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Bounds for double zeta-functions

ISAO KIUCHI AND YOSHIO TANIGAWA

Abstract. In this paper we shall derive the order of magnitude for the double zeta-function of Euler-Zagier type in the region $0 \leq \Re s_j < 1 \ (j = 1, 2)$. First we prepare the Euler-Maclaurin summation formula in a suitable form for our purpose, and then we apply the theory of double exponential sums of van der Corput's type.

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1. Introduction

Let $s_j = \sigma_j + it_j \ (j = 1, 2, \ldots, r)$ be complex variables. The r-ple zeta-function of Euler-Zagier type is defined by

$$\zeta_r(s_1, \ldots, s_r) = \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}},$$

which is absolutely convergent for $\sigma_r > 1, \sigma_r + \sigma_{r-1} > 2, \ldots, \sigma_r + \cdots + \sigma_1 > r$. The function ζ_r has many applications to mathematical physics. In particular, algebraic relations among the values of ζ_r at positive integers have been studied extensively [14]. As a function of the complex variables s_j, the analytic continuation of ζ_r has been dealt already. For $r = 2$, this problem was studied by F. V. Atkinson [3] in his research on the mean value formula of the Riemann zeta-function. For general r, the analytic continuation was proved by S. Akiyama, S. Egami and Y. Tanigawa [1] and J. Q. Zhao [17] independently, and later by K. Matsumoto [13]. The values at negative integers were considered in [2].

On the other hand, the order of magnitude of the zeta-function on some vertical line plays an important role in the theory of zeta-functions, e.g. it is used for the estimation of the sum of arithmetical functions (see below). The aim of this paper is to study such a problem for the double zeta-function of Euler-Zagier type:

$$\zeta_2(s_1, s_2) = \sum_{1 \leq m < n} \frac{1}{m^{s_1} n^{s_2}}. \tag{1.1}$$

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Before stating our results, we shall recall the previous result for the Riemann zeta-function $\zeta(s)$ and the double zeta-function $\zeta_2(s_1, s_2)$. Let $\mu(\sigma)$ denote the infimum of a number $c \geq 0$ such that

$$\zeta(\sigma + it) \ll |t|^c,$$

or alternatively as

$$\mu(\sigma) = \limsup_{t \to \infty} \frac{\log|\zeta(\sigma + it)|}{\log t}. \tag{1.2}$$

As for a classical result for the function $\mu(\sigma)$ it is known that (see A. Ivić [11, Theorem 1.9])

$$\mu(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma \leq 0, \\ 0 & \text{if } \sigma \geq 1, \end{cases}$$

and

$$\mu(\sigma) \leq \frac{1}{2}(1 - \sigma) \quad \text{if } 0 \leq \sigma \leq 1.$$

Furthermore it is well known that

$$\zeta(it) \ll |t|^{\frac{1}{2}} \log|t| \tag{1.3}$$

and

$$\zeta(1 + it) \ll (\log|t|)^{\frac{2}{3}} \tag{1.4}$$

for $|t| \to \infty$ (Ivić [11, p. 144 (6.7)]). In the case of $\sigma = \frac{1}{2}$, which is the most important in the theory of zeta-function, the first non-trivial result

$$\mu\left(\frac{1}{2}\right) \leq \frac{1}{6} \tag{1.5}$$

was obtained by G. H. Hardy and J. E. Littlewood (see [11]). The best estimate hitherto proved is $\mu\left(\frac{1}{2}\right) \leq \frac{89}{570} = 0.156140\ldots$ due to M. N. Huxley [8]. (He announced that he got an improvement $\mu(\frac{1}{2}) \leq \frac{32}{205} = 0.156098\ldots$ [9].)

Concerning the multiple zeta-function, H. Ishikawa and K. Matsumoto used the Mellin-Barnes integral formula to obtain some results on the order of magnitude on the line $\sigma_1 = \sigma_2 = 0$. In fact, they [10] showed that for a fixed $\alpha (\neq \pm 1)$ and any $\varepsilon > 0$,

$$\zeta_2(it, i\alpha t) \ll (1 + |t|)^{\frac{3}{2}+\varepsilon} \tag{1.6}$$

and

$$\zeta_3(-it, it, it) \ll (1 + |t|)^{\frac{5}{2}+\varepsilon}.$$

As they mentioned in [10], it holds that $\zeta_2(it, it) \ll (1 + |t|)^{1+\varepsilon}$ trivially. Hence (1.6) is far from the true order of magnitude for the double zeta-functions. Any other results on the order of magnitude for the double zeta-function (1.1) on the line $\sigma_j = \frac{1}{2}$ are not stated in [10]. In view of this, it is of some interest to try to determine an upper bound for the double zeta-function on the line other than $\sigma_j = 0$.

In this paper, we shall study the order of magnitude of the double zeta-function (1.1) in the region $0 \le \sigma_j < 1$ $(j = 1, 2)$, where we use, instead of Mellin-Barnes integral formula, the theory of double exponential sums of van der Corput's type (see E. Krätzel [12] and E. C. Titchmarsh [15]).

We use the standard notation e.g. $f(x) = O(g(x))$ means that $|f(x)| < Cg(x)$ for $x > x_0$ and some constant $C > 0$ where $f(x)$ is a complex function and $g(x)$ is a positive function. Further $f(x) \ll g(x)$ means the same as $f(x) = O(g(x))$ and $f(x) \asymp g(x)$ means that both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold.

Our main result can be stated as follows.

