Quaternionic maps and minimal surfaces

JINGYI CHEN AND JIAYU LI

Abstract. Let \((M, J^\alpha, \alpha = 1, 2, 3)\) and \((N, J^\alpha, \alpha = 1, 2, 3)\) be hyperkähler manifolds. We study stationary quaternionic maps between \(M\) and \(N\). We first show that if there are no holomorphic 2-spheres in the target then any sequence of stationary quaternionic maps with bounded energy subconverges to a stationary quaternionic map strongly in \(W^{1,2}(M, N)\). We then find that certain tangent maps of quaternionic maps give rise to an interesting minimal 2-sphere. At last we construct a stationary quaternionic map with a codimension-3 singular set by using the embedded minimal \(S^2\) in the hyperkähler surface \(\tilde{M}_2^0\) studied by Atiyah-Hitchin.

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1. Introduction

A Riemannian manifold is called hyperkähler if it possesses covariant constant complex structures \(I, J, K\) which satisfy the quaternionic relation

\[
I^2 = J^2 = K^2 = IJK = -\text{identity}.
\]

Associated to \(I, J, K\) there is a natural family of covariant constant complex structures \(aI + bJ + cK\) where \((a, b, c)\) is a unit vector in \(\mathbb{R}^3\). A hyperkähler manifold is Ricci-flat with dimension \(4k\). Let \(M\) and \(N\) be two hyperkähler manifolds with complex structures \(J^\alpha\) and \(J^\beta\) respectively for \(\alpha, \beta = 1, 2, 3\) which satisfy the quaternionic identities. A smooth map \(f : M \rightarrow N\) is called a quaternionic map if

\[
\sum_{\alpha, \beta = 1}^{3} A_{\alpha\beta} J^\beta \circ df \circ J^\alpha = df
\]

(1.1)

where \(A_{\alpha\beta}\) denote the entries of a constant matrix \(A\) in \(SO(3)\). Since \(SO(3)\) preserves the quaternionic identities, we can always choose complex structures \(J^\alpha\) for \(M\) and \(J^\beta\) for \(N\) such that \(A_{\alpha\beta} = \delta_{\alpha\beta}\) in (1.1).

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Quaternionic maps arise from the higher dimensional gauge theory (cf. [C], [DT], [FKS], [MS], [NN], [PG]). More precisely they naturally arise from the adiabatic limit of Hermitian Yang-Mills connections on $SU(n)$-bundles on a product of two K3 surfaces. Its linear version in dimension four is the so-called Cauchy-Riemann-Fueter equation (or quaternionic d-bar equations):

$$\partial x_1 f - i \partial x_2 f - j \partial x_3 f - k \partial x_4 f = 0$$

for $f : \mathbb{H} \to \mathbb{H}$ where $\mathbb{H}$ is the space of quaternions and $x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$.

Assume $M$ is compact. For any smooth map $u : M \to N$, consider the energy functional

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2$$

and the functional

$$E_T(u) = A_{\alpha\beta} \int_M \langle \omega_{J^\beta}, u^* \omega_{J^\beta} \rangle$$

and set

$$I(u) = \frac{1}{2} \int_M |du - A_{\alpha\beta} J^\beta \circ du \circ J^\alpha|^2.$$ 

It is clear that $I(u) = 0$ if and only if $u$ is a quaternionic map. Since $u$ pulls back the closed 2-form $\omega_{J^\beta}$ to a closed 2-form on $M$ and $\omega_{J^\alpha}$ is closed, $E_T(u)$ is homotopy invariant and depends on $(A_{\alpha\beta})$. The following relation holds (cf. [C], [CL1], [FKS])

$$E(u) + E_T(u) = \frac{1}{4} I(u). \quad (1.2)$$

If $u$ is a quaternionic map, then it minimizes energy in its homotopy class so it is harmonic.

Recall [Sc] that a map in the Sobolev space $W^{1,2}(M, N)$ is a stationary harmonic map if it is a critical point of the energy functional with respect to both of the variations on $M$ and $N$ with compact supports. A stationary harmonic map is smooth away from a closed set of zero $(m - 2)$-dimensional Hausdorff measure where $m = \dim M$. Let $M$ and $N$ be two hyperkähler manifolds. A map $u$ from $M$ to $N$ is called a stationary quaternionic map if it is a stationary harmonic map and it is a quaternionic map outside its singular set.

It is known that the existence harmonic 2-spheres plays an important role in the study of stationary harmonic maps ([SU], [Lin]).

In this note we investigate the special minimal 2-spheres which arise from the stationary quaternionic maps. We first show that if there are no holomorphic 2-spheres in $N$ then any sequence of stationary quaternionic maps with bounded energy subconverges to a stationary quaternionic map strongly in $W^{1,2}(M, N)$. This result was stated and proved in [CL1] when $M$ is of dimension four, and the proof
we shall present here is essentially based on that in [CL1]. We then find that certain tangent maps of quaternionic maps give rise to an interesting minimal 2-sphere equation:

\[ df J_{S^2} = - \sum_{k=1}^{3} x_k J^k df \]

where \( f : S^2 \rightarrow N \), \((x_1, x_2, x_3) \in S^2\) and \( J_{S^2} \) is the standard complex structure on \( S^2 \). We construct a stationary quaternionic map with a codimension-3 singular set by using the embedded minimal \( S^2 \) in the hyperkähler surface \( \tilde{M}_2^0 \) studied by Atiyah-Hitchin [AH], where \( \tilde{M}_2^0 \) is the double cover of the space \( M_2^0 \) of centred 2-monopoles on \( \mathbb{R}^3 \) and it is a complete and simply connected hyperkähler surface.

