots \wedge \omega_{m_k}$ so that $\Theta = \tau_1 \wedge \dots \wedge \tau_k$. Furthermore, $* : \Lambda^k T^*\mathbb{G} \to \Lambda^{n-k} T^*\mathbb{G}$ will denote the Hodge star operator; we explicitly note that $*\tau_1 = \tau_2 \wedge \dots \wedge \tau_k$.

[(2)]We remind that, if $X \in T\mathbb{G}$, then $i(X) : \Lambda^k T^*\mathbb{G} \to \Lambda^{k-1} T^*\mathbb{G}$ denotes the *interior product* with X, i.e. the linear map defined by $i(X)\psi(Y_1, \dots, Y_{k-1}) = \psi(X, Y_1, \dots, Y_{k-1})$; see [50, 53, 54].

Theorem 4.1. *The measure $d\mu_0$ on $UH\mathbb{G}$ turns out to be invariant with respect to the horizontal flow on $H\mathbb{G}$ determined by T_0 and we have that*

$$\Theta \wedge \sigma_{m_1} = \pm \frac{1}{(m_1-1)!} \alpha_0 \wedge (d\alpha_0)^{m_1-1} \wedge (*\tau_1)\,.$$

The proof relies on the following two lemmas.

Lemma 4.2. *With notations as above we have*

$$\begin{aligned}\alpha_0 \wedge (d\alpha_0)^{m_1-1} &= (m_1-1)!\,(-1)^{\frac{(m_1-1)(m_1-2)}{2}}\,\tau_1 \wedge \sigma_{m_1-1}\\ &= (m_1-1)!\sum_{i=1}^{m_1}(-1)^{\frac{(m_1-1)(m_1-2)}{2}}\,(-1)^i a_i\,\omega_1 \wedge \ldots\\ &\qquad \ldots \wedge \omega_{m_1} \wedge da_1 \wedge \ldots \wedge \widehat{da_i} \wedge \ldots \wedge da_{m_1}\,.\end{aligned}$$

Proof. One can prove this lemma by induction on $m_1 (= \dim V_1)$, just using definitions and the expressions of $\alpha_0 = \sum_{i=1}^{m_1} a_i\omega_i$ and $d\alpha_0 = \sum_{i=1}^{m_1} da_i \wedge \omega_i$. □

Lemma 4.3. *If $X \in H\mathbb{G}$ then $\tau_1 \wedge i(X)(d * \tau_1) = 0$.*

Proof. We have that

$$\begin{aligned} d\,(*\tau_1) &= d\,(\tau_2 \wedge \ldots \wedge \tau_k)\\ &= \sum_{i=m_1+1}^{n} (-1)^{i+1}\omega_{m_1+1} \wedge \ldots\\ &\quad \ldots \wedge \omega_{i-1} \wedge d\omega_i \wedge \omega_{i+1} \wedge \ldots \wedge \omega_n\\ &= -\frac{1}{2}\sum_{l=1}^{k}\sum_{i=m_1+1}^{n}\sum_{1\le j,\,h\le h_{l-1}} (-1)^{i+1} c^i_{jh}\,\omega_{m_1+1} \wedge \ldots\\ &\quad \ldots \wedge \omega_{i-1} \wedge (\omega_j \wedge \omega_h) \wedge \omega_{i+1} \wedge \ldots \wedge \omega_n\,.\end{aligned}$$

This formula, which is an easy consequence of Proposition 1.4 and Remark 1.5, enable us to say that $d\,(*\tau_1)$ is a linear combination of $(n-m_1+1)$-forms of the type

$$(\omega_j \wedge \omega_h) \wedge \omega_{m_1+1} \wedge \ldots \wedge \omega_{i-1} \wedge \widehat{\omega_i} \wedge \omega_{i+1} \wedge \ldots \wedge \omega_n$$

for $i = m_1, \ldots, n,\;\; j, h = 1, \ldots, n$ and $i \neq j, h$.

Therefore, by a direct computation it follows that $\tau_1 \wedge i(X)(d * \tau_1)$ is a linear combination of n-forms having the following expression

$$\omega_1 \wedge \ldots \wedge \omega_{s-1} \wedge (\omega_s)^2 \wedge \omega_{s+1} \wedge \ldots \wedge \omega_n$$

for $s = 1, \ldots, n$, and the thesis follows since each of these terms vanishes. □

Proof of Theorem 4.1. We have to show that the Lie derivative along T_0 of $\Theta \wedge \sigma_{m_1-1}$ is equal to 0. From Lemma 4.2 it follows that

$$\Theta \wedge \sigma_{m_1-1} = (-1)^{\frac{(m_1-1)(m_1-2)}{2}} \frac{1}{(m_1-1)!}\, \alpha_0 \wedge (d\alpha_0)^{m_1-1} \wedge (*\tau_1)\,.$$