Theorem 1.1. *Let $|t_1|$ and $|t_2| \ge 2$ be real numbers such that*

$$|t_1| \asymp |t_2| \quad \text{and} \quad |t_1 + t_2| \gg 1.$$

In the case $\sigma_1 = \sigma_2 = 0$, we have

$$\zeta_2(it_1, it_2) \ll |t_1| \log^2 |t_1|. \tag{1.7}$$

Suppose that $0 \le \sigma_j < 1$ $(j = 1, 2)$ and $\sigma_1 + \sigma_2 > 0$. Then we have

$$\zeta_2(\sigma_1+it_1, \sigma_2+it_2) \ll \begin{cases} |t_1|^{1-\frac{2}{3}(\sigma_1+\sigma_2)} \log^2 |t_1| & \left(0 \le \sigma_1 \le \frac{1}{2}, 0 \le \sigma_2 \le \frac{1}{2}\right) \\ |t_1|^{\frac{5}{6}-\frac{1}{3}(\sigma_1+2\sigma_2)} \log^3 |t_1| & \left(\frac{1}{2} < \sigma_1 < 1, 0 \le \sigma_2 \le \frac{1}{2}\right) \\ |t_1|^{\frac{5}{6}-\frac{1}{3}(2\sigma_1+\sigma_2)} \log^3 |t_1| & \left(0 \le \sigma_1 \le \frac{1}{2}, \frac{1}{2} < \sigma_2 < 1\right) \\ |t_1|^{\frac{2}{3}-\frac{1}{3}(\sigma_1+\sigma_2)} \log^4 |t_1| & \left(\frac{1}{2} < \sigma_1 < 1, \frac{1}{2} < \sigma_2 < 1\right). \end{cases} \tag{1.8}$$

As an immediate consequence we have:

Corollary 1.2. *Suppose the same condition on t_1 and t_2 as in the above theorem. Then we have,*

$$\zeta_2\left(\frac{1}{2} + it_1, \frac{1}{2} + it_2\right) \ll |t_1|^{\frac{1}{3}} \log^2 |t_1| \tag{1.9}$$

$$\zeta_2\left(0 + it_1, \frac{1}{2} + it_2\right) \ll |t_1|^{\frac{2}{3}} \log^2 |t_1| \tag{1.10}$$

$$\zeta_2\left(\frac{1}{2} + it_1, 0 + it_2\right) \ll |t_1|^{\frac{2}{3}} \log^2 |t_1|. \tag{1.11}$$

Remark 1.3. Under the condition $|t_1| \asymp |t_2|$ and $|t_1 + t_2| \gg 1$, we can expect that

$$\zeta_2(s_1, s_2) \ll |t_1|^{\mu(\sigma_1)+\mu(\sigma_2)} \log^A |t_1|$$

for some constant A. The non-trivial estimates of the Riemann zeta-function on the imaginary axis and the critical line are $\mu(0) = \frac{1}{2}$ and $\mu\left(\frac{1}{2}\right) \leq \frac{1}{6}$, respectively. The exponents in Corollary 1.2 can be said to correspond to the classical estimates of the Riemann zeta-function.

Our theorem has an application to the modified weighted divisor problem. Let $1 \leq a \leq b$ be fixed integers, and $d(a, b; n)$ the number of representations of n as $n = n_1^a n_2^b$, where n_1 and n_2 are positive integers. This function plays an important role in many problems. J. L. Hafner [7] and A. Ivić [11, Chapter 14] considered the asymptotic behaviour of the sum $\sum_{n\leq x} d(a, b; n)$ whose main term can be obtained by the residue of $\zeta(as)\zeta(bs)$ since

$$\sum_{n=1}^{\infty} \frac{d(a, b; n)}{n^s} = \zeta(as)\zeta(bs) \qquad \Re s > 1/a.$$

The above representation reveals the close connection between the weighted divisor problem and the Riemann zeta-function.

Now let $h(a, b; n)$ be the number of representations of n as $n = n_1^a n_2^b$ with $n_1 < n_2$:

$$h(a, b; n) = \sum_{\substack{n=n_1^a n_2^b \\ n_1<n_2}} 1.$$

In this case we have

$$\sum_{n=1}^{\infty} \frac{h(a, b; n)}{n^s} = \zeta_2(as, bs)$$

for $\Re s > \max\{2/(a + b), 1/b\}$. Our estimate can be applied to the analysis of $\sum_{n\leq x} h(a, b; n)$, which will be considered elsewhere.

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2. Some lemmas on the Riemann zeta-function

Our proof of Theorem 1.1 depends on the expression derived from the Euler-Maclaurin summation formula. Usual formula, however, is not enough for our purpose, so we will give some refinement of it in the following lemma.

We prepare some notation. Let B_r and $B_r(x)$ denote the r-th Bernoulli number and r-th Bernoulli polynomial, respectively. We put $\overline{B}_r(x) = B_r(x - [x])$ as usual. Let $\Gamma(a, z)$ and $\Psi(a, b; z)$ denote the incomplete Gamma function of the second kind and one of the solutions of confluent hypergeometric equation defined by

$$\Gamma(a, z) = \int_z^\infty e^{-t} t^{a-1} dt, \quad \Re(a) > 0,$$

and

$$\Psi(a, b; z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} {}_1F_1(a, b; z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} {}_1F_1(a-b+1, 2-b; z),$$

respectively. These two functions are connected by the relation

$$\Gamma(a, z) = z^a e^{-z} \Psi(1, a+1; z) \tag{2.1}$$

(see A. Erdélyi *et al.* [4, p. 257 (6) and p. 266 (21)]). The integral representation

$$\Psi(a, c; z) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\phi}} e^{-zt} t^{a-1} (1+t)^{c-a-1} dt \tag{2.2}$$

holds true for $\Re a > 0$, $-\pi < \phi < \pi$ and $-\frac{1}{2}\pi < \phi + \arg z < \frac{1}{2}\pi$ ([4, p. 256 (3)]). Finally let $(w)_p$ be the rising factorial defined by

$$(w)_0 = 1, \quad (w)_{p+1} = (w+p)(w)_p$$

for non-negative integer p.

Lemma 2.1. *Let $\Re\mu < 1$ and x be real which satisfies $|x| \geq \frac{\pi}{2}|\Im\mu|$. Then we have*

$$|\Psi(1, \mu+1; ix)| \leq \frac{2}{|x| - |\Im\mu|}. \tag{2.3}$$

Proof. We may suppose that $\Im\mu > 0$ without loss of generality. Let

$$J := \Psi(1, \mu+1, ix) = \int_0^{\infty e^{i\phi}} e^{-ixt} (1+t)^{\mu-1} dt$$

for simplicity, where we used the integral representation of (2.2).

