Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. V (2006), 549-577

Simultaneous unitarizability of $\mathrm{SL}_n\mathbb{C}$-valued maps, and constant mean curvature k-noid monodromy

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Dedicated to Professor Takeshi Sasaki on his sixtieth birthday

Abstract. We give necessary and sufficient local conditions for the simultaneous unitarizability of a set of analytic matrix maps from an analytic 1-manifold into $\mathrm{SL}_n\mathbb{C}$ under conjugation by a single analytic matrix map.

We apply this result to the monodromy arising from an integrable partial differential equation to construct a family of k-noids, genus-zero constant mean curvature surfaces with three or more ends in Euclidean, spherical and hyperbolic 3-spaces.

Mathematics Subject Classification (2000): 53C42 (primary); 53A35, 49Q10 (secondary).

Introduction

In this paper we find necessary and sufficient conditions for the existence of an $\mathrm{SL}_n\mathbb{C}$-valued analytic matrix map on an analytic 1-manifold which simultaneously unitarizes a given set of analytic matrix maps via conjugation. We apply these results to construct families of constant mean curvature (CMC) immersions with arbitrarily many ends into ambient 3-dimensional space forms (Theorem 8.5). The ends are conjectured to be asymptotic to half-Delaunay surfaces.

We show that the existence of a global unitarizer is equivalent to the existence of analytic unitarizers defined only on local neighborhoods (Theorem 1.10). In the case of $\mathrm{SL}_2\mathbb{C}$ the necessary and sufficient conditions for global simultaneous unitarization are local diagonalizability, pointwise simultaneous unitarizability, and pairwise infinitesimal irreducibility (Theorem 3.4). This latter condition means that at each point of the 1-manifold, the coefficient of the leading term of the series expansion of the commutator has full rank. For general $\mathrm{SL}_n\mathbb{C}$, global unitarizability is equivalent to pointwise unitarizability together with a graph condition (Theorem 2.7). These results are proven by linearizing the unitarization problem and applying analytic Cholesky decompositions.

The Unitarization Theorem 2.7 is a refinement of the variant r-unitarization Theorem 4.4 (see [22, 6] for the case of $\mathrm{SL}_2\mathbb{C}$). In Theorem 4.4, the analytic curve

Received May 31, 2006; accepted in revised form October 23, 2006.

is the standard unit circle $\mathbb{S}^1$, and an analytic simultaneous unitarizer C is found on a radius-r circle for some r less than 1.

While the unitarized loops extend holomorphically to $\mathbb{S}^1$, the unitarizer C generally has branch points. The conditions of the Unitarization Theorem 2.7 are the obstructions to extending C holomorphically to $\mathbb{S}^1$.

One application of the Unitarization Theorem 2.7 is to solving the monodromy problem arising in the construction of CMC surfaces via the extended Weierstrass representation. In this construction, using integrable system methods, the problem of closing the surface is solved by unitarizing the monodromy group, whose elements are defined on a loop. The unitarization theorem provides a construction of a global analytic simultaneous unitarizer once it is known that the monodromy group is pointwise simultaneously unitarizable along the loop.

Hence in the second part of the paper we apply the unitarization theorem to the construction of CMC genus-zero surfaces with arbitrary numbers $k \geq 3$ of ends which are asymptotic to half-Delaunay surfaces [12], lying in ambient 3-dimensional space forms (see Figures 1.1 and 1.2). We call these surfaces k-noids, and trinoids when $k = 3$. For $k = 2$ these are the well-known Delaunay surfaces, CMC surfaces of revolution with translational periodicity.

Trinoids in $\mathbb{R}^3$, either embedded or non-Alexandrov-embedded, were first constructed by Kapouleas [11] using techniques that glue parts of CMC surfaces together via the study of Jacobi operators, and later there has been further work in this direction [16, 19], but this approach only gives examples that are in some sense "close" to the boundary of the moduli space of the surfaces.