Thus we need to compute the Lie derivative along T_0 of $\alpha_0 \wedge (d\alpha_0)^{m_1-1} \wedge (*\tau_1)$ and using Cartan's identity and the invariance of α_0 under the horizontal flow induced by T_0 we get

$$\begin{aligned}
&\mathfrak{L}_T\Big(\alpha_0 \wedge (d\alpha_0)^{m_1-1} \wedge (*\tau_1)\Big)\\
&= \mathfrak{L}_T\Big(\alpha_0 \wedge (d\alpha_0)^{m_1-1}\Big) \wedge (*\tau_1) + \Big(\alpha_0 \wedge (d\alpha_0)^{m_1-1}\Big) \wedge \mathfrak{L}_T(*\tau_1)\\
&= \Big(\alpha_0 \wedge (d\alpha_0)^{m_1-1}\Big) \wedge \Big(i(T)(d * \tau_1) + d\,(i(T) * \tau_1)\Big)\\
&= \Big(\alpha_0 \wedge (d\alpha_0)^{m_1-1}\Big) \wedge \Big(i(T)(d * \tau_1)\Big)
\end{aligned}$$

and the thesis now follows from Lemma 4.2 and Lemma 4.3. □

Let $\mathcal{D} \subset \mathbb{G}$ be a smooth, relatively compact domain (open and connected) and let

$$UH\mathcal{D} = \Big\{X \in UH\mathbb{G} : \pi_{|UH\mathbb{G}}(X) \in \mathcal{D}\Big\}\,,$$

i.e. the restriction to $\mathcal{D}$ of the structure of unit horizontal bundle. If $(x; X) \in UH\mathcal{D}$ we set

$$\ell_x(X) := \sup\Big\{s \in \mathbb{R}_+ : \gamma_X(t) \in \mathcal{D},\ \forall\, t \in (0, s)\Big\}\,,$$

where γ_X is the (unique) horizontal line satisfying $\gamma_X(0) = \pi_{|UH\mathbb{G}}(X)$, $\dot{\gamma}_X(0) = X$. Notice that

$$\ell_x(X) = \mathcal{H}^1_c\Big(\gamma_X\Big(\,]0, \ell_x(X)[\,\Big)\Big)\,.$$

By the boundedness of $\mathcal{D}$ we have $\ell_x(X) < \infty$, everywhere in $\mathcal{D}$. Moreover $\gamma_X(\ell_x(X))$ is the first point of the horizontal line γ_X starting from $x = \pi_{|UH\mathbb{G}}(X)$ to hit the boundary of $\mathcal{D}$.

Let now $\nu_{\mathcal{D}}$ be the unit inward $\mathbb{G}$-normal to $\partial\mathcal{D}$ and let us set

$$UH^+\partial\mathcal{D} := \Big\{X \in UH\overline{\mathcal{D}} : \pi_{|UH\mathbb{G}}(X) \in \partial\mathcal{D},\ \langle X, \nu_{\mathcal{D}}\rangle_{H_x\mathbb{G}} > 0\Big\}\,. \qquad (60)$$

This is the set of inward pointing unit horizontal vectors along the boundary $\partial\mathcal{D}$ and, identifying the generic fiber with $\mathbb{S}^{m_1-1}$, we may think it as the hemisphere determined by $\nu_{\mathcal{D}}$ which will be denoted by $\mathbb{U}^{m_1-1}$. We also provide $UH^+\partial\mathcal{D}$ with the following measure

$$d\,\sigma(x; X) := d\,\mu_{0_x}(X)\, d\,|\partial\mathcal{D}|_{\mathbb{G}}(x) \quad \forall\,(x; X) \in UH^+\partial\mathcal{D}\,.$$

Clearly, $d\,\mu_{0x}$ will be concentrated on the hemisphere $\mathbb{U}^{m_1-1} \cong UH_x^+\partial\mathcal{D}$.

Below we shall denote by $\mathrm{Car}(\partial\mathcal{D})$ the so-called *characteristic set* of $\partial\mathcal{D}$ (see for instance [7, 39, 40, 41, 42, 57]), i.e.

$$\mathrm{Car}(\partial\mathcal{D}) := \Big\{x \in \partial\mathcal{D} : \langle \mathrm{n}(x), X(x)\rangle = 0 \ \forall\, X \in H_x\mathbb{G}\Big\}.$$

Moreover we shall set

$$\mathcal{D}^* = \Big\{x \in \mathcal{D} : \exists X \in H\mathbb{G}_x \ s.t.\ \gamma_X(\ell_x(X)) \in \mathrm{Car}(\partial\mathcal{D})\Big\}.$$