(I) The case $x > 0$.
Since $\arg(ix) = \frac{\pi}{2}$, we can take $-\pi < \phi < 0$. We introduce a new variable u by $t = e^{i\phi}u \ \ (u \geq 0)$, thereby we have

$$J = e^{i\phi\mu} \int_0^\infty e^{-e^{i(\phi+\frac{\pi}{2})}xu}(u + e^{-i\phi})^{\mu-1}du.$$

Putting

$$u + e^{-i\phi} = re^{i\xi} \ \ (r \geq 1, \ 0 < \xi \leq -\phi)$$

and noting that $\Re\mu < 1$ by the assumption of this lemma, we have

$$|J| \leq e^{-\phi\,\Im\mu} \int_0^\infty e^{-xu\cos(\frac{\pi}{2}+\phi)} e^{-\xi\,\Im\mu}du. \tag{2.4}$$

Now we take $\phi = -\frac{\pi}{2}$, then we have $\cot\xi = u$ for this choice. To evaluate the integral (2.4), we divide the range of integration into two parts at 1. For $0 \leq u \leq 1$, using the following inequality

$$\xi = \mathrm{arccot} u \geq \frac{\pi}{2} - u,$$

we have

$$\int_0^1 e^{-xu-\xi\,\Im\mu}du \leq \int_0^1 e^{-xu-(\frac{\pi}{2}-u)\Im\mu}du \leq \frac{e^{-\frac{\pi}{2}\Im\mu}}{x - \Im\mu}. \tag{2.5}$$

For $u \geq 1$, we have

$$\int_1^\infty e^{-xu-\xi\,\Im\mu}du \leq \int_1^\infty e^{-xu}du = \frac{e^{-x}}{x}. \tag{2.6}$$

Hence (2.5) and (2.6) give us

$$|J| \leq e^{\frac{\pi}{2}\Im\mu}\left(\frac{e^{-\frac{\pi}{2}\Im\mu}}{x - \Im\mu} + \frac{e^{-x}}{x}\right) \leq \frac{2}{x - \Im\mu}. \tag{2.7}$$

(II) The case $x < 0$.
Since $\arg(ix) = -\frac{\pi}{2}$, we can take $0 < \phi < \pi$ in this case. We put $u + e^{-i\phi} = re^{-i\xi} \ \ (0 < \xi \leq \phi)$ and as in the previous case, we can easily see that

$$|J| \leq \int_0^\infty e^{-|x|u\cos(-\frac{\pi}{2}+\phi)} e^{-(\phi-\xi)\Im\mu}du \leq \frac{1}{|x|\cos(-\frac{\pi}{2}+\phi)}.$$

Hence, taking $\phi = \frac{\pi}{2}$, we have

$$|J| \leq \frac{1}{|x|}. \tag{2.8}$$

This completes the proof of the lemma. □

Lemma 2.2. *Let $s=\sigma+it$, $|t|>1$. For $N>\frac{1}{4}|t|$, $m\geq 1$ and $\sigma>-2m-1$, we have*

$$\zeta(s)=\sum_{n\leq N}\frac{1}{n^s}+\frac{N^{1-s}}{s-1}-\frac{N^{-s}}{2}+\sum_{k=1}^{2m}\frac{B_{k+1}}{(k+1)!}(s)_k N^{-(s+k)} + O\left(|t|^{2m+1}N^{-\sigma-2m-1}\right), \tag{2.9}$$

where the implied constant does not depend on t.

To prove our theorem, we apply Lemma 2.2 in the case $m=1$ which we present as a corollary.

Corollary 2.3. *Let $s=\sigma+it$, $|t|>1$. For $N>\frac{1}{4}|t|$ and $\sigma>-3$, we have*

$$\zeta(s)=\sum_{n\leq N}\frac{1}{n^s}+\frac{N^{1-s}}{s-1}-\frac{N^{-s}}{2}+\frac{s}{12}N^{-s-1}+O\left(|t|^3N^{-\sigma-3}\right), \tag{2.10}$$

where the implied constant does not depend on t.

Proof of Lemma 2.2. We start with the well-known formula for the Riemann zeta-function which is derived by the Euler-Maclaurin summation formula:

$$\zeta(s)=\sum_{n\leq N}\frac{1}{n^s}+\frac{N^{1-s}}{s-1}-\frac{1}{2N^s}+\sum_{k=1}^{M-1}\frac{B_{k+1}}{(k+1)!}(s)_k N^{-(s+k)}+R_{M,N}, \tag{2.11}$$

where N and M are positive integers and

$$R_{M,N}=-\frac{(s)_M}{M!}\int_N^\infty \overline{B}_M(x)x^{-s-M}dx. \tag{2.12}$$

We take $M=2m+1$. The function $\overline{B}_{2m+1}(x)$ is a periodic function with period 1 whose Fourier expansion is given by

$$\overline{B}_{2m+1}(x)=2(2m+1)!(-1)^{m-1}\sum_{\nu=1}^{\infty}\frac{\sin 2\pi\nu x}{(2\pi\nu)^{2m+1}}. \tag{2.13}$$

Substituting (2.13) into (2.12), we have

$$R_{2m+1,N}=2(-1)^m(s)_{2m+1}\sum_{\nu=1}^{\infty}(2\pi\nu)^{s-1}\int_{2\pi\nu N}^{\infty}x^{-s-2m-1}\sin x\,dx. \tag{2.14}$$

Now the last integral of (2.14) can be written as

$$\int_{2\pi\nu N}^{\infty}x^{\mu-1}\sin x dx=\frac{i}{2}\left\{e^{-\frac{\pi i\mu}{2}}\Gamma(\mu,2\pi i\nu N)-e^{\frac{\pi i\mu}{2}}\Gamma(\mu,-2\pi i\nu N)\right\}$$

for $\Re\mu < 1$ (I. S. Gradshteyn and I. M. Ryzhik [5, 3.761-2]), hence putting $\mu = -s - 2m$ and using Lemma 2.1, we have

$$\int_{2\pi\nu N}^{\infty} x^{-s-2m-1} \sin x dx \ll \frac{(\nu N)^{-\sigma-2m}}{2\pi\nu N - |t|} \tag{2.15}$$

for $N \geq \frac{1}{4}|t|$. Therefore we get

$$\begin{aligned} R_{2m+1,N} &\ll |(s)_{2m+1}| N^{-\sigma-2m} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2m+1}} \frac{1}{2\pi\nu N - |t|} \\ &\ll \frac{|t|^{2m+1}}{N^{\sigma+2m+1}} \end{aligned} \tag{2.16}$$

for $N \geq \frac{1}{4}|t|$. This completes the proof of Lemma 2.2. □

Remark 2.4. (i) If we evaluate the integral (2.12) directly, we only get

$$R_{2m+1,N} \ll \frac{|t|^{2m+1}}{N^{\sigma+2m}},$$

which is not sufficient for our purpose.