There are interesting results on decomposition of differential forms in quaternionic geometry using representations of special groups (e.g. [Bo], [K], [Sa], [Sw], [W], etc). It is commented in [W] that the quaternionic maps between hyperkähler manifolds can be described by the splitting of \( Sp(1) \)-representations. The authors thank the referee for his bringing this point and the related references in quaternionic geometry to their attention.

2. Compactness of stationary quaternionic maps

A sequence of stationary harmonic maps with bounded energies subconverges to a stationary harmonic map strongly in \( W^{1,2} \) topology if there are no harmonic 2-spheres in the target manifold [L]. For stationary quaternionic maps, the absence of holomorphic 2-spheres is sufficient to conclude the strong convergence.

**Theorem 2.1.** Let \( M \) and \( N \) be compact hyperkähler manifolds with \( \dim M = m \). Suppose that \( u_k \) is a sequence of stationary quaternionic maps with bounded energies. If \( N \) does not admit holomorphic \( S^2 \)'s with respect to the complex structure \( a_i J^i \) on \( \mathbb{R}^2 \) restricted to \( S^2 \) and the complex structure \( a_i J^i \) on \( N \) for some constants \( a_i \) \((i = 1, 2, 3)\) with \( \sum a_i^2 = 1 \), then there is a subsequence of \( \{u_k\} \) which converges strongly to a stationary quaternionic map \( u \).

**Proof.** We can always assume that \( u_k \rightharpoonup u \) weakly in \( W^{1,2}(M, N) \) and that \( |\nabla u_k|^2 dx \rightharpoonup |\nabla u|^2 dx + \nu \) in the sense of measure as \( k \to \infty \). Here \( \nu \) is a nonnegative Radon measure on \( M \) with support in \( \Sigma \), and \( \Sigma \) is the blow-up set of the sequence \( u_k \) which is \( m - 2 \) rectifiable [L]. We will prove the Hausdorff measure \( H^{m-2}(\Sigma) = 0 \) which implies the strong convergence in \( W^{1,2}(M, N) \). Assume \( H^{m-2}(\Sigma) \neq 0 \). Then [L] there is a nonconstant harmonic map \( v : \mathbb{R}^m \to N \) with finite energy and \( \nabla \Sigma v = 0 \). Here we have identified the tangent space of \( \Sigma \) at \( 0 \in \mathbb{R}^m = \mathbb{R}^{m-2} \times \mathbb{R}^2 \) with \( \mathbb{R}^{m-2} \times \{0\} \) so \( \nabla \Sigma \) means the differentiation along \( \mathbb{R}^{m-2} \times \{0\} \). The rescaling process for constructing \( v \) is taken place around smooth points of \( u_k \) which approach 0, therefore \( v \) is also a smooth quaternionic map (cf. [CT]).

At the point \( 0 \in \mathbb{R}^m \), suppose that \( e \) is in the normal direction of \( \Sigma \). Let \( K \) be the linear space spanned by \( J^\alpha e \) for \( \alpha = 1, 2, 3 \), so \( K \perp e \). Since rank \( dv = 2 \), we have...
\(dv(e) \neq 0\). This implies, from the quaternionic map equation, \(\sum_{i=1}^{3} J^i dv(J^i e) \neq 0\) and in turn \(dv(J^i e) \neq 0\) for some \(i\). Hence \(\dim dv(K) = 1\). It then follows that there are real constants \(a_1, a_2, a_3\) with \(a_1^2 + a_2^2 + a_3^2 = 1\) such that \(a_i J^i e \in [0] \times \mathbb{R}^2\) and \(dv(a_i J^i e) \neq 0\). Notice that we then have three vectors \(a_i J^i e = a_j J^j e, i \neq j\) which are perpendicular to \(e\) and to \(\sum_{i=1}^{3} a_i J^i e\), so they belong to \(T \Sigma\). We therefore have \((a_2 J^1 - a_1 J^2)e \in \text{Ker} (dv), (a_3 J^1 - a_1 J^3)e \in \text{Ker} (dv), (a_2 J^3 - a_3 J^2)e \in \text{Ker} (dv)\), \(J^a dv J^a = dv\), thus \(dv(\sum_i a_i J^i e)\) can only have components on \(J^a (dv(e))\). By a simple calculation, one easily checks that

\[
dv \left( \sum_{i=1}^{3} a_i J^i e \right) = \sum_{i,j=1}^{3} J^j dv(a_i J^i J^j e)
\]

\[
= - \sum_{i=1}^{3} a_i J^i dv(e) + J^1 dv(a_2 J^1 J^2 + a_3 J^1 J^3) e
\]

\[
+ J^2 dv(a_1 J^2 J^1 + a_3 J^2 J^3) e + J^3 dv(a_1 J^3 J^1 + a_2 J^3 J^2) e
\]

\[
= - \sum_{i=1}^{3} a_i J^i dv(e).
\]