Along the lines of [15, 65] and 69] we may prove the following

Theorem 4.4. *Let $\mathcal{D}$ be a smooth relatively compact domain. For all $f \in L^1(UH\mathcal{D})$ we have*

$$\begin{aligned}
&\int_{UH\mathcal{D}} f(y;Y)\,d\,\mu_0(y;Y)\\
&= \int_{UH^+\partial\mathcal{D}} \int_0^{\ell_x(X)} f(\gamma_X(t);X)\,\langle X, \nu_{\mathcal{D}}\rangle_{H_x\mathbb{G}}\,dt\,d\,\sigma(x;X) \qquad (61)\\
&= \int_{\partial\mathcal{D}} \int_{UH_x^+\partial\mathcal{D}} \int_0^{\ell_x(X)} f(\gamma_X(t);X)\,\langle X, \nu_{\mathcal{D}}\rangle_{H_x\mathbb{G}}\,dt\,d\,\mu_{0x}(X)\,d\,|\partial\mathcal{D}|_{\mathbb{G}}(x)\,.
\end{aligned}$$

Proof. Let us first consider the following map

$$\mathbb{R}_+ \times UH^+\partial\mathcal{D} \ni (t,(x;X)) \longmapsto (\gamma_X(t);X) \in UH\mathbb{G}\,,$$

that is nothing but the restriction to $UH^+\partial\mathcal{D}$ of the horizontal flow. Denoting by $\Phi_t(X)$ this flow, we shall see how $\Phi_t(X)$ acts on the measure $d\,\mu_0$. To this end we have to compute the pull back by $\Phi_t(X)$ of the volume form of $UH\mathbb{G}$. Observing that $(\Phi_t(X))^*\sigma_{m_1-1} = \sigma_{m_1-1}$, we get

$$\Big(\Phi_t(X)\Big)^*(\Theta \wedge \sigma_{m_1-1}) = \Big((\gamma_X(t))^*\Theta\Big) \wedge \sigma_{m_1-1}(X)\,.$$

Notice that we have already performed this computation in the proof of Lemma 2.8 by means of a local parametrization and so we have just to reformulate it. We have

$$(\gamma_X(t))^*\Theta = (i(X)\Theta)_x \wedge dt\,,$$

and explicitly this means that

$$(\gamma_X(t))^* d\,\mathcal{L}^n = \langle X, \nu_{\mathcal{D}}\rangle_{H_x\mathbb{G}}\,d\,t\,d\,|\partial\mathcal{D}|_{\mathbb{G}}(x), \quad \text{for } t>0 \text{ and } x \in \pi_{|UH\mathbb{G}}\big(UH^+\partial\mathcal{D}\big)\,.$$

Therefore

$$(\Phi_t(X))^* d\mu_0 = \langle X, \nu_{\mathcal{D}}\rangle_{H_x\mathbb{G}}\,d\,t\,d\,|\partial\mathcal{D}|_{\mathbb{G}}(x)\,d\mu_{0x}(X)\,. \qquad (62)$$

Since $\mathcal{D}$ is a relatively compact domain, we can univocally associate to any $(y; Y) \in UH(\mathcal{D} \setminus \mathcal{D}^*)$ the *time* $t = \ell_y(-Y) < \infty$ and the point $(x; X) = (\gamma_{-Y}(\ell_y(-Y)); -Y)$, so that x is the first point on the boundary of $\mathcal{D}$ reachable from y along the (unique) horizontal Y-line passing through y; furthermore $t < \ell_x(X)$. Thus we have that the map $\Phi_t(X)$ which takes $(t, (x; X))$ onto $(y; Y)$ is a diffeomorphism of the open set $\{(t, (x; X)) : 0 < t < \ell_x(X)\}$ of $\mathbb{R}_+ \times UH^+\partial\mathcal{D}$ onto $UH(\mathcal{D} \setminus \mathcal{D}^*)$.

Finally, if $\mu_0(UH(\mathcal{D}^*)) = 0$ we get the thesis by multiplying both sides of (62) by f and then integrating. But the last assertion follows from the classical Area formula [31], by applying again the same computations of Lemma 2.8. □

Remark 4.5. From Theorem 4.4 we easily deduce the following integral geometric formula for the volume of a smooth relatively compact domain in a Carnot group:

$$\mathcal{L}^n(\mathcal{D}) = \frac{1}{O_{m_1-1}} \int_{\partial\mathcal{D}} \int_{UH^+\partial\mathcal{D}_x} \ell_x(X) \, \langle X, \nu_{\mathcal{D}} \rangle_{H_x\mathbb{G}} d\,\mu_{0x}(X) \, d\,|\partial\mathcal{D}|_{\mathbb{G}}(x) \,, \qquad (63)$$

where O_{m_1-1} denotes the $(m_1 - 1)$-dimensional surface measure of the sphere $\mathbb{S}^{m_1-1}$.

We will give some applications of this theorem. To this end we need some preliminaries.