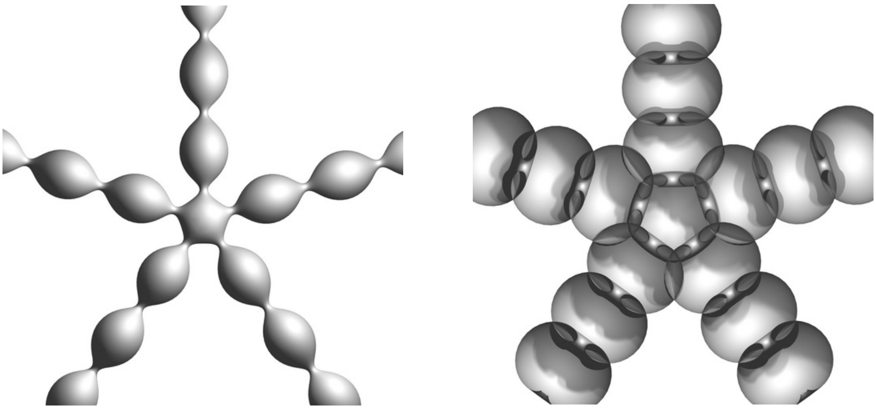

Figure 1.1. Symmetric CMC 5-noids in $\mathbb{R}^3$, one with unduloidal ends and one with nodoidal ends (cutaway view). The images were produced by CMCLab [20].

A construction that gives a broader collection of Alexandrov embedded trinoids [9, 10] and k-noids [8] in $\mathbb{R}^3$ with embedded ends asymptotic to Delaunay unduloids was found by Große-Brauckmann, Kusner and Sullivan, using an isometric correspondence between minimal surfaces in the 3-dimensional sphere $\mathbb{S}^3$ and CMC 1 surfaces in $\mathbb{R}^3$. The family of k-noids we construct here also includes surfaces with asymptotically Delaunay ends [12].

Constant mean curvature trinoids in $\mathbb{R}^3$ with embedded ends via loop group techniques are constructed in [22, 5] by methods derived from integrable systems techniques. Developed initially by Dorfmeister, Pedit and Wu [4], this construction employs the r-Unitarization Theorem together with the r-Iwasawa decomposition [17], a generalization of the Iwasawa decomposition on the unit circle to a radius-r circle with $r < 1$.

These trinoids extend to larger classes of non-Alexandrov-embedded trinoids: In one way, Kilian, Sterling and the second author [13] dressed these trinoids into "bubbleton" versions, also conformal to thrice-punctured spheres; these bubbleton versions have ends asymptotic to embedded Delaunay unduloids [12], and computer graphics suggest that they are not Alexandrov embedded. In another way, Kilian, Kobayashi and the authors [22] extended the class of trinoids to CMC 1 surfaces in $\mathbb{R}^3$ whose potentials are perturbations of nonembedded Delaunay nodoid potentials, and hence are not Alexandrov embedded [12]. Also, in [22], trinoids in $\mathbb{S}^3$ and hyperbolic 3-space $\mathbb{H}^3$ were proved to exist, including examples that are not Alexandrov embedded.

In this work, we establish the closing conditions for trinoids and symmetric k-noids via loop group techniques by a more elementary approach using only the 1-unitarization theorem and 1-Iwasawa decomposition.

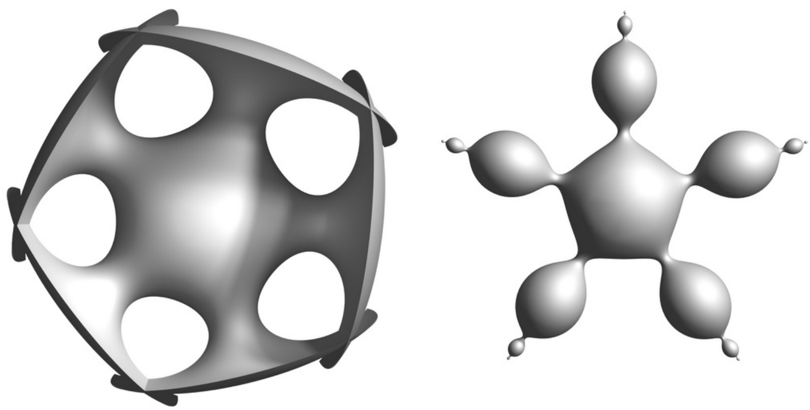

Figure 1.2. Symmetric CMC 5-noids in $\mathbb{S}^3$ and $\mathbb{H}^3$. Here, $\mathbb{S}^3$ has been stereographically projected to $\mathbb{R}^3 \cup \{\infty\}$, and $\mathbb{H}^3$ is shown in the Poincaré model.