(ii) The approximate functional equation in the simplest form can be written as

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} + O(x^{-\sigma}) \tag{2.17}$$

for $0 < \sigma_0 \leq \sigma \leq 2, x \geq |t|/\pi$ (A. Ivić [11, Theorem 1.8]). Lemma 2 can be regarded as a refinement of this formula.

For the estimate of finite zeta sum and the zeta-function, we shall use the following lemma.

Lemma 2.5. *Let $t > 2$, $N \leq N_1 \leq 2N$ and $N \ll t$, then we have*

$$\sum_{N<n\leq N_1} \frac{1}{n^{1/2+it}} \ll t^{\frac{1}{6}}, \tag{2.18}$$

$$\sum_{N<n\leq N_1} \frac{1}{n^{it}} \ll t^{\frac{1}{2}}, \tag{2.19}$$

$$\sum_{N<n\leq N_1} \frac{1}{n^s} \ll \begin{cases} t^{\frac{1}{2}-\frac{2}{3}\sigma} & \left(0 < \sigma < \frac{1}{2}\right) \\ t^{\frac{1}{3}-\frac{1}{3}\sigma} \log t & \left(\frac{1}{2} < \sigma \leq 1\right), \end{cases} \tag{2.20}$$

and

$$\zeta(\sigma+it) \ll \begin{cases} t^{\frac{1}{2}-\frac{2}{3}\sigma}\log t & \left(0 \le \sigma \le \frac{1}{2}\right) \\ t^{\frac{1}{3}-\frac{1}{3}\sigma}\log^2 t & \left(\frac{1}{2} < \sigma \le 1\right). \end{cases} \tag{2.21}$$

Proof. For the proof of (2.18) and (2.19), see E. C. Titchmarsh [16, Theorem 5.12] and S. W. Graham and G. Kolesnik [6, Theorem 2.2]. As for (2.20), we use (2.18), (2.19), trivial estimate $\sum_{n\le N}\frac{1}{n^{1+it}} \ll \log N$ and the Phragmén-Lindelöf convexity principal.

From Corollary 2.3, we get

$$\zeta(s) = \sum_{n\le N}\frac{1}{n^s} + O\left(t^{-\sigma}\right) \quad \text{for} \quad \frac{t}{4} < N \ll t.$$

Hence by dividing the range $1 \le n \le N$ into $O(\log t)$ subsums of the form of the left-hand side of the above, we obtain (2.21). □

3. Double exponential sums

Next we shall recall the simplest result for double exponential sum, which is given by E. Krätzel [12, p. 61] and E. C. Titchmarsh [15]. Throughout this paper the following conditions are always assumed to be true:

(A) Suppose that D is a subset of the rectangle

$$D_1 = \{(x_1, x_2) \mid a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2\}$$

with $c_j := b_j - a_j \ge 1\,(j = 1, 2)$, where D denotes a bounded plane domain with the area $|D|$.

(B) Any straight line parallel to any of coordinate axis intersects D in a bounded number of line segment. For the sake of simplicity we only consider such domains D where these straight lines intersects the boundary of D in at most two points or in one line segments. We can do this without loss of generality, because each such general domain can be divided into a finite number of these special domains.

(C) Let $f(x_1, x_2)$ be a real function in D_1 with continuous partial derivatives of as many orders as may be required. Suppose that the functions $f_{x_1}(x_1, x_2)$ and $f_{x_2}(x_1, x_2)$ are monotonic in x_1 and x_2, respectively.

(D) Intersections of D with domains of the type $f_{x_j}(x_1, x_2) \le c$ or $f_{x_2}(x_1, x_2) \le c\,(j = 1, 2)$ must satisfy condition (B) as well.

(E) The boundary of D can be divided into a bounded number of parts. In each part the curve of boundary is given by $x_2 =$ constant or a function $x_1 = \rho(x_2)$, which is continuous in the closed intervals described above.

We need the following lemmas.

Lemma 3.1 (Titchmarsh [15, Lemma γ]). *Let $f(x_1, x_2)$ be a real differentiable function of x_1 and x_2. Let $f_{x_1}(x_1, x_2)$, as a function of x_1 for each fixed value of x_2, have a finite number of maxima and minima, and let $f_{x_2}(x_1, x_2)$ satisfy a similar condition as a function of x_2 for each fixed value of x_1. Let $0 < \delta < 1$ be a fixed number and let*

$$|f_{x_1}(x_1, x_2)| \le \delta, \quad |f_{x_2}(x_1, x_2)| \le \delta$$

for $(x_1, x_2) \in D$. Then

$$\sum_{(n_1,n_2)\in D} e^{2\pi i f(n_1,n_2)} = \int\int_D e^{2\pi i f(x_1,x_2)} dx_1 dx_2 + O(c_1) + O(c_2).$$

Lemma 3.2 (Krätzel [12, Theorem 2.21]). *Let $f(x_1, x_2)$ be a real function in D', and let H_1, H_2 be integers with $1 \le H_j \le c_j$ $(j = 1, 2)$. Let*

$$W = \sum_{(n_1,n_2)\in D} e^{2\pi i f(n_1,n_2)}.$$

Then we have

$$W \ll \frac{|D'|}{\sqrt{H_1 H_2}} + \left\{ \frac{|D'|}{H_1 H_2} \sum_{h_1=1}^{H_1-1} \sum_{h_2=0}^{H_2-1} |W_1| \right\}^{1/2} + \left\{ \frac{|D'|}{H_1 H_2} \sum_{h_1=0}^{H_1-1} \sum_{h_2=1}^{H_2-1} |W_2| \right\}^{1/2},$$

where

$$W_1 = \sum_{\substack{(n_1,n_2)\in D \\ (n_1+h_1,n_2+h_2)\in D}} e^{2\pi i (f(n_1+h_1,n_2+h_2)-f(n_1,n_2))},$$

and

$$W_2 = \sum_{\substack{(n_1,n_2)\in D \\ (n_1+h_1,n_2-h_2)\in D}} e^{2\pi i (f(n_1+h_1,n_2-h_2)-f(n_1,n_2))}.$$

Further, we denote the Hessian of the function $f(x_1, x_2)$ by

$$H(f) = \frac{\partial(f_{x_1}, f_{x_2})}{\partial(x_1, x_2)} = f_{x_1x_1}(x_1, x_2) f_{x_2x_2}(x_1, x_2) - f_{x_1x_2}^2(x_1, x_2).$$