At any other point \((0, x)\) in \(\mathbb{R}^{m-2} \times \mathbb{R}^2\), the vectors \(e\) and \(\sum_{i=1}^{3} a_i J^i e\) still belong to \([0] \times \mathbb{R}^2\), and the vectors \((a_1 J^2 - a_2 J^1) e, (a_2 J^2 - a_3 J^2) e, (a_1 J^3 - a_3 J^1) e\) lie in \(\mathbb{R}^{m-2} \times \{x\}\) hence in the kernel of \(dv\) at \((0, x)\), so we can repeat the argument above to conclude \(v\) is holomorphic at \((x, 0)\) with respect to the same complex structures \(\sum a_i J^i\) and \(\sum a_i J^i\). It follows that \(v\) induces a holomorphic map from \(S^2\) to \(N\). But no such holomorphic map can exist by assumption. So we must have \(H^{m-2}(\Sigma) = 0\) and in turn \(u_k\) converge strongly to \(u\) in \(W^{1,2}\) norm.

\[\square\]

**Remark 2.2.** The strong convergence is equivalent to \(H^{m-2}(\Sigma) = 0\) and is equivalent to that the Hausdorff dimension of the singular set \(\text{sing}(u)\) of \(u\) is no bigger than \(m - 3\). Moreover \(\text{sing}(u)\) is rectifiable since \(N\) real analytic [Si].

### 3. Quaternionic minimal surfaces via quaternionic maps

In this section we study a special class of minimal surfaces which arise from certain tangent maps of the quaternionic maps.

Assume that \(M\) is a 4-dimensional hyperkähler manifold and \(N\) is a 4\(n\)-dimensional hyperkähler manifold. We can choose a coordinate system around a point \(x\) in \(M\) so that the matrix expressions of the complex structures on \(M\) take the following form:

\[
J^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\]
Note that the three Kähler forms $\omega_{Ji}$, $i = 1, 2, 3$ have variable coefficients in these coordinates. For $f : M \to N$, if we denote $\frac{\partial f}{\partial x_k}$ by $f_k$ for $k = 1, 2, 3, 4$ in the coordinate system we have just chosen, the quaternionic map equation (1.1) reads

$$f_1 - a_\alpha a_3 J^\alpha f_2 + a_\alpha a_2 J^\alpha f_3 + a_\alpha a_1 J^\alpha f_4 = 0 \quad (3.1)$$

where we take summation over $\alpha$.

Now assume that $f$ is a homogeneous degree-0 quaternionic map from $\mathbb{R}^4$ to $N$ and satisfies $f(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3, 0)$. So $f$ is singular along the $x_4$-axis or it is constant. Note that such an $f$ is just a tangent map, with a line of singularities, of a quaternionic map from $M$ to $N$.

As a radially independent harmonic map, $f$ induces a smooth harmonic map from $S^2$ to $N$: $\phi(x) = f(x, x_4)$ for $x \in S^2 \subset \mathbb{R}^3$.

**Lemma 3.1.** With $f$ and $\phi$ as above, then

$$d\phi J_{S^2} = -a_{\alpha\beta} x_\beta J^\alpha d\phi. \quad (3.2)$$

**Proof.** Because $f$ is a homogeneous degree-0 map,

$$\sum_{k=1}^4 x_k f_k = 0$$

and this combined with (3.1) leads to

$$(x_2 + x_1 a_\alpha a_3 J^\alpha) f_2 + (x_3 - x_1 a_\alpha a_2 J^\alpha) f_3 = 0.$$ 

In the spherical coordinates

$$\begin{cases} x_1 = r \sin \alpha \cos \theta \\ x_2 = r \sin \alpha \sin \theta \\ x_3 = r \cos \alpha, \end{cases}$$

it reads

$$(x_2 + x_1 a_\alpha a_3 J^\alpha) \left( \cos \alpha \sin \theta f_\alpha + \cos \theta \frac{f_\theta}{\sin \alpha} \right) + (x_3 - x_1 a_\alpha a_2 J^\alpha) (-\sin \alpha f_\alpha) = 0.$$ 

Multiplying this equation by $\sin(\alpha)$ yields

$$(x_2 + x_1 a_\alpha a_3 J^\alpha) \left( x_3 x_2 f_\alpha + x_1 \frac{f_\theta}{\sin \alpha} \right) - (x_3 - x_1 a_\alpha a_2 J^\alpha)(x_1^2 + x_2^2) f_\alpha = 0$$

i.e.