1. Simultaneous unitarizability for $\mathrm{SL}_n\mathbb{C}$

1.1. Preliminaries

An *analytic curve* is a connected real analytic one-dimensional manifold without boundary. On an analytic curve $\mathcal{C}$ we denote by $\mathcal{H}_\mathcal{C}\mathcal{V}$ the set of analytic maps $\mathcal{C} \to \mathcal{V}$ into a space $\mathcal{V}$, and by $\mathcal{M}_\mathcal{C}\mathcal{V}$ the set of analytic maps $\mathcal{C} \to \mathcal{V}$ with possible poles.

$M \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ is *locally diagonalizable at* $p \in \mathcal{C}$ if there exists a neighborhood $\mathcal{U} \subseteq \mathcal{C}$ of p and $V \in \mathcal{H}_{\mathcal{U}}\mathrm{SL}_n\mathbb{C}$ such that VMV^{-1} is diagonal.

$M_1, \ldots, M_q \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ are *locally simultaneously unitarizable at* $p \in \mathcal{C}$ if there exists a neighborhood $\mathcal{U} \subseteq \mathcal{C}$ of p and $V \in \mathcal{H}_{\mathcal{U}}\mathrm{SL}_n\mathbb{C}$ such that $VM_1V^{-1}, \ldots$ $\ldots, VM_qV^{-1} \in \mathcal{H}_{\mathcal{U}}\mathrm{SU}_n\mathbb{C}$.

$M_1, \ldots, M_q \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ are *simultaneously unitarizable* on $\mathcal{C}$ if there exists $V \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ such that $VM_1V^{-1}, \ldots, VM_qV^{-1} \in \mathcal{H}_{\mathcal{C}}\mathrm{SU}_n\mathbb{C}$.

For $f \in \mathcal{M}_{\mathcal{C}}\mathbb{C}$, define $f^* = \overline{f}$, and for $M \in \mathcal{M}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$, define $M^* = \overline{M}^t$.

For $0 < r < s < \infty$, let $\mathcal{A}_{r,s} \subset \mathbb{C}$ denote the open annulus $\mathcal{A}_{r,s} = \{\lambda \in \mathbb{C} \mid r < |\lambda| < s\}$.

A subset $\mathcal{G} \subset \mathrm{M}_{n\times n}\mathbb{C}$ is *reducible* if there exists a proper non-zero subspace $V \subset \mathbb{C}^n$ such that $GV \subset V$ for all $G \in \mathcal{G}$. In the case $\mathcal{G} = \{A, B\}$ is a set of two elements, we say $A, B \in \mathrm{M}_{n\times n}\mathbb{C}$ are reducible.

We have the following Schur-type lemma.

Lemma 1.1. *If $A, B \in \mathrm{M}_{n\times n}\mathbb{C}$ are irreducible and $X \in \mathrm{M}_{n\times n}\mathbb{C}$ commutes with A and B, then X is a multiple of the identity matrix* $\mathrm{I} \in \mathrm{M}_{n\times n}\mathbb{C}$.

Proof. Suppose $X \notin \mathbb{C}\,\mathrm{I}$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of X. Let $V = \{v \in \mathbb{C}^n \mid Xv = \lambda v\}$, so $V \neq \{0\}$. Then $V \subset \mathbb{C}^n$ is a proper subspace because $X \notin \mathbb{C}\,\mathrm{I}$.

For any $v \in V$, we have $XAv = AXv = \lambda Av$, so $Av \in V$. Hence $AV \subseteq V$. Similarly, $BV \subseteq V$, so A and B would be reducible. □

1.2. Linearizing the simultaneous unitarization problem

We begin with an elementary proof of a specialization of standard results in the theory of holomorphic vector bundles, constructing a global kernel of a suitable bundle map which depends holomorphically (or real analytically) on a parameter.

Lemma 1.2. *Let $\mathcal{D}$ be an analytic* 1*-manifold or* 2*-manifold, $N, M \in \mathbb{N}$, and $L : \mathcal{D} \to \mathrm{Hom}(\mathbb{C}^{\mathrm{N}}, \mathbb{C}^{\mathrm{M}})$ a holomorphic map. Suppose* $\dim\ker L = 1$ *on $\mathcal{D}$ away from a subset $S \subset \mathcal{D}$ of isolated points. Then*

(i) $\dim\ker L \geq 1$ *on* $\mathcal{D}$.
(ii) *There exists a holomorphic map $X : \mathcal{D} \to \mathbb{C}^N$ which is not identically* 0 *such that $X \in \ker L$, that is, for each $p \in \mathcal{D}$, $X(p) \in \ker L(p)$.*

Proof. Since L has rank $N-1$ on $\mathcal{D} \setminus S$, all the $N \times N$ minor determinants of L are holomorphic on $\mathcal{D}$ and zero on $\mathcal{D} \setminus S$, and hence 0 on $\mathcal{D}$. Hence $\dim\ker L \geq 1$ on $\mathcal{D}$.