Lemma 3.3 (Krätzel [12, Lemma 2.6]). *Suppose that*

$$\lambda_j \le |f_{x_j x_j}(x_1, x_2)| \ll \lambda_j \ (j = 1, 2), \qquad |f_{x_1x_2}(x_1, x_2)| \ll \sqrt{\lambda_1 \lambda_2}$$

and

$$H(f) \gg \lambda_1 \lambda_2$$

throughout the rectangle D_1*. For all parts of the curve of boundary let* $x_2 = \text{const}$ *or* $x_1 = \rho(x_2)$*, where* $\rho(x)$ *is partly twice differential and* $|\rho''(x)| \ll r$*. Then we have*

$$\iint_D e^{2\pi i f(x_1,x_2)} dx_1 dx_2 \ll \frac{1 + \log|D_1| + |\log\lambda_1| + |\log\lambda_2|}{\sqrt{\lambda_1\lambda_2}} + \frac{c_2 r}{\lambda_2}. \tag{3.1}$$

Lemma 3.4 (Krätzel [12, Theorem 2.16]). *Suppose that*

$$|f_{x_j x_j}(x_1, x_2)| \asymp \lambda_j \ (j = 1, 2), \quad |f_{x_1 x_2}(x_1, x_2)| \ll \sqrt{\lambda_1\lambda_2}$$

and

$$|H(f)| \gg \lambda_1\lambda_2$$

throughout the rectangle D_1*. For all parts of the curve of boundary let* $x_2 = \mathit{const}$ *or* $x_1 = \rho(x_2)$*, where* $\rho(x)$ *is partly twice differentiable and* $|\rho''(x)| \ll r$*. If* R *is defined by*

$$R = 1 + \log|D_1| + |\log\lambda_1| + |\log\lambda_2| + c_2 r\sqrt{\frac{\lambda_1}{\lambda_2}},$$

then we have

$$\begin{aligned} &\sum_{(n_1,n_2)\in D} e(f(n_1, n_2)) \\ &\ll \left(c_1\lambda_1 + c_2\sqrt{\lambda_1\lambda_2} + 1\right)\left(c_2\lambda_2 + c_1\sqrt{\lambda_1\lambda_2} + 1\right)\frac{R}{\sqrt{\lambda_1\lambda_2}}. \end{aligned} \tag{3.2}$$

Let M and N be positive integers such that $M < N$. In the next lemma we shall give the partial summation formula for the double sum $\sum_{M<m\le n\le N} f(m, n)g(m, n)$ where $f(x, y)$ is a C^2-function on $[M, N] \times [M, N]$ and $g(m, n)$ is an arithmetical function on the same domain. Let

$$G(x, y) = \sum_{x<m\le n\le y} g(m, n).$$

Lemma 3.5. *Let the notation be as above. Suppose that*

$$\begin{aligned} |G(x, y)| &\le G, & |f_x(x, y)| &\le \kappa_1, \\ \left|f_y(x, y)\right| &\le \kappa_2, & |f_{xy}(x, y)| &\le \kappa_3 \end{aligned}$$

for any $M \le x,\ y \le N$.

Then we have

$$\left|\sum_{M<m\le n\le N} f(m,n)g(m,n)\right| \\ \le G\Big(|f(M,N)| + (\kappa_1+\kappa_2)(N-M) + \kappa_3(N-M)^2\Big). \tag{3.3}$$

Proof. We shall apply partial summation twice. Let

$$V(n) = \sum_{M<m\le n} f(m,n)g(m,n).$$

Then we can write

$$J := \sum_{M<m\le n\le N} f(m,n)g(m,n) = \sum_{M<n\le N} V(n). \tag{3.4}$$

By using partial summation to the sum $V(n)$, we have

$$V(n) = f(n,n)H(n,n) - \int_M^n f_x(x,n)H(x,n)dx \tag{3.5}$$

with

$$H(x,n) = \sum_{M<m\le x} g(m,n).$$

Substituting (3.5) into (3.4), we have

$$J = \sum_{M<n\le N} f(n,n)H(n,n) - \sum_{M<n\le N} \int_M^n f_x(x,n)H(x,n)dx \\ = J_1 - J_2,$$

say. We apply partial summation again in the sums of J_1 and J_2, namely we have

$$J_1 = f(N,N)\sum_{M<n\le N} H(n,n) - \int_M^N \frac{d}{dx}f(x,x)\Big(\sum_{M<n\le x} H(n,n)\Big)dx \\ = f(N,N)G(M,N) - \int_M^N \frac{d}{dx}f(x,x)G(M,x)dx \\ = f(N,N)G(M,N) - \int_M^N \Big(f_x(x,x)+f_y(x,x)\Big)G(M,x)dx,$$

and

$$\begin{aligned}
J_2 &= \int_M^N \sum_{x<n\le N} f_x(x,n)H(x,n)dx \\
&= \int_M^N \left\{ f_x(x,N) \sum_{x<n\le N} H(x,n) - \int_x^N f_{xy}(x,y)\Big(\sum_{x<n\le y} H(x,n)\Big) dy \right\} dx \\
&= \int_M^N \Big\{ f_x(x,N)\Big(G(M,N)-G(M,x)-G(x,N)\Big) \\
&\quad - \int_x^N f_{xy}(x,y)\Big(G(M,y)-G(M,x)-G(x,y)\Big)dy \Big\} dx \\
&= G(M,N)\Big(f(N,N)-f(M,N)\Big) \\
&\quad - \int_M^N \Big(f_x(x,N)G(x,N)+f_x(x,x)G(M,x)\Big)dx \\
&\quad - \int_M^N \int_x^N f_{xy}(x,y)\Big(G(M,y)-G(x,y)\Big)dy\,dx.
\end{aligned}$$

Hence we have

$$\begin{aligned}
J = f(M,N)G(M,N) &+ \int_M^N \Big(f_x(x,N)G(x,N) - f_y(x,x)G(M,x)\Big)dx \\
&+ \int_M^N \int_x^N f_{xy}(x,y)\Big(G(M,y)-G(x,y)\Big)dydx.
\end{aligned} \tag{3.6}$$

Our assertion follows by taking the absolute value in the right-hand side of (3.6). □

4. Proof of Theorem 1.1

Let $s_j = \sigma_j + it_j$ $(j = 1, 2)$ be complex variables with $|t_1| \asymp |t_2|$. We take a parameter τ such that $\max\{|t_1|, |t_2|, |t_1 + t_2|\} + 2 \le \tau \ll |t_1|$.

