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A simple proof of the propagation of singularities for solutions of Hamilton-Jacobi equations

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Abstract. In Albano-Cannarsa [1] the authors proved that, under some conditions, the singularities of the semiconcave viscosity solutions of the Hamilton-Jacobi equation propagate along generalized characteristics. In this note we will provide a simple proof of this interesting result.

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$. Throughout this note we assume that $u \in C(\bar{\Omega})$ is a semiconcave viscosity solution of the following Hamilton-Jacobi equation

$$H(Du, u, x) = 0 \quad \text{in } \Omega. \tag{1.1}$$

For $x \in \Omega$ we set

$$D^{+}u(x) = \{p \in \mathbb{R}^n \mid u(y) \leq u(x) + p \cdot (y - x) + \mathrm{o}(|x - y|)\}.$$

As in [1], we assume that $H \in C^1(\mathbb{R}^n \times \mathbb{R} \times \Omega)$ and satisfies

(A1) $H(\cdot, z, x)$ is convex for each $(z, x) \in \mathbb{R} \times \Omega$;
(A2) For each $(z, x) \in \mathbb{R} \times \Omega$, the 0-level set $\{p \mid H(p, z, x) = 0\}$ does not contain any line segment.

We want to remark that under the convexity assumption (A1), u is a semiconcave viscosity solution of equation (1.1) if and only if u is semiconcave and satisfies equation (1.1) almost everywhere. For $K \subset \mathbb{R}^n$, we denote $\mathrm{co}(K)$ as the convex hull of K. Using some results in Albano-Cannarsa [2] about the propagation of the singularities for semiconcave functions, Albano and Cannarsa proved the following interesting theorem in [1].

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Theorem 1.1. *If $x_0 \in \Sigma(u)$ and $0 \notin \mathrm{co}(D_pH(D^+u(x_0), u(x_0), x_0))$ then there exists $\sigma > 0$ and a Lipschitz continuous curve $\xi(s) : [0, \sigma] \to \Sigma(u)$ such that*

$$\begin{cases} \dot{\xi}(s) \in \mathrm{co}(D_pH(D^+u(\xi(s)), u(\xi(s)), \xi(s))) \not\ni 0 & \text{for a.e. } s \in [0, \sigma] \\ \xi(0) = x_0, \end{cases}$$

and

$$\max_{0\le s\le\sigma} \min_{p\in D^+u(\xi(s))} H(p, u(\xi(s)), \xi(s)) < 0,$$

where

$$\Sigma(u) = \{x \in \Omega|\ u \text{ is not differentiable at } x\}.$$

The proof in [1] is very technical. The techniques and methods used there are important for studying the singularities for general semiconcave functions. For the semiconcave viscosity solution u of equation (1.1), we can in fact give a simple proof of Theorem 1.1 by approximating u with smooth functions. See [1] for more backgrounds and comments in the singularities of semiconcave viscosity solutions of Hamilton-Jacobi equations. We also refer to [2], Ambrosio-Cannarsa-Soner [3] and Cannarsa-Sinestrari [4] for detailed discussions about singularities of semiconcave functions.

2. Proofs

Since semiconcave functions are locally Lipschitz continuous, in this section, we assume that

$$\operatorname*{esssup}_{\Omega} |Du| \le C \text{ and } D^2u \le CI_n \quad \text{in } \Omega,$$

where I_n is the $n \times n$ identity matrix. We first prove the following lemma.

Lemma 2.1. *Let V be an open subset such that $x_0 \in V \subset \bar{V} \subset \Omega$. If $x_0 \in \Sigma(u)$, then there exist a sequence of smooth functions $\{u_m(x)\}_{m\ge1}$ in Ω such that*

(i) $\lim_{m\to+\infty} u_m = u$, *uniformly in* $\bar{V}$;

(ii) $\max_{\bar{V}} |Du_m| \le C$, $D^2u_m \le CI_m$ *in* V;

(iii) $\lim_{m\to+\infty} Du_m(x_0) = q$ *for some* $q \in D^+u(x_0)$ *satisfying* $H(q, u(x_0), x_0) < 0$.

Proof. Let

$$u_\epsilon(x) = \frac{1}{\epsilon^n} \int_\Omega u(y)\eta\left(\frac{x-y}{\epsilon}\right) dy,$$

where $\eta \in C_0^\infty(\overline{B_1(0)})$ and satisfies

$$\eta > 0 \text{ in } B_1(0) \text{ and } \int_{B_1(0)} \eta(x)\, dx = 1.$$

Then u_ϵ is smooth and

$$\lim_{\epsilon\to 0} u_\epsilon = u \quad \text{uniformly in } \bar{V}.$$

When ϵ is small enough, we have that

$$|Du_\epsilon| \le C \text{ and } D^2 u_\epsilon \le C I_n \quad \text{in } V.$$