Since L has rank $N-1$ on $\mathcal{D} \setminus S$, then there exists an $(N-1) \times (N-1)$ minor determinant of L which is not identically 0 on $\mathcal{D}$. Hence by a permutation we may assume without loss of generality that

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathcal{H}_{\mathcal{D}}\mathrm{M}_{(N-1)\times(N-1)}\mathbb{C}$ with $a := \det A \not\equiv 0$ on $\mathcal{D}$, $B \in \mathcal{H}_{\mathcal{D}}\mathrm{M}_{(N-1)\times 1}\mathbb{C}$, $C \in \mathcal{H}_{\mathcal{D}}\mathrm{M}_{(M-N+1)\times(N-1)}\mathbb{C}$, and $D \in \mathcal{H}_{\mathcal{D}}\mathrm{M}_{(M-N+1)\times 1}\mathbb{C}$.

Define $X = (-aA^{-1}B,\ a)^t$. Then aA^{-1} is holomorphic on $\mathcal{D}$ because its entries are polynomials in the entries of A. Hence X is holomorphic on $\mathcal{D}$. Moreover, $X \not\equiv 0$ because $a \not\equiv 0$.

It is clear that the upper $(N-1)\times 1$ block of LX is 0. To show that the lower $(M-N+1)\times 1$ block of LX is zero, let $A_k \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{1\times(N-1)}\mathbb{C}$ and $B_k \in \mathcal{H}_{\mathcal{C}}\mathbb{C}$ be the respective k'th rows of A and B ($k \in \{1,\dots,N-1\}$). Then since $B - AA^{-1}B = 0$,

$$B_k - A_k A^{-1} B = 0, \quad k \in \{1,\dots,N-1\}. \tag{1.1}$$

Fix $j \in \{1,\dots,M-N+1\}$ and let $C_j \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{1\times(N-1)}\mathbb{C}$ and $D_j \in \mathcal{H}_{\mathcal{C}}\mathbb{C}$ be the respective j'th rows of C and D. Because all $N\times N$ minor determinants of L are 0, then $C_j = \sum r_k A_k$ and $D_j = \sum r_k B_k$ for some holomorphic scalar functions $r_1,\dots,r_{N-1}$. By (1.1), $C_j A^{-1} B = D_j$. Since this holds for all $j \in \{1,\dots,M-N+1\}$, we have $D - CA^{-1}B = 0$. Hence the lower $(M-N+1)\times 1$ block $a(D - CA^{-1}B)$ of LX is 0, $LX = 0$, and $X \in \ker L$. □

A global simultaneous unitarizer of a suitable set of analytic maps $M_1,\dots,M_q$ on an analytic curve is constructed in two steps. First, a global analytic solution X to the linear system $XM_kX^{-1} = M_k^{*-1}$, $k \in \{1,\dots,q\}$, is found which is Hermitian positive definite. Then the analytic Cholesky decomposition gives $X = V^*V$, and V is a simultaneous unitarizer of $M_1,\dots,M_q$.

We continue by defining a linear map L whose kernel will contain X.

Definition 1.3. For $M_1,\dots,M_q \in \mathrm{SL}_n\mathbb{C}$ ($n \geq 1$), define $L = \mathcal{L}(M_1,\dots,M_q)$ as the linear map $L : \mathrm{M}_{n\times n}\mathbb{C} \to (\mathrm{M}_{n\times n}\mathbb{C})^n$

$$L(X) = \left(XM_1 - M_1^{*-1}X, \dots, XM_q - M_q^{*-1}X\right).$$

L is similarly defined for $M_1,\dots,M_q \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$.

The linear map $L(M_1,\dots,M_q)$ has the following easily computed properties. The first of these properties motivates the definition of L.

Lemma 1.4. *With $n \geq 2$, let $M_1,\dots,M_q \in \mathrm{SL}_n\mathbb{C}$ and let $L = \mathcal{L}(M_1,\dots,M_q)$ be as in Definition* 1.3. *Then*

(i) *If $A \in \mathrm{SL}_n\mathbb{C}$ and $A^*A \in \ker L$, then A simultaneously unitarizes $M_1,\dots,M_q$.*
(ii) *If $X \in \ker L$, then $X^* \in \ker L$.*
(iii) *Let $C \in \mathrm{GL}_n\mathbb{C}$ and let $\widetilde{L} = \mathcal{L}(CM_1C^{-1},\dots,CM_qC^{-1})$. Then $X \in \ker\widetilde{L}$ if and only if $C^*XC \in \ker L$.*

We will require the following further properties of L.