$$-x_1 (x_2 + x_1 a_\alpha a_3 J^\alpha) \frac{f_\theta}{\sin \alpha} = \left( x_2 x_3 (x_2 + x_1 a_\alpha a_3 J^\alpha) - (x_3 - x_1 a_\alpha a_2 J^\alpha)(x_1^2 + x_2^2) \right) f_\alpha.$$
Multiplying \( x_2 - x_1 a_3 J^\alpha \) from left on both sides of the equation above, we obtain,

\[
-x_1 (x_1^2 + x_2^2) \frac{f_\theta}{\sin \alpha} = x_1 (x_1^2 + x_2^2) \left( x_1 a_\alpha J^\alpha + x_2 a_\alpha J^\alpha + x_3 a_3 J^\alpha \right) f_\alpha
\]

here we have used \( a_\alpha J^\alpha \cdot a_\beta J^\beta = a_\gamma J^\gamma \) with the summation convention over repeated indices applied. So we see \( \phi \) satisfies the equation:

\[
d\phi J^\alpha \circ u = -a_{\alpha \beta} x_\beta J^\alpha d\phi.
\]

This finishes the proof. \( \square \)

Note that \( a_{\alpha \beta} x_\beta J^\alpha \) is only defined along the image surface \( f(\mathbb{S}^2) \) and \( f \) cannot be holomorphic with respect to any complex structure in the 2-sphere family of complex structures on \( N \).

Let \( \Sigma \) be a Riemann surface, \( N^{4n} \) a hyperkähler manifold with the complex structures \( \mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^3 \) which satisfy the quaternion relation \( \mathcal{J}^1 \mathcal{J}^2 = \mathcal{J}^3 \). Let \( \tilde{a} = (a_1, a_2, a_3) \) be smooth functions \( \Sigma \to \mathbb{S}^2 \).

**Definition 3.2.** Let \( f : \Sigma \to N^{4n} \) be a smooth immersion which satisfies

\[
df J_\Sigma = -\sum_{k=1}^{3} a_k \mathcal{J}^k df,
\]

where \( \tilde{a} = (a_1, a_2, a_3) : \Sigma \to \mathbb{S}^2 \). We say \( f \) is a quaternionic surface in \( N^{4n} \). If in addition \( f \) is harmonic, we say \( f \) is a quaternionic minimal surface.

Condition (3.3) requires the image of \( df \) lying in the span of \( \mathcal{J}^1 df, \mathcal{J}^2 df, \mathcal{J}^3 df \). In the twistor space approach to minimal surfaces and harmonic maps, this condition is called "inclusive" (see [AM], [ES], [R], [Sa] and the references therein).

It is not difficult to see that if \( f \) satisfies (3.3) then \( f \) is conformal. Furthermore, any conformal immersion from \( (\Sigma, J_\Sigma) \) to a 4-dimensional hyperkähler manifold satisfies the equation (3.3). In fact, suppose that \( e_1, e_2 \) is an orthonormal frame of \( \Sigma \). Because \( f \) is conformal and \( df(e_1) \perp df(e_2) \), we have

\[
df(e_1) = c_i J^i df(e_2) \quad \text{and} \quad df(e_2) = d_i J^i df(e_1)
\]

with \( \sum_i c_i^2 = 1 \) and \( \sum_i d_i^2 = 1 \). It is clear that

\[
c_i |df(e_2)|^2 = \langle df(e_1), J^i df(e_2) \rangle = -\langle J^i df(e_1), df(e_2) \rangle = -d_i |df(e_1)|^2.
\]

Since \( |df(e_2)|^2 = |df(e_1)|^2 = 1/2 |df|^2 \), we have \( c_i = -d_i \) hence (3.3) holds.

**Lemma 3.3.** Let \( u : \Sigma_1 \to \Sigma_2 \) be a holomorphic map between two Riemann surfaces with complex structures \( J_{\Sigma_1} \) and \( J_{\Sigma_2} \) respectively. Then for any smooth map \( f : \Sigma_2 \to N \) which satisfies (3.3) with \( \tilde{a} : \Sigma_1 \to \mathbb{S}^2 \), \( f \circ u : \Sigma_1 \to N \) satisfies (3.3) with \( \tilde{a} \circ u : \Sigma_1 \to \mathbb{S}^2 \). If \( f(\Sigma_2) \) is a quaternionic minimal surface, then \( f \circ u(\Sigma_1) \) is also a quaternionic minimal surface.
Proof. Then for any \( x \in \Sigma_1 \)
\[
d(f \circ u)_x J_{\Sigma_1}(x) = df_{u(x)} \circ du_x J_{\Sigma_1}(x)
= df_{u(x)} \circ J_{\Sigma_2}(u(x)) du_x
= -a_i(u(x)) J^i_{u(x)} df_{u(x)} \circ du_x
= -a_i(u(x)) J^i_{u(x)} d(f \circ u)_x.
\]

If \( f \) is harmonic and \( u \) is holomorphic, \( f \circ u \) is harmonic. \( \square \)

**Proposition 3.4.** A quaternionic surface in \( N^{4n} \) is a minimal surface if and only if \( \overset{\sim}{a} \) is holomorphic with respect to the complex structure on \( \Sigma \) which makes the metric \( g \) Hermitian and the standard complex structure on \( \mathbb{S}^2 \). \( \overset{\sim}{a} \) is constant if and only if the quaternionic surface is a holomorphic curve.