Let $(x; \widetilde{X}) \in UH\mathbb{G}$ be fixed and denotes by $UH_x^+\mathbb{G}$ the hemisphere determined by $\widetilde{X}$, i.e.

$$UH_x^+\mathbb{G} := \left\{ X \in UH_x\mathbb{G} : \langle \widetilde{X}, X \rangle_{H_x\mathbb{G}} > 0 \right\}.$$

Lemma 4.6.

$$\int_{UH_x^+\mathbb{G}} \langle X, \widetilde{X} \rangle_{H_x\mathbb{G}} \, d\,\mu_{0x}(X) = \frac{O_{m_1-2}}{m_1 - 1} \,. \qquad (64)$$

Proof. It is enough to observe that this integral is the measure of the projection of the $(m_1 - 1)$-dimensional hemisphere $\mathbb{U}^{m_1-1} \cong UH_x^+\mathbb{G}$ onto a diametral plane and so we may perform the computation using spherical coordinates. □

As above, let $\mathcal{D}$ be a smooth, relatively compact, open subset of $\mathbb{G}$ and denotes by $\mathrm{diam}_H(\mathcal{D})$ its *horizontal diameter,* that is the quantity defined by

$$\mathrm{diam}_H(\mathcal{D}) := \sup_{(y;Y) \in UH^+\partial\mathcal{D}} \ell_y(Y) \,.$$

Denoting by $\mathrm{diam}_c(\mathcal{D})$ the diameter of $\mathcal{D}$ with respect to the Carnot-Carathéodory distance d_c, we obviously have

$$\mathrm{diam}_H(\mathcal{D}) \leq \mathrm{diam}_c(\mathcal{D}) \,.$$

Corollary 4.7. *Let $\mathcal{D} \subset \mathbb{G}$ be a smooth and relatively compact domain. Then we have*

$$\frac{\mathcal{L}^n(\mathcal{D})}{|\partial\mathcal{D}|_{\mathbb{G}}(\mathbb{G})} \leq \frac{O_{m_1-2}}{O_{m_1-1} \cdot (m_1 - 1)} \cdot \mathrm{diam}_c(\mathcal{D}) \,,$$

where, in general, O_k denotes the k-dimensional surface measure of the unit sphere $\mathbb{S}^k$ of $\mathbb{R}^{k+1}$.

Proof. From Remark 4.5 we get

$$\begin{aligned}\mathcal{L}^n(\mathcal{D}) &\le \frac{\mathrm{diam}_H(\mathcal{D})}{O_{m_1-1}}\int_{UH^+\partial\mathcal{D}} \langle X, \nu_{\mathcal{D}}\rangle_{H_x\mathbb{G}}d\,\sigma(X)\\ &\le \frac{\mathrm{diam}_c(\mathcal{D})}{O_{m_1-1}}\int_{\partial\mathcal{D}} d\,|\partial\mathcal{D}|_{\mathbb{G}}(x)\int_{UH_x^+\mathbb{G}} \langle X, \nu_{\mathcal{D}}\rangle_{H_x\mathbb{G}}d\,\mu_{0_x}(X)\end{aligned}$$

and, using the foregoing lemma, we get the asserted inequality. □

Finally, we will show some applications of Theorem 4.4 to Analysis in Carnot groups. For these topics, we refer the reader to [12, 22, 74, 75]. More precisely, we will give two explicit lower bounds for the first eigenvalue of the Dirichlet problem for the Carnot sub-Laplacian. To this end we will use Theorem 4.4 by adapting some arguments of Riemannian geometry, for which we refer the reader to [15, 20, 21, 25]. We have to remark that in these inequalities, as well as in the previous Corollary 4.7, we do not characterize the equality cases and, in general, they are non-sharp.

We recall that, with our notations, the Carnot sub-Laplacian of $\mathbb{G}$ is defined as

$$\Delta_{\mathbb{G}} := \sum_{j=1}^{m_1} X_j^2, \qquad \Delta_{\mathbb{G}}\psi(x) = \sum_{j=1}^{m_1} \frac{d^2}{dt^2}\bigg|_{t=0} \psi(x\cdot \mathrm{Exp}(tX_j)) \quad \forall\ \psi \in \mathbf{C}^{\infty}(\mathbb{G})\,.$$

Now let us consider the *Dirichlet eigenvalue problem* for $\Delta_{\mathbb{G}}$ on a smooth bounded domain $\mathcal{D}$, i.e. we find all real number λ for which there exist non trivial solutions $\phi \in W^{1,2}_{\mathbb{G}}(\mathcal{D})$ (the *horizontal* Sobolev space) of the problem

$$\Delta_{\mathbb{G}}\phi + \lambda\phi = 0 \quad (x \in \mathcal{D}) \tag{65}$$

satisfying the boundary condition $\phi_{|\partial\mathcal{D}} = 0$. One can prove that all *eigenvalues* λ of this problem are real and strictly positive and that all *eigenfunctions* ϕ can be choose to be real-valued. Moreover all eigenfunctions corresponding to distinct eigenvalues are orthogonal in $L^2(\mathcal{D})$ with respect to the usual inner product on $L^2(\mathcal{D})$. The main result that we use in what follows is the variational characterization of the *first eigenvalue* of (65) that we denote by $\lambda_1(\mathcal{D})$, i.e.