Lemma 1.5. *With $n \geq 2$, let $M_1,\dots,M_q \in \mathrm{SL}_n\mathbb{C}$ and $L = \mathcal{L}(M_1,\dots,M_q)$. If for some $i, j \in \{1,\dots,q\}$, M_i and M_j are irreducible, and if $M_1,\dots,M_q$ are simultaneously unitarizable, then* $\dim\ker L = 1$.

Proof. Let C be a simultaneous unitarizer of $M_1, \ldots, M_q$, define $P_k = CM_kC^{-1} \in \mathrm{SU}_n$ ($k \in \{1, \ldots, n\}$), and set $\widetilde{L} = \mathcal{L}(P_1, \ldots, P_r)$. Since $P_k = P_k^{*-1}$, then $\ker \widetilde{L}$ is the set of $X \in \mathrm{M}_{n\times n}\mathbb{C}$ such that $[X, P_k] = 0$ ($k \in \{1, \ldots, q\}$). Since M_i and M_j are irreducible, then P_i and P_j are irreducible. By Lemma 1.1, $\ker \widetilde{L} = \mathbb{C} \otimes \mathrm{I}$. By Lemma 1.4(iii), $\ker L = \mathbb{C} \otimes (C^*C)$, so $\dim \ker L = 1$. $\square$

Lemma 1.6. *Let $M_1, M_2 \in \mathrm{SL}_n\mathbb{C}$ and $L = \mathcal{L}(M_1, M_2)$. Suppose M_1 and M_2 are irreducible and simultaneously unitarizable. Let $X \in \mathrm{M}_{n\times n}\mathbb{C} \setminus \{0\}$ and suppose $X \in \ker L$ and $X^* = X$. Then X is positive or negative definite.*

Proof. First assume the special case that $M_1, M_2 \in \mathrm{SU}_n$. Then $X \in \ker L$ means $[X, M_1] = 0$ and $[X, M_2] = 0$. Since M_1 and M_2 are irreducible, by Lemma 1.1, $X = r\,\mathrm{I}$ for some $r \in \mathbb{C}^*$. Since X is Hermitian, then $r \in \mathbb{R}^*$. Hence X is positive or negative definite.

To show the general case, let $C \in \mathrm{SL}_n\mathbb{C}$ be a simultaneous unitarizer of M_1 and M_2, and let $\widetilde{L} = \mathcal{L}(CM_1C^{-1}, CM_2C^{-1})$. Then $\widetilde{X} = (C^{-1})^* XC^{-1}$ is Hermitian, and $\widetilde{X} \in \ker \widetilde{L}$ by Lemma 1.4(iii). By the special case above, $\widetilde{X}$ is positive or negative definite. Hence X is positive or negative definite. $\square$

Lemma 1.7. *Let $\mathcal{C}$ be an analytic curve, and $M_1, \ldots, M_q \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ ($q \geq 2$), and let $L = \mathcal{L}(M_1, \ldots, M_q)$. Suppose for some subset $S \subset \mathcal{C}$ of isolated points, M_1 and M_2 are irreducible on $\mathcal{C} \setminus S$, and $M_1, \ldots, M_q$ are pointwise simultaneously unitarizable on $\mathcal{C} \setminus S$. Then there exists $X \in (\mathcal{H}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}) \cap \ker L$ with $X^* = X$ and a subset $S' \subset \mathcal{C}$ of isolated points such that X is positive definite on $\mathcal{C} \setminus S'$.*

Proof. By Lemma 1.5, $\dim \ker L = 1$ on $\mathcal{C} \setminus S$. By Lemma 1.2, $\dim \ker L \geq 1$ on $\mathcal{C}$ and there exists an analytic map $X_1 \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$ such that $X_1 \in (\ker L) \setminus \{0\}$.

If X_1 is Hermitian, define $X_2 = X_1$; otherwise define $X_2 = i(X_1 - X_1^*)$. Then $X_2 \not\equiv 0$, X_2 is Hermitian, and $X_2 \in \ker L$ by Lemma 1.4(ii).