Case 1. If $\lim_{\epsilon\to 0} Du_\epsilon(x_0)$ does not exist, then there exist two subsequence $\epsilon_m \to 0$ and $\delta_m \to 0$ such that

$$\lim_{m\to+\infty} Du_{\epsilon_m}(x_0) = p_1 \neq p_2 = \lim_{m\to+\infty} Du_{\delta_m}(x_0).$$

We have that $p_1, p_2 \in D^+u(x_0)$. Owing to (A1), $H(p_1, u(x_0), x_0) \le 0$ and $H(p_2, u(x_0), x_0) \le 0$. Let

$$u_m(x) = \frac{1}{2}(u_{\epsilon_m}(x) + u_{\delta_m}(x)).$$

By (A2), we get the desired $\{u_m\}_{m\ge 1}$.

Case 2. If $\lim_{\epsilon\to 0} Du_\epsilon(x_0)$ exists, we denote

$$q = (q_1, ..., q_n) = \lim_{\epsilon\to 0} Du_\epsilon(x_0).$$

According to (A1), $H(q, u(x_0), x_0) \le 0$. If $H(q, u(x_0), x_0) = 0$, we claim that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^n} \int_{B_\epsilon(x_0)} |Du(y) - q| \eta\left(\frac{x_0 - y}{\epsilon}\right) dy = 0.$$

If not, then there exists a $\delta > 0$ and a subsequence $\epsilon_k \to 0^+$ as $k \to +\infty$ such that

$$\lim_{k\to+\infty} \frac{1}{\epsilon_k^n} \int_{B_{\epsilon_k}(x_0)} |Du(y) - q| \eta\left(\frac{x_0 - y}{\epsilon_k}\right) dy \ge 2\sqrt{n}\delta.$$

For $i = 1, ..., n$, we denote

$$A_i = \{x \in B_{\epsilon_k}(x_0) | \, |u_{x_i}(x) - q_i| \ge \delta\}.$$

Then we must have that

$$\varlimsup_{k\to+\infty} \frac{1}{\epsilon_k^n} \int_{\cup_{i=1}^n A_i} \eta\left(\frac{x_0 - y}{\epsilon_k}\right) dy > 0. \tag{2.1}$$

Since

$$A_i = \{x \in B_{\epsilon_k}(x_0) | \, u_{x_i}(x) - q_i \ge \delta\} \cup \{x \in B_{\epsilon_k}(x_0) | \, u_{x_i}(x) - q_i \le -\delta\},$$

upon passing if necessary to a subsequence, according to (2.1), without loss of generality, we may assume that

$$\lim_{k\to+\infty}\frac{1}{\epsilon_k^n}\int_{A_1^+}\eta\left(\frac{x_0-y}{\epsilon_k}\right)dy=\tau>0$$

and

$$\lim_{k\to+\infty}\frac{1}{\tau\epsilon_k^n}\int_{A_1^+}Du(y)\eta\left(\frac{x_0-y}{\epsilon_k}\right)dy=q'=(q_1',\ldots,q_n')$$

where

$$A_1^+=\{x\in B_{\epsilon_k}(x_0)|\ u_{x_1}(x)-q_1\geq\delta\}.$$

Since $q=\lim_{\epsilon\to 0}Du_\epsilon(x_0)$, we have that $\tau<1$. Otherwise, we have that

$$\lim_{k\to+\infty}u_{\epsilon_k,x_1}(x_0)-q_1\geq\delta.$$

Therefore

$$\lim_{k\to+\infty}\frac{1}{\epsilon_k^n}\int_{B_{\epsilon_k}(x_0)\setminus A_1^+}\eta\left(\frac{x_0-y}{\epsilon_k}\right)dy=1-\tau>0$$

and

$$\lim_{k\to+\infty}\frac{1}{(1-\tau)\epsilon_k^n}\int_{B_{\epsilon_k}(x_0)\setminus A_1^+}Du(x)\eta\left(\frac{x_0-y}{\epsilon_k}\right)dy=\frac{q-\tau q'}{1-\tau}=q''.$$

Owing to (A1), we have that

$$H(q',u(x_0),x_0)\leq 0,\ \ H(q'',u(x_0),x_0)\leq 0. \tag{2.2}$$

Also,

$$q=\tau q'+(1-\tau)q''.$$

By the definition of A_1^+, $q_1'-q_1\geq\delta$. Hence $q'\neq q$ and $q'\neq q''$. Since $H(q,u(x_0),x_0)=0$, (2.2) implies that the 0-level set $\{p|\ H(p,u(x_0),x_0)=0\}$ contains the line segment connecting q' and q''. This contradicts the assumption (A2). So our claim holds. Since $\eta>0$ in $B_1(0)$, we have that x_0 is a Lebesgue point of Du. So u is differentiable at x_0. This is a contradiction. Therefore $H(q,u(x_0),x_0)<0$. So in this case we can choose u_ϵ as the desired sequence of smooth functions. □

Remark 2.2. Lemma 2.1 is still true by replacing q in (iii) with any $p\in D^+u(x_0)$. To prove it, we need to choose more delicate mollification of u instead of the standard mollification. For our purpose, Lemma 2.1 is enough.