$$\lambda_1(\mathcal{D}) = \inf_{\varphi\in\mathbf{C}_0^{\infty}(\mathcal{D})} \frac{\displaystyle\int_{\mathcal{D}} |\nabla_{\mathbb{G}}\varphi|^2_{H\mathbb{G}}\, d\mathcal{L}^n}{\displaystyle\int_{\mathcal{D}} |\varphi|^2\, d\mathcal{L}^n}\,. \tag{66}$$

We point out that to prove (66) one uses the following Green's identity

$$\int_{\mathcal{D}} \{\varphi\,\Delta_{\mathbb{G}}\psi + \langle\nabla_{\mathbb{G}}\varphi, \nabla_{\mathbb{G}}\psi\rangle_{H\mathbb{G}}\}\, d\mathcal{L}^n = 0$$

whenever $\varphi,\ \psi : \mathcal{D} \longrightarrow \mathbb{R}$ are smooth and with at least one of them compactly supported in $\mathcal{D}$. The above identity can easily be proved by means of Proposition 1.10.

Proposition 4.8. *Let $\mathcal{D} \subset \mathbb{G}$ be a smooth, relatively compact domain and let $\lambda_1(\mathcal{D})$ be the first eigenvalue of* (65). *Then we have*

$$\lambda_1(\mathcal{D}) \geq \frac{\pi^2 \cdot m_1}{\mathrm{diam}_H(\mathcal{D})^2} \geq \frac{\pi^2 \cdot m_1}{\mathrm{diam}_c(\mathcal{D})^2}.$$

Proof. We have to prove only the first inequality since the second one is trivial. To this end, we first notice that for any $\varphi \in \mathbf{C}_0^\infty(\mathcal{D})$ we have

$$|\nabla_{\mathbb{G}}\varphi|^2_{H_x\mathbb{G}} = \frac{m_1}{O_{m_1-1}} \int_{UH_x\mathbb{G}} (X\varphi)^2 d\,\mu_{0x}(X).$$

Moreover the *fixed-endpoint* version of the one dimensional Wirtinger's inequality says that

$$\int_0^l \dot{h}(t)^2 dt \geq \frac{\pi^2}{l^2} \int_0^l h(t)^2 dt \quad \forall\, h \in \mathbf{C}^1([0,l]),\ \ h(0) = h(l) = 0.$$

Using these remarks and Theorem 4.4 we have that

$$\begin{aligned}
&\int_{\mathcal{D}} |\nabla_{\mathbb{G}}\varphi|^2_{H_x\mathbb{G}}\, d\mathcal{L}^n(x) = \frac{m_1}{O_{m_1-1}} \int_{UH\mathcal{D}} (X\varphi)^2 d\,\mu_0(x; X)\\
=&\frac{m_1}{O_{m_1-1}} \int_{UH^+\partial\mathcal{D}} \int_0^{\ell_y(X)} \left[\frac{d}{dt}\varphi(\gamma_X(t))\right]^2 \langle X, \nu_{\mathcal{D}}\rangle_{H_y\mathbb{G}}\, dt\, d\,\sigma(y; X)\\
\geq&\frac{m_1}{O_{m_1-1}} \int_{UH^+\partial\mathcal{D}} \frac{\pi^2}{\ell_y^2(X)} \int_0^{\ell_y(X)} \left[\varphi(\gamma_X(t))\right]^2 \langle X, \nu_{\mathcal{D}}\rangle_{H_y\mathbb{G}}\, dt\, d\,\sigma(y; X)\\
\geq&\frac{\pi^2 \cdot m_1}{O_{m_1-1} \cdot \mathrm{diam}_H^2(\mathcal{D})} \int_{UH^+\partial\mathcal{D}} \int_0^{\ell_y(X)} \left[\varphi(\gamma_X(t))\right]^2 \langle X, \nu_{\mathcal{D}}\rangle_{H_y\mathbb{G}}\, dt\, d\,\sigma(y; X)\\
=&\frac{\pi^2 \cdot m_1}{O_{m_1-1} \cdot \mathrm{diam}_H^2(\mathcal{D})} \int_{UH\mathcal{D}} \left[\varphi(x)\right]^2 d\,\mu_0(x; X)\\
=&\frac{\pi^2 \cdot m_1}{\mathrm{diam}_H(\mathcal{D})^2} \int_{\mathcal{D}} |\varphi(x)|^2\, d\mathcal{L}^n(x).
\end{aligned}$$

□

Similarly, along the lines of [21], we can prove the following inequality; see also [15] and [25].