Assuming that $\Re s_j = \sigma_j > 1$ $(j = 1, 2)$, we divide the double series (1.1) as

$$\begin{aligned}
\zeta_2(s_1,s_2) &= \sum_{m<n\le\tau} \frac{1}{m^{s_1}n^{s_2}} + \sum_{\substack{m<n \\ n>\tau}} \frac{1}{m^{s_1}n^{s_2}} \\
&=: S_1(s_1,s_2) + S_2(s_1,s_2),
\end{aligned} \tag{4.1}$$

say. After analytic continuation of the infinite sum $S_2(s_1, s_2)$, we consider the order of magnitude of these sums in the range

$$0 \le \sigma_j < 1 \ \ (j = 1, 2).$$

4.1. Evaluation of $S_2(s_1, s_2)$

First we shall consider the estimate of $S_2(s_1, s_2)$. Since n runs over the integers greater than $\tau > |t_1|$, we can use Corollary 2.3 to obtain

$$\begin{aligned} S_2(s_1, s_2) &= \sum_{n>\tau} \frac{1}{n^{s_2}} \left(\sum_{m \le n} \frac{1}{m^{s_1}} - \frac{1}{n^{s_1}} \right) \\ &= \zeta(s_1) \sum_{n>\tau} \frac{1}{n^{s_2}} + \frac{1}{1-s_1} \sum_{n>\tau} \frac{1}{n^{s_1+s_2-1}} - \frac{1}{2} \sum_{n>\tau} \frac{1}{n^{s_1+s_2}} \\ &\quad - \frac{s_1}{12} \sum_{n>\tau} \frac{1}{n^{s_1+s_2+1}} + O\left(|s_1|^3 \sum_{n>\tau} \frac{1}{n^{\sigma_1+\sigma_2+3}} \right) \\ &=: I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned} \tag{4.2}$$

say.

Since $\sigma_1 + \sigma_2 \ge 0$, the sum in I_5 converges absolutely, and we have

$$I_5 \ll \tau^{1-(\sigma_1+\sigma_2)}. \tag{4.3}$$

The analytic continuation and the estimate of the sum $\sum_{n>\tau} \frac{1}{n^w}$ in the range $\Re w \le 1$ are also given by Corollary 2.3 under the condition $\tau > |\Im w|/4$. For the estimate of I_1, we have

$$\begin{aligned} I_1 &= \zeta(s_1) \left\{ \frac{\tau^{1-s_2}}{s_2 - 1} - \frac{1}{2} \tau^{-s_2} + \frac{s_2}{12} \tau^{-s_2-1} + O\left(\tau^{-\sigma_2}\right) \right\} \\ &\ll |\zeta(s_1)| \tau^{-\sigma_2}, \end{aligned} \tag{4.4}$$

and for I_2, we have

$$\begin{aligned} I_2 = \frac{1}{1-s_1} &\left\{ \frac{\tau^{2-(s_1+s_2)}}{s_1+s_2-2} - \frac{1}{2} \tau^{1-(s_1+s_2)} \right. \\ &\left. \quad + \frac{s_1+s_2-1}{12} \tau^{-(s_1+s_2)} + O\left(\tau^{1-(\sigma_1+\sigma_2)}\right) \right\} \\ \ll \tau^{1-(\sigma_1+\sigma_2)}&. \end{aligned} \tag{4.5}$$

Similarly we have

$$I_j \ll \tau^{1-(\sigma_1+\sigma_2)}, \quad (j = 3, 4). \tag{4.6}$$

Combining (4.3), (4.4), (4.5) and (4.6), we have

$$S_2(s_1, s_2) \ll \tau^{\max\{\mu(\sigma_1), 1-\sigma_1\} - \sigma_2},$$

in particular, for $\sigma_1 < 1$,

$$S_2(s_1, s_2) \ll \tau^{1-(\sigma_1+\sigma_2)}. \tag{4.7}$$

4.2. Evaluation of $S_1(s_1, s_2)$

We shall consider the estimate of $S_1(s_1, s_2)$. Let $2 \le M \le \tau/2$. We define, for $\sigma_j \ge 0$ $(j = 1, 2)$,

$$T(s_1, s_2; M) = \sum_{M<m<n\le 2M} \frac{1}{m^{s_1} n^{s_2}}$$

and

$$U(s_1, s_2; M) = \sum_{m\le M} \frac{1}{m^{s_1}} \sum_{M<n\le 2M} \frac{1}{n^{s_2}}.$$

Since S_1 can be written as

$$S_1(s_1, s_2) = \sum_{j=1}^{\left[\frac{\log 2\tau}{\log 2}\right]} \left\{T(s_1, s_2; 2^{-j}\tau) + U(s_1, s_2; 2^{-j}\tau)\right\}, \tag{4.8}$$

it is enough to consider the estimates for $T(s_1, s_2; M)$ and $U(s_1, s_2; M)$.

First we consider the case $\sigma_1 = \sigma_2 = 0$. Applying Lemma 3.4 to the function $f(x_1, x_2) = -\frac{1}{2\pi}(t_1 \log x_1 + t_2 \log x_2)$ and noting that $\tau \asymp |t_j|$ and $M \le \tau/2$, we have

$$T(it_1, it_2; M) \ll \tau \log \tau. \tag{4.9}$$

As for the term $U(it_1, it_2; M)$, we have from (2.19)

$$U(it_1, it_2; M) \ll \tau \log \tau. \tag{4.10}$$

From (4.8), (4.9) and (4.10), we have

$$S_1(it_1, it_2) \ll \tau \log^2 \tau. \tag{4.11}$$

The proof of (1.7) follows from (4.11) in conjunction with (4.7).

Next we consider the case of $\sigma_1 + \sigma_2 > 0$.