Proof. Since \( f \) is conformal, a quaternionic surface in \( N^{4n} \) is a minimal surface if and only if \( f \) is a harmonic map from \( \Sigma \) to \( N \). Let \( e_1, e_2 \) be an orthonormal frame on \( \Sigma \) which satisfies \( I e_1 = e_2, I e_2 = -e_1 \). Note that, by the definition,

\[
f_1 := df(e_1) = \sum_{i=1}^{3} a_i J^i f_2, \quad f_2 := df(e_2) = -\sum_{i=1}^{3} a_i J^i f_1.
\]

Taking the normal coordinates centred at \( x \) and \( f(x) \), we have

\[
\Delta f = -\nabla_2 \left( \sum_{i=1}^{3} a_i J^i \right) f_1 + \nabla_1 \left( \sum_{i=1}^{3} a_i J^i \right) f_2
= \left( -\sum_{i=1}^{3} \nabla_2 a_i J^i - \left( \sum_{i=1}^{3} \nabla_1 a_i J^i \right) \left( \sum_{i=1}^{3} a_i J^i \right) \right) f_1
= (-\nabla_2 a_1 - a_3 \nabla_1 a_2 + a_2 \nabla_1 a_3) J^1 f_1
+ (-\nabla_2 a_2 - a_1 \nabla_1 a_3 + a_3 \nabla_1 a_1) J^2 f_1
+ (-\nabla_2 a_3 - a_2 \nabla_1 a_1 + a_1 \nabla_1 a_2) J^3 f_1. \tag{3.4}
\]

Since \( f \) is harmonic, it follows that

\[
\begin{align*}
\nabla_2 a_1 + a_3 \nabla_1 a_2 - a_2 \nabla_1 a_3 &= 0 \\
\nabla_2 a_2 + a_1 \nabla_1 a_3 - a_3 \nabla_1 a_1 &= 0 \\
\nabla_2 a_3 + a_2 \nabla_1 a_1 - a_1 \nabla_1 a_2 &= 0.
\end{align*}
\tag{3.5}
\]

Solving (3.5) and using \( a_1 \nabla_2 a_1 + a_2 \nabla_2 a_2 + a_3 \nabla_2 a_3 = 0 \), one gets

\[
\begin{align*}
\nabla_1 a_1 + a_2 \nabla_2 a_2 - a_3 \nabla_2 a_3 &= 0 \\
\nabla_1 a_2 + a_3 \nabla_2 a_1 - a_1 \nabla_2 a_3 &= 0 \\
\nabla_1 a_3 + a_1 \nabla_2 a_2 - a_2 \nabla_2 a_1 &= 0.
\end{align*}
\tag{3.6}
\]
We can rewrite (3.5) as
\[ \nabla_2 \vec{a} = \vec{a} \times \nabla_1 \vec{a}, \]
and rewrite (3.6) as
\[ \nabla_1 \vec{a} = -\vec{a} \times \nabla_2 \vec{a}. \]
Noting that the standard complex structure on \( S^2 \) at \( \vec{a} \) is \( \vec{a} \times \), we can see that \( \vec{a} \) satisfies the equations (3.5) and (3.6) if and only if it is a holomorphic map with respect to the complex structure on \( \Sigma \) which makes the metric \( g \) Hermitian and the standard complex structure on \( S^2 \).

Remark that if we write the equation in \( b_i = -a_i \) then \( \vec{b} \) is anti-holomorphic and if \( N \) is 4-dimensional the above result was obtained in [ES] and by S.S. Chern if \( N = \mathbb{R}^4 \).

In particular, when a quaternionic surface is minimal, the mapping \( \vec{a} \) satisfies the harmonic map equation to the standard sphere:
\[ \Delta \vec{a} + |\nabla \vec{a}|^2 \vec{a} = 0. \tag{3.7} \]

The following theorem is known to be true for minimal surface in a Kähler-Einstein manifold of real dimension 4 (cf. [CW]) by noticing that \( a_k = \cos \alpha_k \) where \( \alpha_k \) is the Kähler angle of the surface \( f(\Sigma) \) with respect to the Kähler form \( \omega_J^k \) in \( N \).

**Theorem 3.5.** If a quaternionic surface in \( N^{4n} \) is a minimal surface with \( \vec{a} = (a_1, a_2, a_3) : \Sigma \to S^2 \), then
\[ \Delta a_k + 2 \frac{|\nabla a_k|^2 a_k}{1 - a_k^2} = 0. \]

**Proof.** We only need to prove the result for \( a_1 \). First we compute the Laplacian of \( a_1 \) as follows. Again we take the normal coordinates centred at \( x \in M \) and at \( f(x) \in N \). Differentiating in \( \nabla_2 \) of
\[ \nabla_2 a_1 = a_2 \nabla_1 a_3 - a_3 \nabla_1 a_2 \]
yields
\[ \nabla^2_{22} a_1 = \nabla_2 a_2 \nabla_1 a_3 + a_2 \nabla^2_{12} a_3 - \nabla_2 a_3 \nabla_1 a_2 - a_3 \nabla^2_{12} a_2. \]