Theorem 4.9. *Let $\mathcal{D} \subset \mathbb{G}$ and $\lambda_1(\mathcal{D})$ be defined as above. Then we have*

$$\lambda_1(\mathcal{D}) \geq \frac{m_1 \cdot \pi^2}{O_{m_1-1}} \cdot \inf_{x \in \mathcal{D}} \int_{UH_x\mathbb{G}} \frac{1}{\ell_x^2(X)}\, d\mu_{0x}(X).$$

Proof. Analogously to the previous proof, we have

$$
\begin{aligned}
&\int_{\mathcal{D}} |\nabla_{\mathbb{G}}\varphi|^2_{H_x\mathbb{G}}\, d\mathcal{L}^n(x)\\
\geq &\frac{m_1}{O_{m_1-1}} \int_{UH^+\partial\mathcal{D}} \frac{\pi^2}{\ell_y^2(X)} \int_0^{\ell_y(X)} \left[\varphi(\gamma_X(t))\right]^2 \langle X, \nu_{\mathcal{D}}\rangle_{H_y\mathbb{G}}\, dt\, d\sigma(y; X)\\
= &\frac{m_1\cdot\pi^2}{O_{m_1-1}} \int_{UH^+\partial\mathcal{D}} \int_0^{\ell_y(X)} \frac{\varphi^2(\gamma_X(t))}{\ell_y^2(X)} \langle X, \nu_{\mathcal{D}}\rangle_{H_y\mathbb{G}}\, dt\, d\sigma(y; X)\\
= &\frac{m_1\cdot\pi^2}{O_{m_1-1}} \int_{UH\mathcal{D}} \frac{\varphi^2(x)}{\ell_x^2(X)}\, d\mu(x; X)\\
= &\frac{m_1\cdot\pi^2}{O_{m_1-1}} \int_{\mathcal{D}} \varphi^2(x) \left[\int_{UH_x\mathbb{G}} \frac{1}{\ell_x^2(X)}\, d\mu_{0x}(X)\right] d\mathcal{L}^n(x)\\
\geq &\frac{m_1\cdot\pi^2}{O_{m_1-1}} \cdot \left[\inf_{x\in\mathcal{D}} \int_{UH_x\mathbb{G}} \frac{1}{\ell_x^2(X)}\, d\mu_{0x}(X)\right] \cdot \int_{\mathcal{D}} |\varphi(x)|^2\, d\mathcal{L}^n(x)
\end{aligned}
$$

and the thesis follows. □

References

[1] L. AMBROSIO, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*, Adv. Math. **159** (2001), 51–67.

[2] L. AMBROSIO, N. FUSCO and D. PALLARA, "Functions of Bounded Variation and Free Discontinuity Problems", Oxford University Press, 2000.

[3] L. AMBROSIO and B. KIRCHHEIM, *Rectifiable sets in metric and Banach spaces*, Math. Ann. **318** (2000), 527–555.

[4] L. AMBROSIO and B. KIRCHHEIM, *Currents in metric spaces*, Acta Math. **185** (2000), 1–80.

[5] L. AMBROSIO and V. MAGNANI, *Some fine properties of BV functions on sub-Riemannian groups*, Math. Z. **245** (2003).

[6] L. AMBROSIO and P. TILLI, "Selected topics on Analysis in Metric Spaces", Quaderni della Scuola Normale Superiore, Pisa, 2000.

[7] Z. M. BALOGH, *Size of characteristic sets and functions with prescribed gradients*, J. Reine Angew. Math. **564** (2003), 63–83.

[8] Z. M. BALOGH and M. RICKLY, *Regularity of convex functions on Heisenberg groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **2** (2003), 847–868.

[9] Z. M. BALOGH, J. J MANFREDI and J. TYSON, *Fundamental solution for the Q-Laplacian and sharp Moser-Trudinger inequality in Carnot groups*, J. Funct. Anal. **204** (2003), 35–49.

[10] A. BELLAÏCHE, *The tangent space in subriemannian geometry*, In: "Subriemannian Geometry", A. Bellaiche and J. Risler (eds.), Progress in Mathematics 144, Birkhauser Verlag, Basel, 1996.

[11] A. L. BESSE, "Manifolds all of whose Geodesics are Closed", Springer Verlag, Berlin, 1978.

[12] I. BIRINDELLI and J. PRAJAPAT, *Monotonicity and simmetry results for degenerate elliptic equations on nilpotent Lie groups*, Pacific J. Math. **204** (2002), 1–17.

[13] YU. D. BURAGO and V. A. ZALGALLER, "Geometric Inequalities", Springer Verlag, Berlin, 1980.

[14] L. CAPOGNA, D. DANIELLI and N. GAROFALO, *The geometric Sobolev embedding for vector fields and the isoperimetric inequality*, Comm. Anal. Geom. **2** (1994), 203–215.

[15] I. CHAVEL, "Riemannian Geometry: a modern introduction", Cambridge University Press, 1994.

[16] I. CHAVEL, "Isoperimetric Inequalities", Cambridge University Press, 2001.

[17] J. CHEEGER, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), 428–517.