Estimation of $T(s_1, s_2; M)$. To consider the upper bounds for $T(s_1, s_2; M)$, we divide the region into three parts:

$$M \ll \tau^{\frac{1}{3}}, \ \tau^{\frac{1}{3}} \ll M \ll \tau^{\frac{2}{3}}, \ \tau^{\frac{2}{3}} \ll M \ll \tau.$$

Let $j_0 = [\log \tau / 3 \log 2]$ and $N = 2^{-j_0}\tau \asymp \tau^{2/3}$. To reduce the evaluation of $T(\sigma_1 + it_1, \sigma_2 + it_2; M)$ into that of $T(it_1, it_2; M)$, we apply Lemma 3.5 with $f(x, y) = \frac{1}{x^{\sigma_1} y^{\sigma_2}}$ and $g(m, n) = e^{-i(t_1 \log m + t_2 \log n)}$. Thus we have

$$\begin{aligned} &T(s_1, s_2; M) \\ &= \sum_{M<m\le n\le 2M} \frac{1}{m^{\sigma_1+it_1} n^{\sigma_2+it_2}} - \sum_{M<n\le 2M} \frac{1}{n^{\sigma_1+\sigma_2+i(t_1+t_2)}} \\ &\ll M^{-\sigma_1-\sigma_2}\left\{\max_{M<x<y\le 2M}\left|\sum_{x<m\le n\le y} \frac{1}{m^{it_1} n^{it_2}}\right| + \max_{M<u\le 2M}\left|\sum_{M<n\le u} \frac{1}{n^{i(t_1+t_2)}}\right|\right\}, \end{aligned}$$

and hence

$$\sum_{j \le j_0} T(s_1, s_2; 2^{-j}\tau) \ll \frac{1}{N^{\sigma_1+\sigma_2}} \tau \log^2 \tau \ll \tau^{1-\frac{2}{3}(\sigma_1+\sigma_2)} \log^2 \tau \tag{4.12}$$

by (2.19) and (4.9).

On the other hand, for $M \ll \tau^{1/3}$, it follows that

$$T(s_1, s_2; M) = \sum\sum_{M<m<n\le 2M} \frac{1}{m^{s_1} n^{s_2}} \ll M^{2-\sigma_1-\sigma_2} \log M.$$

We take $j_1 = [2\log\tau/3\log 2]$, then

$$\sum_{j > j_1} T(s_1, s_2; 2^{-j}\tau) \ll \tau^{\frac{2}{3}-\frac{1}{3}(\sigma_1+\sigma_2)} \log \tau. \tag{4.13}$$

We use Lemma 3.2 with $H_1 = H_2 = H$ to consider the estimate of $T(s_1, s_2; M)$ for $\tau^{1/3} \ll M \ll \tau^{2/3}$, where H is chosen later. Let $M < M' \le 2M$ and

$$W = \sum\sum_{M<m<n\le M'} m^{it_1} n^{it_2} = \sum\sum_{M<m<n\le M'} e^{2\pi i f(m,n)},$$

where we put

$$f(x_1, x_2) = \frac{1}{2\pi}(t_1 \log x_1 + t_2 \log x_2).$$

For each $1 \le h_j \le H$ $(j = 1, 2)$, we define

$$D_{h_1,h_2} = \{(m, n) \in \mathbb{Z}^2 \mid M < m < n \le M', M < m + h_1 < n + h_2 < M'\}$$

and

$$D'_{h_1,h_2} = \{(m, n) \in \mathbb{Z}^2 \mid M < m < n \le M', M < m + h_1 < n - h_2 < M'\}.$$

By Lemma 3.2, we have

$$W \ll \frac{M^2}{H} + \frac{M}{H}\left\{\left(\sum_{h_1=1}^{H-1}\sum_{h_2=0}^{H-1} |W_1(h_1, h_2)|\right)^{\frac{1}{2}} + \left(\sum_{h_1=0}^{H-1}\sum_{h_2=1}^{H-1} |W_2(h_1, h_2)|\right)^{\frac{1}{2}}\right\}, \tag{4.14}$$

where

$$W_1(h_1, h_2) = \sum_{(m,n)\in D_{h_1,h_2}} e^{2\pi i (f(m+h_1, n+h_2) - f(m,n))}$$

and

$$W_2(h_1, h_2) = \sum_{(m,n)\in D'_{h_1,h_2}} e^{2\pi i (f(m+h_1, n-h_2) - f(m,n))}.$$

Now we treat the sum W_1. Denote

$$g(x_1, x_2) = f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2),$$

then

$$g_{x_1}(x_1, x_2) = \frac{t_1}{2\pi}\left(\frac{1}{x_1 + h_1} - \frac{1}{x_1}\right), \quad g_{x_2}(x_1, x_2) = \frac{t_2}{2\pi}\left(\frac{1}{x_2 + h_2} - \frac{1}{x_2}\right)$$

$$g_{x_1x_1}(x_1, x_2) = \frac{t_1}{2\pi}\frac{h_1(2x_1 + h_1)}{x_1^2(x_1 + h_1)^2}, \qquad g_{x_2x_2}(x_1, x_2) = \frac{t_2}{2\pi}\frac{h_2(2x_2 + h_2)}{x_2^2(x_2 + h_2)^2}.$$

Consider the case $h_2 \neq 0$ firstly. We divide the triangular region D_{h_1,h_2} into the squares of side l, or parts of such squares

$$\Delta_{p,q} = \{(x, y) \mid M + pl < x \le M + (p+1)l,\ M + ql < y \le M + (q+1)l\} \cap D_{h_1,h_2}$$

with

$$l = \frac{AM^3}{\tau H} \tag{4.15}$$

where A is a small constant.

For a fixed $(\alpha, \beta) \in \Delta_{p,q}$, we get

$$g_{x_1}(x_1, x_2) - g_{x_1}(\alpha, \beta) \ll \frac{\tau H}{2\pi}\frac{lM}{M^2} = A,$$

and if A is small enough, the total variation of g_{x_1} in $\Delta_{p,q}$ is smaller than $\frac{3}{4}$, and so is g_{x_2}. Hence there are two integers P and Q such that

$$\left|g_{x_1}(x_1, x_2) - P\right| \le \frac{3}{4} \quad \text{and} \quad \left|g_{x_2}(x_1, x_2) - Q\right| \le \frac{3}{4}$$

for any $(x_1, x_2) \in \Delta_{p,q}$. Now putting

$$G(x_1, x_2) = g(x_1, x_2) - 2\pi(Px_1 + Qx_2),$$

then we have, from Lemma 3.1,

$$\sum_{(m,n)\in\Delta_{p,q}} e^{2\pi i g(m,n)} = \iint_{\Delta_{p,q}} e^{2\pi i G(x_1,x_2)} dx_1 dx_2 + O(l).$$

Since

$$G_{x_1 x_1}(x_1, x_2) \asymp \lambda_j = \frac{\tau h_j}{M^3} \quad (j = 1, 2),$$

by applying Lemma 3.3, we have

$$\iint_{\Delta_{p,q}} e^{2\pi i G(x,y)} dx dy \ll \frac{\log\tau}{\tau}\frac{M^3}{\sqrt{h_1 h_2}}.$$

Now the number of $\Delta_{p,q}$ is at most $O(M^2/l^2)$, thus we have

$$\begin{aligned} W_1(h_1,h_2) &\ll \sum_{p,q}\left|\iint_{\Delta_{p,q}} e^{2\pi i G(x,y)}dxdy + O(l)\right| \\ &\ll \left(\frac{\log\tau}{\tau}\frac{M^3}{\sqrt{h_1h_2}}+l\right)\frac{M^2}{l^2} \\ &\ll \tau\log\tau\frac{H^2}{\sqrt{h_1h_2}}\frac{1}{M}. \end{aligned}$$

by (4.15).