Multiplying \( a_3, a_2, a_1 \) accordingly to the following three equations
\[ a_3 \nabla_1 a_1 = \nabla_2 a_2 + a_1 \nabla_1 a_3 \]
\[ a_2 \nabla_1 a_1 = a_1 \nabla_1 a_2 - \nabla_2 a_3 \]
\[ a_1 \nabla_1 a_1 = -a_2 \nabla_1 a_2 - a_3 \nabla_1 a_3 \]
then summing them up leads to
\[ \nabla_1 a_1 = a_3 \nabla_2 a_2 - a_2 \nabla_2 a_3. \]
Differentiating in $\nabla_1$ gives

$$\nabla_1^2 a_1 = \nabla_1 a_3 \nabla_2 a_2 + a_3 \nabla_2^2 a_2 - \nabla_1 a_2 \nabla_2 a_3 - a_2 \nabla_2^2 a_3.$$ 

Now we conclude

$$\Delta a_1 = 2(\nabla_1 a_3 \nabla_2 a_2 - \nabla_1 a_2 \nabla_2 a_3)$$

and we may write the right hand side in terms which only involve $\nabla_1$ as follows:

$$\nabla_1 a_3 \nabla_2 a_2 - \nabla_1 a_2 \nabla_2 a_3 = \nabla_1 a_3 (a_3 \nabla_1 a_1 - a_1 \nabla_1 a_3)$$

$$- \nabla_1 a_2 (a_1 \nabla_1 a_2 - a_2 \nabla_1 a_1)$$

$$= a_3 \nabla_1 a_1 \nabla_1 a_3 - a_1 |\nabla_1 a_3|^2$$

$$- a_1 |\nabla_1 a_2|^2 + a_2 \nabla_1 a_1 \nabla_1 a_2$$

$$= -a_1 (|\nabla_1 a_1|^2 + |\nabla_1 a_2|^2 + |\nabla_1 a_3|^2).$$

So we have just shown

$$\Delta a_1 = -2a_1 (|\nabla_1 a_1|^2 + |\nabla_1 a_2|^2 + |\nabla_1 a_3|^2). \quad (3.8)$$

On the other hand, we have

$$|\nabla a_1|^2 = |\nabla_1 a_1|^2 + |\nabla_2 a_1|^2$$

$$= |\nabla_1 a_1|^2 + (a_2 \nabla_1 a_3 - a_3 \nabla_1 a_2)^2$$

$$= |\nabla_1 a_1|^2 + a_2^2 |\nabla_1 a_3|^2 + a_3^2 |\nabla_1 a_2|^2 - 2a_2 a_3 \nabla_1 a_2 \nabla_1 a_3.$$ 

However,

$$(1 - a_1^2)(|\nabla_1 a_1|^2 + |\nabla_1 a_2|^2 + |\nabla_1 a_3|^2) - |\nabla a_1|^2$$

$$= -a_1^2 |\nabla_1 a_1|^2 + (1 - a_1^2 - a_2^2) |\nabla_1 a_2|^2 + (1 - a_1^2 - a_3^2) |\nabla_1 a_3|^2 + 2a_2 a_3 \nabla_1 a_2 \nabla_1 a_3$$

$$= -a_1^2 |\nabla_1 a_1|^2 + a_2^2 |\nabla_1 a_2|^2 + a_3^2 |\nabla_1 a_3|^2 + 2a_2 a_3 \nabla_1 a_2 \nabla_1 a_3$$

$$= 0 \quad (3.9)$$

by recalling $a_1 \nabla_1 a_1 = a_2 \nabla_1 a_2 + a_3 \nabla_1 a_3.$

Putting (3.8) and (3.9) together, we have

$$\Delta a_1 = -2 \frac{|\nabla a_1|^2 a_1}{1 - a_1^2},$$

which completes the proof. 

Theorem 3.6. Suppose that $f$ is a minimal quaternionic surface in $N^4$. Then either $f$ is constant or the Euler characteristic number $\frac{1}{2\pi} \chi (N f (\Sigma))$ of the normal bundle of $f (\Sigma)$ is $2g - 2 - 2 \deg \bar{a}$. In particular, if $f \in C^2 (\mathbb{S}^2, N^4)$ satisfies the equation

$$df J_{\mathbb{S}^2} = - \sum_{i=1}^3 x_i J^i df,$$ 

(3.10)

where $x \in \mathbb{S}^2 \subset \mathbb{R}^3$, then either $f$ is constant or the Euler characteristic number of the normal bundle of $f (\mathbb{S}^2)$ is $-4.$
Proof. Let $\Sigma_0 = f(\Sigma)$. $\Sigma_0$ is a minimal surface in $N$ because $f$ is harmonic and conformal. Proposition 4.2 and Proposition 4.3 in [CT] assert, for a compact minimal surface in a Kähler-Einstein surface $N$, that the generalized adjunction formula

$$\chi(T \Sigma_0) + \chi(N \Sigma_0) = \int_\Sigma \Omega_{12} + \Omega_{34} - \frac{1}{2} \int_\Sigma |\nabla J_{\Sigma_0}|^2$$

$$= 2\pi \int_\Sigma \alpha c_1(N) - \frac{1}{2} \int_\Sigma |\nabla J_{\Sigma}|^2$$

holds for some function $\alpha$ on $\Sigma_0$, where $\Omega_{12}, \Omega_{34}$ are the curvature tensors of $N$ along the tangential and normal directions of $\Sigma_0$ respectively. The term $|\nabla J_{\Sigma_0}|^2$ is equal to $2|h_{12}^4 - h_{11}^3|^2 + 2|h_{22}^4 - h_{12}^3|^2$ where $h_{ij}^k$ are the second fundamental forms of $\Sigma_0$ in $N$.