Proof of Theorem 1.1.

Step I. Choose an open set V such that $x_0 \in V \subset \bar{V} \subset \Omega$. Let $\{u_m\}_{m\geq 1}$ be the sequence of smooth functions from Lemma 2.1. By a compactness argument, it is easy to show that for any fixed $x \in V$

$$\operatorname*{Sup}_{\{k\geq m, |y-x|\leq\delta\}} d(Du_k(y), D^+u(x)) \to 0 \quad \text{as } m \to +\infty \text{ and } \delta \to 0, \tag{2.3}$$

and

$$\lim_{\delta\to 0} \operatorname*{Sup}_{\{|y-x|\leq\delta\}} d(Du^+(y), D^+u(x)) \to 0.$$

Since $0 \notin \mathrm{co}(D_pH(D^+u(x_0), u(x_0), x_0))$, without loss of generality, we may assume that there exists a $\delta > 0$ such that

$$B_\delta(0)\cap\mathrm{co}\Big\{D_pH(Du^+(x),u(x),x), D_pH(Du_m(x),u_m(x),x)|x\in V, m\geq 1\Big\} \\ = \Phi. \tag{2.4}$$

Hence there exits a $\sigma > 0$ such that for each $m \geq 1$, there exists a C^1 curve $\xi_m(s) : [0, \sigma] \to V$ such that

$$\begin{cases} \dot{\xi}_m(s) = D_pH(Du_m(\xi_m(s)), u_m(\xi_m(s)), \xi_m(s)) \neq 0 \\ \xi_m(0) = x_0. \end{cases}$$

Step II. We claim that

$$H(Du_m(\xi_m(s)), u_m(\xi_m(s)), \xi_m(s)) \leq H(Du_m(x_0), u_m(x_0), x_0) + Cs, \tag{2.5}$$

where C is some constant depending only on H and u. Since $D^2u_m \leq CI_m$, we have that

$$\frac{d}{ds}H(Du_m(\xi_m(s)),u_m(\xi_m(s)), \xi_m(s)) = H_{p_i}H_{p_j}u_{m,x_ix_j} + H_{p_i}H_zu_{m,x_i} + H_{x_i}H_{p_i}$$

$$\leq C|D_pH|^2+|D_pH||Du_m||H_z|+|D_xH||D_pH| \leq C.$$

So our claim holds. We assume that $\lim_{m\to+\infty} Du_m(x_0) = q$. According to the choice of u_m, $q \in D^+u(x_0)$ and $H(q, u(x_0), x_0) < 0$. Owing to (2.5), if we choose $\sigma > 0$ small enough, without loss of generality, we may assume that for $m \geq 1$ and $s \in [0, \sigma]$

$$H(Du_m(\xi_m(s)), u_m(\xi_m(s)), \xi_m(s)) \leq \frac{1}{2}H(q, u(x_0), x_0) < 0. \tag{2.6}$$

Step III. Since $\{\xi_m\}_{m\geq 1}$ is uniformly Lipschitz continuous, passing to a subsequence if it is necessary, we assume that

$$\lim_{m\to+\infty} \xi_m(s) = \xi(s) \quad \text{uniformly in } [0,\sigma].$$

Hence

$$\dot{\xi}_m = D_p H(Du_m(\xi_m(s)), u_m(\xi_m(s)), \xi_m(s)) \rightharpoonup \dot{\xi}(s) \quad \text{weakly in } L^2[0,\sigma]. \tag{2.7}$$

Owing to (2.7), a subsequence of convex combinations of $\dot{\xi}_m(s)$ converges to $\dot{\xi}(s)$ a.e. in $[0,\sigma]$. Hence by (2.3) and (2.4),

$$\dot{\xi}(s) \in \mathrm{co}(D_p H(D^+u(\xi(s)), u(\xi(s)), \xi(s))) \not\ni 0 \quad \text{for a.e. } s \in [0,\sigma].$$

Owing to (2.3) and (2.6), we derive that

$$\max_{s\in[0,\sigma]} \min_{p\in D^+u(\xi(s))} H(p, u(\xi(s)), \xi(s)) \leq \frac{1}{2} H(q, u(x_0), x_0) < 0.$$

Hence $\xi([0,\sigma]) \subset \Sigma(u)$ and

$$\dot{\xi}(s) \in \mathrm{co}(D_p(H(D^+u(\xi(s)), u(\xi(s)), \xi(s)))) \not\ni 0 \quad \text{for a.e. } s \in [0,\sigma]. \qquad \square$$

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