[18] G. CITTI, M. MANFREDINI and A. SARTI, *Minimum of the Mumford-Shah functional in a conctact manifold on the Heisenberg space*, Preprint 2003.

[19] L. J. CORVIN and F. P. GREENLEAF, "Representations of nilpotent Lie groups and their applications", Cambridge University Press, 1984.

[20] C. B. CROKE, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. (4) **13** (1980), 419–435.

[21] C. B. CROKE and A. DERDZIŃSKI, *A lower bound for λ_1 on manifolds with boundary*, Comment. Math. Helv. **62** (1987), 106–121.

[22] T. COULHON and L. SALOFF-COSTE, *Isopérimétrie pour les groupes et les variétésf*, Rev. Math. Iberoamericana **9** (1993), 293–314.

[23] D. DANIELLI, N. GAROFALO and D. M. NHIEU, *Notions of convexity in Carnot groups*, Comm. Anal. Geom. **11** (2003), 263–341.

[24] G. DAVID and S. SEMMES, "Fractured Fractals and Broken Dreams. Self-Similar Geometry through Metric and Measure", Oxford University Press, 1997.

[25] E. B. DAVIES, "Heat Kernels and Spectral Theory", Cambridge University Press, 1989.

[26] E. DE GIORGI, *Sulla proprietà isoperimetrica dell'ipersfera*, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. Sez. I (8) **5** (1958), 33–44.

[27] E. DE GIORGI, *Un progetto di teoria delle correnti, forme differenziali e varietà non orientate in spazi metrici*, In: "Variational Methods, Non Linear Analysys and Differential Equations in Honour of J. P. Cecconi", M. Chicco et al. (eds.), ECIG, Genova, 1993, 67–71.

[28] E. DE GIORGI, *Un progetto di teoria unitaria delle correnti, forme differenziali, varietà ambientate in spazi metrici, funzioni a variazione limitata*, Manuscript (1995).

[29] E. DE GIORGI, *Problema di Plateau generale e funzionali geodetici*, Atti Sem. Mat. Fis. Univ. Modena **43** (1995), 285–292.

[30] L. C. EVANS and R. F. GARIEPY, "Measure Theory and Fine Properties of functions", CRC Press, Boca Raton, 1992.

[31] H. FEDERER, "Geometric Measure Theory", Springer Verlag, 1969.

[32] G. B. FOLLAND, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13** (1975), 161–207.

[33] G. B. FOLLAND and E. M. STEIN, "Hardy spaces on homogeneous groups", Princeton University Press, 1982.

[34] B. FRANCHI, S. GALLOT and R. L. WHEEDEN, *Sobolev and isoperimetric inequalities for degenerate metrics*, Math. Ann. **300** (1994), 557–571.

[35] B. FRANCHI and E. LANCONELLI, *Hölder regularity theorem for a class of non uniformly elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10** (1983), 523–541.

[36] B. FRANCHI, G. LU and R. L. WHEEDEN, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields*, Ann. Inst. Fourier, Grenoble **45** (1995), 577–604.

[37] B. FRANCHI, R. SERAPIONI and F. S. CASSANO, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*, Houston J. Math. **22** (1996), 859–890.

[38] B. FRANCHI, R. SERAPIONI and F. S. CASSANO, *Approximation and imbedding theorems for wheighted Sobolev spaces associated with Lipschitz continous vector fields*, Boll. Unione Mat. Ital. 11-B **7** (1997).

[39] B. FRANCHI, R. SERAPIONI and F. S. CASSANO, *Rectifiability and Perimeter in the Heisenberg Group*, Math. Ann. **321** (2001), 479–531.

[40] B. FRANCHI, R. SERAPIONI and F. S. CASSANO, *Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups*, Comm. Anal. Geom. **11** (2003).

[41] B. FRANCHI, R. SERAPIONI and F. S. CASSANO, *On the structure of finite perimeter sets in step 2 Carnot groups*, J. Geom. Anal. **13** (2003).

[42] N. GAROFALO and D. M. NHIEU, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), 1081–1144.

[43] M. GOBBINO, *Finite difference approximation of the Mumford Shah functional*, Comm. Pure. Appl. Math. **51** (1998).

[44] N. GOODMAN, “Nilpotent Lie groups”, Springer Lecture notes in Mathematics, Vol. 562, 1976.

[45] M. GROMOV, *Carnot-Carathéodory spaces seen from within*, In: “Subriemannian Geometry”, Progress in Mathematics, **144**, A. Bellaiche and J. Rislered (eds.), Birkhauser Verlag, Basel, 1996.

[46] M. GROMOV, “Metric Structures for Riemannian and Non Riemannian Spaces”, Progress in Mathematics **153**, Birkhauser Verlag, Boston, 1999.

[47] C. E. GUTIERREZ and A. MONTANARI, *On the second order derivatives of convex functions on the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), 349–366.