Since $c_1(N) = 0$, we have

$$\chi(T \Sigma_0) + \chi(N \Sigma_0) = -\frac{1}{2} \int_{\Sigma_0} |\nabla J_{\Sigma_0}|^2. \quad (3.11)$$

In particular, an embedded holomorphic $\mathbb{S}^2$ has self-intersection number $-2$ in $M$ with $C_1(M) = 0$.

On the other hand, for any solution of (3.5), by Proposition 3.4 and Theorem 3.5 and Proposition 3.2 in [CL2] (specializing the general formula for cosine of the Kähler angle along the mean curvature flow to minimal surface) and (3.7), we always have

$$|\nabla J_{\Sigma_0}|^2 = |\nabla \tilde{a}|^2 = \frac{2|\nabla a_i|^2}{1-a_i^2} \quad (3.12)$$

for $i = 1, 2, 3$. One then has

$$\frac{1}{2\pi} \chi(N \Sigma) = -\frac{1}{4\pi} \int_{\Sigma_g} |\nabla \tilde{a}|^2 + 2g - 2 = 2g - 2 - 2\deg \tilde{a}.$$

Here we recall for holomorphic $\tilde{a}$ to $\mathbb{S}^2$,

$$\deg \tilde{a} = \frac{1}{\text{vol}(\mathbb{S}^2)} \int_{\Sigma_g} Jac(\tilde{a}) = \frac{1}{4\pi} \int_{\Sigma_g} |\nabla \tilde{a}|^2 = \frac{1}{8\pi} \int_{\Sigma_g} |\nabla \tilde{a}|^2.$$

Now if $\Sigma = \mathbb{S}^2$ and $\tilde{a}(x) = (x_1, x_2, x_3), f : \mathbb{S}^2 \rightarrow N$ is harmonic because $\tilde{a} : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is the identity map. We conclude

$$\frac{1}{2\pi} \chi(N \Sigma) = -2 - \frac{1}{4\pi} \int_{\mathbb{S}^2} |\nabla x|^2 = -4.$$

This completes the proof. $\square$
Based on the results we obtained so far, we next construct an example of stationary quaternionic map from \( \mathbb{R}^4 \) with a line of singularities. For any smooth map \( \phi : S^2 \rightarrow N \), we have an extension \( f(x, x_4) := \phi(x/|x|) \) for any \( x \in \mathbb{R}^3 \setminus \{0\} \). Moreover, the proof of Lemma 3.1 can be reversed to produce a quaternionic map with the \( x_4 \)-axis as its singular set from a map \( \phi \) which satisfies (3.2).

In the monograph [AH], Atiyha and Hitchin considered the space \( M^0_2 \) of centred 2-monopoles on \( \mathbb{R}^3 \) with finite action. It is a complete hyperkähler manifold of dimension 4. \( SO(3) \) acts on \( M^0_2 \) isometrically and this action lifts to a double (also Riemannian universal) covering \( \tilde{M}^0_2 \). The space of axisymmetric monopoles, which constitute a special class of solutions to the monopole equations, defines an embedded minimal \( \mathbb{R}P^2 \) in \( M^0_2 \). This \( \mathbb{R}P^2 \) lifts to an embedded minimal \( S^2 \) in the hyperkähler manifold \( \tilde{M}^0_2 \).

**Corollary 3.7.** There does exist a nontrivial minimal quaternionic sphere \( \phi \) in the hyperkähler manifold \( \tilde{M}^0_2 \) with \( \tilde{a} = (x_1, x_2, x_3) \). The extended map \( f \) from \( \phi \) is a stationary quaternionic map from \( \mathbb{R}^4 \) to \( \tilde{M}^0_2 \) with the entire \( x_4 \)-axis as singular set.

**Proof.** We take the nontrivial embedded minimal \( S^2 \) in \( \tilde{M}^0_2 \) discussed above. The Euler characteristic number of the normal bundle of this minimal 2-sphere is \(-4\) as shown in [AH].

By Theorem 3.6, we know that the minimal 2-sphere is a minimal quaternionic sphere \( \phi_0 \) with a function \( \tilde{a}_0 \) in its definition, and \( \deg \tilde{a}_0 = 1 \). Since \( \tilde{a}_0 : S^2 \rightarrow S^2 \) is holomorphic and of degree 1, it is diffeomorphic because the sum of orders of the zeros of \( |\tilde{\partial}\tilde{a}_0| \) is \( -\deg(\tilde{a}_0)(2 \cdot 0 - 2) + (2 \cdot 0 - 2) = 0 \), \( |\tilde{\partial}\tilde{a}_0| \) has no zeros, and therefore the inverse \( \tilde{a}_0^{-1} \) of \( \tilde{a}_0 \) exists and is holomorphic. So, \( \phi := \phi_0 \circ \tilde{a}_0^{-1} \) is a nontrivial minimal quaternionic sphere with \( \tilde{a} = (x_1, x_2, x_3) \) by Lemma 3.3.