By Lemma 1.6, X_2 is (pointwise) either positive definite or negative definite except at the set of isolated points at which $X_2 = 0$ or where $M_1, \ldots, M_q$ all commute.

Let $v \in \mathbb{C}^n \setminus \{0\}$ be any non-zero constant vector and define $f = v^* X_2 v$. Then $f \not\equiv 0$ and $X = fX_2$ is positive definite on $\mathcal{C}$ away from a set of isolated points. $\square$

1.3. Analytic Cholesky decompositions

We prove two analytic versions of the Cholesky decomposition theorem. The first (Proposition 1.8) is for Hermitian positive definite maps on an analytic curve, used in the Unitarization Theorem 1.10. The second (Proposition 4.2), used in the r-unitarization Theorem 4.4, is for meromorphic maps on $\mathbb{S}^1$ which are Hermitian positive definite except at a finite set of points.

Proposition 1.8 (Holomorphic Cholesky decomposition). *Let $\mathcal{C}$ be an analytic curve and let $X \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ be Hermitian positive definite. Then*

(i) *There exists $V \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ such that $X = V^*V$.*
(ii) *V is unique up to left multiplication by elements of $\mathcal{H}_{\mathcal{C}}\mathrm{SU}_n$.*

Proof. We first prove the following analytic version of the LDU-decomposition: if $X \in \mathcal{H}_{\mathcal{C}}\mathrm{GL}_n\mathbb{C}$ is Hermitian positive definite, then there exists $R, D \in \mathcal{H}_{\mathcal{C}}\mathrm{GL}_n\mathbb{C}$ such that $X = R^*DR$, where R is upper triangular with diagonal elements $\equiv 1$, and D is diagonal with diagonal elements in $\mathcal{H}_{\mathcal{C}}\mathbb{R}^{>0}$. The proof is by induction on n. The case $n = 1$ is clear, with $R = \mathrm{I}$, $D = X$.

Now assume the statement is true for $n - 1$ and write

$$X = \begin{pmatrix} X_0 & Y_0 \\ Y_0^* & z \end{pmatrix},$$

with $X_0 \in \mathcal{H}_{\mathcal{C}}\mathrm{GL}_{n-1}\mathbb{C}$, $Y_0 \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{(n-1)\times 1}\mathbb{C}$ and $z \in \mathcal{H}_{\mathcal{C}}\mathbb{C}$. Then X_0 is Hermitian positive definite, so let $X_0 = R_0^* D_0 R_0$ be the decomposition given by the induction hypothesis. Then D_0 and R_0^* are invertible on $\mathcal{C}$, so we can define $R, D \in \mathcal{H}_{\mathcal{C}}\mathrm{GL}_n\mathbb{C}$ by

$$R = \begin{pmatrix} R_0 & S_0 \\ 0 & 1 \end{pmatrix}, \quad S_0 = (R_0^* D_0)^{-1} Y_0, \quad D = \begin{pmatrix} D_0 & 0 \\ 0 & d \end{pmatrix}, \quad d = z - S_0^* D_0 S_0.$$

Then we have $X = R^*DR$. Taking the determinant yields $\det X = \det D = d \det D_0$, showing that d takes values in $\mathbb{R}^{>0}$. This proves the statement for n.

To show the first part of the theorem, take Hermitian positive definite $X \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$, and let $X = R^*DR$ be its analytic LDU factorization. With $D = \mathrm{diag}(d_1, \dots, d_n)$, let $E = \mathrm{diag}(\sqrt{d_1}, \dots, \sqrt{d_n})$, choosing positive square roots. Let $V = ER$, so $X = V^*V$. It is clear that $\det V \equiv 1$, so $V \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$, proving (i).

To show uniqueness (ii), suppose $V, W \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ with $V^*V = W^*W$. Let $U = WV^{-1} \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$. Then $U^* = U^{-1}$, so $U \in \mathcal{H}_{\mathcal{C}}\mathrm{SU}_n$. □

Lemma 1.9. *If $X_1, X_2 \in \mathrm{SL}_n\mathbb{C}$ are positive definite, and X_2 is a multiple of X_1, then $X_1 = X_2$.*

Proof. Let $X_2 = cX_1$. Since X_1 and X_2 are positive definite, then $c > 0$. Taking the determinant, $c^n = 1$, so c is an n'th root of 1. Hence $c = 1$, so $X_1 = X_2$. □

1.4. Simultaneous unitarization

We are now prepared to prove the following unitarization theorem. Conditions equivalent to condition (ii) of this theorem (local simultaneous unitarizability) are found in Sections 2.1–3.