When $h_2=0$, we have, by E. Krätzel [12, Theorem 2.1],

$$\begin{aligned} W_1(h_1,0) &= \sum_{M+h_1<n\le M'}\ \sum_{M<m<n} e^{it_1(\log(m+h_1)-\log m)} \\ &\ll \sum_{M+h_1<n\le M'}\frac{\tau h_1/M^2}{\sqrt{\tau h_1/M^3}} \\ &\ll \sqrt{\tau M h_1} \end{aligned}$$

Similar estimates can be obtained for $W_2(h_1,h_2)$.

Therefore, we obtain

$$\begin{aligned} W &\ll \frac{M^2}{H}+\frac{M}{H}\left\{\left(\sum_{h_1=1}^{H-1}\sum_{h_=1}^{H-1}\frac{\tau\log\tau}{M}\frac{H^2}{\sqrt{h_1h_2}}\right)^{\frac12}\right. \\ &\qquad\left. +\left(\sum_{h_1=1}^{H-1}\sqrt{\tau M h_1}\right)^{\frac12}+\left(\sum_{h_2=1}^{H-1}\sqrt{\tau M h_2}\right)^{\frac12}\right\} \\ &\ll \frac{M^2}{H}+(MH\tau\log\tau)^{\frac12}+\left(\frac{\tau M^5}{H}\right)^{\frac14}. \end{aligned}$$

Taking $H=M/\tau^{1/3}$, we get

$$W \ll M\tau^{\frac13}(\log\tau)^{\frac12}. \tag{4.16}$$

By Lemma 3.5 and (4.16), we have

$$T(s_1,s_2;M)\ll M^{1-\sigma_1-\sigma_2}\tau^{\frac13}\log^{\frac12}\tau.$$

It follows that

$$\sum_{j_0<j\le j_1} T(s_1,s_2;2^{-j}\tau)\ll \begin{cases} \tau^{1-\frac23(\sigma_1+\sigma_2)}\log^{\frac32}\tau & \sigma_1+\sigma_2\le 1 \\ \tau^{\frac23-\frac13(\sigma_1+\sigma_2)}\log^{\frac32}\tau & \sigma_1+\sigma_2>1. \end{cases} \tag{4.17}$$

From (4.12), (4.13) and (4.17) we have

$$\sum_j T(s_1, s_2; 2^{-j}\tau) \ll \left(\tau^{1-\frac{2}{3}(\sigma_1+\sigma_2)} + \tau^{\frac{2}{3}-\frac{1}{3}(\sigma_1+\sigma_2)}\right) \log^2 \tau. \tag{4.18}$$

Remark 4.1. In the case $W_1(h_1, 0)$, we note that $g_{x_2x_2} = 0$, $H(g) = 0$, since $g(x_1, x_2) = \frac{t_1}{2\pi}(\log(x_1 + h_1) - \log x_1)$. This is the reason that we used E. Krätzel [12, Theorem 2.1]. The situation is the same for $W_2(0, h_2)$.

Estimation of $U(s_1, s_2; M)$

To treat the sum of $U(s_1, s_2, M)$, we shall apply Lemma 2.5. Noting that $\tau \asymp |t_j|$, we have

$$\sum_{1\le m\le M} \frac{1}{m^{s_1}} \ll \begin{cases} \tau^{\frac{1}{2}-\frac{2}{3}\sigma_1} \log \tau & \left(0 \le \sigma_1 \le \frac{1}{2}\right) \\ \tau^{\frac{1}{3}-\frac{1}{3}\sigma_1} \log^2 \tau & \left(\frac{1}{2} < \sigma_1 < 1\right) \end{cases}$$

and

$$\sum_{M<n\le 2M} \frac{1}{n^{s_2}} \ll \begin{cases} \tau^{\frac{1}{2}-\frac{2}{3}\sigma_2} & \left(0 \le \sigma_2 \le \frac{1}{2}\right) \\ \tau^{\frac{1}{3}-\frac{1}{3}\sigma_2} \log \tau & \left(\frac{1}{2} < \sigma_2 < 1\right). \end{cases}$$

Collecting these estimates, we obtain that

$$\sum_j U(s_1, s_2; 2^{-j}\tau) \ll \begin{cases} \tau^{1-\frac{2}{3}(\sigma_1+\sigma_2)} \log^2 \tau & \left(0\le\sigma_1\le\frac{1}{2}, 0\le\sigma_2\le\frac{1}{2}\right) \\ \tau^{\frac{5}{6}-\frac{1}{3}(\sigma_1+2\sigma_2)} \log^3 \tau & \left(\frac{1}{2}<\sigma_1<1, 0\le\sigma_2\le\frac{1}{2}\right) \\ \tau^{\frac{5}{6}-\frac{1}{3}(2\sigma_1+\sigma_2)} \log^3 \tau & \left(0\le\sigma_1\le\frac{1}{2}, \frac{1}{2}<\sigma_2 < 1\right) \\ \tau^{\frac{2}{3}-\frac{1}{3}(\sigma_1+\sigma_2)} \log^4 \tau & \left(\frac{1}{2}<\sigma_1<1, \frac{1}{2}<\sigma_2<1\right). \end{cases} \tag{4.19}$$

From (4.7), (4.18) and (4.19), we get the assertion (1.8).

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Department of Mathematical Sciences,
Faculty of Science,
Yamaguchi University,
Yoshida 1677-1 Yamaguchi 753-8512, Japan
kiuchi@yamaguchi-u.ac.jp

Graduate School of Mathematics,
Nagoya University,
Nagoya, 464-8602, Japan
tanigawa@math.nagoya-u.ac.jp