[48] P. HAJŁASZ and P. KOSKELA, “Sobolev Met Poincaré”, Mem. Amer. Math. Soc. 688, Providence, RI, 2000.

[49] J. HEINONEN, *Calculus on Carnot groups*, In: “Fall School in Analysis” (Jyväskyla, 1994), Report, 68, Univ. Jyväskyla, Jyväskyla, 1995, 1–31.

[50] S. HELGASON, “Differential Geometry, Lie Groups, and Symmetric Spaces”, Academic Press, New York, 1978.

[51] D. JERISON, *The Poincaré inequality for vector fields satisfying Hörmander condition*, Duke Math. J. **53** (1986), 503–523.

[52] A. KORÁNYI and H. M. REIMANN, *Foundation for the theory of quasiconformal mapping on the Heisenberg group*, Adv. Math. **111** (1995), 1–85.

[53] S. LANG, “Differential and Riemannian Manifolds”, Springer Verlag, 1994.

[54] J. M. LEE, “Introduction to Smooth Manifolds”, Springer Verlag, 2003.

[55] G. LU, J. J. MANFREDI and B. STROFFOLINI, *Convex functions on the Heisenberg group*, Calc. Var. Partial Differential Equations **19** (2004), 1–22.

[56] V. MAGNANI, *Differentiability and Area Formula on Statified Lie Groups*, Houston J. Math. **27** (2001), 297–323.

[57] V. MAGNANI,*Characteristic point, rectifiability and perimeter measure on stratified groups*, Preprint 2003.

[58] G. A. MARGULIS and G. D. MOSTOW, *The differential of a quasi-conformal mapping of a Carnot-Carathéodory spaces*, Geom. Functional Anal. **5** (1995), 402–433.

[59] P. MATTILA, "Geometry of Sets and Measures in Euclidean Spaces", Cambridge University Press, 1995.

[60] J. MITCHELL, *On Carnot-Carathèodory metrics*, J. Differential Geom. **21** (1985), 35–45.

[61] R. MONTGOMERY, "A Tour of Subriemannian Geometries, Their Geodesics and Applications", American Mathematical Society, Math. Surveys and Monographs, vol. 91, 2002.

[62] R. MONTI, "Distances, Boundaries and surface measures in Carnot Carathéodory Spaces", UTMPhDTS, **31**, Ph. D. Thesis Series, Dip. Mat. Univ. Trento, Nov. 2001.

[63] R. MONTI and F. SERRA CASSANO, *Surface measures in Carnot-Carathéodory spaces*, Calc. Var. Partial Differential Equations **13** (2001), 339–376.

[64] A. NAGEL, E. M. STEIN and S. WAINGER, *Balls and metrics defined by vector fields I: Basic properties*, Acta Math. **155** (1985), 103–147.

[65] P. PANSU, "Geometrie du Group d'Heisenberg", These pour le titre de Docteur, 3ème cycle, Universite Paris VII, 1982.

[66] P. PANSU, *Métriques de Carnot Carathéodory et quasi-isométries des espaces symmétriques de rang un*, Ann. of Math. 2 **129** (1989), 1–60.

[67] D. PREISS and J. TISÊR, *On Besicovitch* $1/2$*-problem*, J. London Math. Soc. **45** (1992), 279–287.

[68] L. P. ROTHSCHILD and E. M. STEIN, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.

[69] L. A. SANTALÓ, "Integral Geometry and Geometric Probability", Addison-Wesley, Reading, Mass., 1976.

[70] E. M. STEIN, "Harmonic Analysis", Princeton University Press, 1993.

[71] R. S. STRICHARTZ, *Sub-Riemannian geometry*, J. Differential Geom. **24** (1986), 221–263. Corrections: J. Differential Geom. **30** (1989), 595–596.

[72] G. TALENTI, *The standard isoperimetric theorem*, In: "Handbook of Convexity", Vol. A, P. M. Gruber and J. M. Wills (eds.), 73–123, Amsterdam, North Holland, 1993.

[73] V. S. VARADARAJAN, "Lie Groups, Lie Algebras, and their Representations", Springer, 1984.

[74] N. TH. VAROPOULOS, *Analysis on Lie groups*, J. Funct. Anal. **76** (1988), 346–410.

[75] N. TH. VAROPOULOS, L. SALOFF-COSTE and T. COULHON, "Analysis and Geometry on Groups", Cambridge University Press, 1992.

[76] S. K. VODOP'YANOV, $\mathcal{P}$*-differentiability on Carnot Groups in different topologies and related topics*, Proc. on Analysis and Geometry, pp. 603–670, Sobolev Inst. Press, Novosibirsk, 2000.

[77] W. P. ZIEMER, "Weakly Differentiable Functions", Springer Verlag, 1989.

Dipartimento di Matematica
Università degli Studi di Bologna
Piazza di P. ta S. Donato, 5
40126 Bologna, Italia
montefal@dm.unibo.it