Recall that action of the complex structure \( J \) on \( S^2 \) at \( x \in S^2 \) is given by the standard cross product \( x \times \). Write \( \tilde{a}_0 = (a_{01}, a_{02}, a_{03}) \). Then

\[
a_{0i}(x) = -\frac{(d\phi_0(x \times e), J^i d\phi_0(e))_x}{|d\phi_0(e)|^2}
\]

and \( d\phi_0 \) at \( x \) is the same as \( d\phi \) at \(-x\) because \( \phi_0 \) is the lift from \( \mathbb{R}P^2 \). We then conclude

\[
\tilde{a}_0(-x) = -\tilde{a}_0(x), \quad \tilde{a}_0^{-1}(-x) = -\tilde{a}_0^{-1}(x).
\]

The chain rule implies

\[
|\nabla \phi(-x)|^2 = |\nabla \phi_0(\tilde{a}_0^{-1}(-x))||\nabla \tilde{a}_0^{-1}(-x)|^2
\]

\[
= |\nabla \phi_0(-\tilde{a}_0^{-1}(x))|^2 - |\nabla \tilde{a}_0^{-1}(x)|^2
\]

\[
= |\nabla \phi_0(\tilde{a}^{-1}(x))||\nabla \tilde{a}_0^{-1}(x)|^2 = |\nabla \phi(x)|^2
\]
because $\phi_0$ is the lift from $\mathbb{R}P^2$. Therefore for $i = 1, 2, 3,$

$$\int_{\mathbb{S}^2} x_i |\nabla \phi|^2 = 0.$$  

The fact that the extended map $f$ is stationary follows from the lemma below.  

The lemma below is known to experts. For the sake of completeness, we present a proof of it.

**Lemma 3.8.** Let $\phi$ be a smooth harmonic map from $\mathbb{S}^2$ to a Riemannian manifold $N$. Then the extended map $f$ of $\phi$, which is defined by $f(x, x') = \phi(x/|x|)$ for $x = (x_1, x_2, x_3) \neq (0, 0, 0), x' \in \{0\} \times \mathbb{R}^{m-3} \subset \mathbb{R}^m$, is a stationary harmonic map if and only if $\phi$ satisfies

$$\int_{\mathbb{S}^2} x_i |\nabla \phi|^2 = 0, \quad i = 1, 2, 3, \quad (x_1, x_2, x_3) \in \mathbb{S}^2.$$  

**Proof.** In fact, we have

$$\nabla_x f = 0, \quad \frac{\partial f}{\partial r} = 0, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$  

Define a cut-off function by

$$\eta_\epsilon(r, \alpha, \beta, x') = \begin{cases} 1 & r \geq \epsilon \\ \frac{2}{\epsilon} (r - \epsilon) & \epsilon/2 < r < \epsilon \\ 0 & r \leq \epsilon/2 \end{cases}$$  

where $x_1 = r \sin \alpha \cos \beta, x_2 = r \sin \alpha \sin \beta, x_3 = r \cos \beta$.

For any smooth vector field $X = (X_1, \cdots, X_m)$ in $\mathbb{R}^m$ with compact support, because $f$ is smooth away from $\{0\} \times \mathbb{R}^{m-3}$, we have

$$0 = \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2 \nabla_i f \nabla_j f) \nabla_j (\eta_\epsilon X_i)$$  

$$= \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2 \nabla_i f \nabla_j f) \nabla_j \eta_\epsilon X_i$$  

$$+ \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2 \nabla_i f \nabla_j f) \eta_\epsilon \nabla_j X_i.$$  

It then follows

$$\int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2 \nabla_i f \nabla_j f) \nabla_j X_i = \lim_{\epsilon \to 0} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2 \nabla_i f \nabla_j f) \eta_\epsilon \nabla_j X_i$$  

$$= - \lim_{\epsilon \to 0} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2 \nabla_i f \nabla_j f) \nabla_j \eta_\epsilon X_i.$$
Therefore, \( f \) is stationary if and only if
\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}^m} (|\nabla f|^2 \delta_{ij} - 2 \nabla_i f \nabla_j f) \nabla_j \eta_\epsilon X_i = 0.
\]

Direct computation shows that the above condition is equivalent to
\[
\int_{\mathbb{R}^{m-3}} \int_{S^2} |\nabla \phi|^2 \sum_{i=1}^{3} x_i X_i(0, x') d\sigma dx' = 0.
\]

Since \( X \) is arbitrary, we see the desired statement holds. \( \square \)

References


Department of Mathematics
The University of British Columbia
Vancouver, BC, Canada V6T 1Z2
jychen@math.ubc.ca

Math. Group
The abdus salam ICTP
Trieste 34100 Italy
and
Academy of Mathematics
and Systems Sciences
Chinese Academy of Sciences
Beijing 100080, P. R. of China.
jyli@ictp.it