Theorem 1.10. *Let $\mathcal{C}$ be an analytic curve and $M_1, \dots, M_q \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ ($q \geq 2$). Suppose M_1 and M_2 are irreducible on $\mathcal{C}$ except at a subset of isolated points. Then the following are equivalent:*

(i) *$M_1, \dots, M_q$ are globally simultaneously unitarizable on $\mathcal{C}$.*
(ii) *$M_1, \dots, M_q$ are locally simultaneously unitarizable at each $p \in \mathcal{C}$.*

In this case, any simultaneous unitarizer V is unique up to left multiplication by an element of $\mathcal{H}_{\mathcal{C}}\mathrm{SU}_n$.

Proof. Clearly (i) implies (ii). Conversely, suppose (ii). A global analytic simultaneous unitarizer V is constructed as follows.

Fix $p \in \mathcal{C}$. By (ii), there exists a neighborhood $\mathcal{U}_p$ of p and $W \in \mathcal{H}_{\mathcal{U}}\mathrm{SL}_n\mathbb{C}$ such that $WM_jW^{-1} \in \mathcal{H}_{\mathcal{U}_p}\mathrm{SU}_n$, $j \in \{1, \dots, q\}$. Define $X_p = W^*W$. Then $X_p \in \mathcal{H}_{\mathcal{U}_p}\mathrm{SL}_n\mathbb{C} \cap \ker L$ and X_p is Hermitian positive definite. By Lemma 1.9, X_p is the unique map in a neighborhood of p which takes values in $\mathrm{SL}_n\mathbb{C}$, is in $\ker L$, and is Hermitian positive definite.

Define $X : \mathcal{C} \to \mathrm{SL}_n\mathbb{C}$ by $X(p) = X_p(p)$. Then for each $p \in \mathcal{C}$, X is analytic at p because it coincides with the local analytic map X_p on $\mathcal{U}_p$, again by Lemma 1.9. This defines the unique map $X \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C} \cap \ker L$ which is Hermitian positive definite on $\mathcal{C}$.

By the Cholesky Decomposition Proposition 1.8, there exists $V \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ such that $X = V^*V$. Then for $k \in \{1, \dots, q\}$, $VM_kV^{-1} \in \mathcal{H}_{\mathcal{C}}\mathrm{SU}_n$ by Lemma 1.4(i), so V is the required simultaneous unitarizer.

To show uniqueness, suppose $W \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ is another simultaneous unitarizer. Then W^*W is a Hermitian positive definite element of $\mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C} \cap \ker L$. By Lemma 1.9, $W^*W = X = V^*V$. The uniqueness result follows by the uniqueness result in the Cholesky Decomposition Proposition 1.8. □

2. Local conditions for simultaneous unitarizability

Theorem 1.10 effectively reduces global simultaneous unitarizability to local simultaneous unitarizability. We will now give more explicit conditions for local simultaneous unitarizability.

2.1. Ξ_n-graphs

Given two matrix maps $M_1,\ M_2 \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$ with M_1 diagonal, the local simultaneous unitarizability of M_1 and M_2 at p is equivalent to the equalities of the orders of certain corresponding entries of M_2 and M_2^{*-1} at p. These relations are naturally expressed in terms of graphs on $\{1, \dots, n\}$ (Definition 2.1).

Definition 2.1. A Ξ_n*-graph* is a directed graph with vertices $V = \{1, \dots, n\}$. A Ξ_n-graph is not a multigraph — it may not have two instances of the same edge. However, it may have an edge connecting a vertex to itself. A Ξ_n-graph is *connected* if it is connected as a undirected graph.

Let $\mathcal{C}$ be an analytic curve, let $p \in \mathcal{C}$, and let $A,\ B \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$. For $\mu, \nu \in \{1, \dots, n\}$, let $A_{\mu\nu}$ denote the entry of A lying in row μ and column ν, and similarly for B.

Let G be a Ξ_n-graph. A is *G-non-zero at p* if for every directed edge $(\mu,\ \nu) \in V^2$ from μ to ν of G we have $\operatorname{ord}_p A_{\mu\nu} < \infty$. (Since $\mathcal{C}$ is connected and A is holomorphic, this order condition is equivalent to the condition $A_{\mu\nu} \not\equiv 0$.)

Let G be a Ξ_n-graph. A and B are *G-compatible at p* if for every directed edge $(\mu,\ \nu) \in V^2$ of G, we have $\operatorname{ord}_p A_{\mu\nu} = \operatorname{ord}_p B_{\mu\nu} < \infty$.

Let $e_1 = (1,\ 0, \dots, 0), \dots, e_n = (0, \dots, 0,\ 1) \in \mathbb{C}^n$ be the standard basis for $\mathbb{C}^n$. Fixing n, we have the natural correspondence between the nonempty subsets of $\{1, \dots, n\}$ and the nonzero subspaces of $\mathbb{C}^n$ generated by standard basis elements, where $K \subset \{1, \dots, n\}$ corresponds to the subspace

$$E_K = \operatorname{span}\{e_k \mid k \in K\} \subseteq \mathbb{C}^n.$$

This correspondence induces a natural one-to-one correspondence between the set of partitions of $\{1, \dots, n\}$ all of whose terms are non-empty, and the set of direct sum decompositions of $\mathbb{C}^n$ relative to its standard basis, all of whose terms are non-zero.

The following lemma expresses in terms of Ξ_n-graphs a notion closely related to block-diagonalizability.

Lemma 2.2. *Let $\mathcal{C}$ be an analytic curve, let $A \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$, and let $p \in \mathcal{C}$. Then the following are equivalent:*

(i) *Every Ξ_n-graph H for which A is H-non-zero at p is disconnected.*
(ii) *There exists a direct decomposition $\mathbb{C}^n = V_1 \oplus V_2$ into non-zero summands V_1 and V_2 generated by standard basis elements of $\mathbb{C}^n$ such that $AV_1 \subseteq V_1$ and $AV_2 \subseteq V_2$ near p.*

Proof. Suppose (ii) and let $G_1 \sqcup G_2 = \{1, \dots, n\}$ be the corresponding partition. The conditions $AV_1 \subseteq V_1$ and $AV_2 \subseteq V_2$ are equivalent to the condition that $A_{\mu\nu} \equiv 0$ for all $(\mu,\ \nu) \in (G_1 \times G_2) \cup (G_2 \times G_1)$. Let G be a Ξ_n-graph for which A is G-non-zero at p. Then G does not contain any of the edges in $(G_1 \times G_2) \cup (G_2 \times G_1)$. Hence the two subsets G_1 and G_2 of the vertex set of G are disconnected, proving (i).

Conversely, suppose (i) and let G be the maximal Ξ_n-graph among the Ξ_n-graphs H for which A is H-non-zero at p. Then G is disconnected, so let $G_1 \sqcup G_2 = \{1, \dots, n\}$ be a partition with G_1 and G_2 nonempty such that no edge of G is in $(G_1 \times G_2) \cup (G_2 \times G_1)$. Since G is maximal, then $A_{\mu\nu} \equiv 0$ for all $(\mu,\ \nu) \in (G_1 \times G_2) \cup (G_2 \times G_1)$. Let $\mathbb{C}^n = V_1 \oplus V_2$ be the direct sum decomposition corresponding to the partition $G_1 \sqcup G_2$. Then $AV_1 \subseteq V_1$ and $AV_2 \subseteq V_2$, proving (ii). □

We shall say that $X \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$ is *infinitesimally invertible at p* if the leading term in its series expansion at p in some local coordinate on $\mathcal{C}$ near p has full rank. Equivalently, there exists a local meromorphic scalar function f near p such that fX is holomorphic at p and $\operatorname{rank}(fX)(p) = n$.

Lemma 2.3. *Let $\mathcal{C}$ be an analytic curve, let $p \in \mathcal{C}$ and let $A,\ B \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$. Let $X \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$ be diagonal with $X \not\equiv 0$, and suppose $XA = BX$. Then the following are equivalent:*

(i) *A and B are G-compatible at p for some connected Ξ_n-graph G.*
(ii) *X is infinitesimally invertible at p, and neither of the equivalent conditions of Lemma 2.2 hold for A at p.*

Proof. First suppose (ii) holds. Since X is diagonal, the infinitesimal invertibility of X is equivalent to

$$\operatorname*{ord}_p X_{11} = \cdots = \operatorname*{ord}_p X_{nn} < \infty. \tag{2.1}$$

The components of $XA - BX$ are

$$0 = (XA - BX)_{\mu\nu} = X_{\mu\mu}A_{\mu\nu} - X_{\nu\nu}B_{\mu\nu}. \tag{2.2}$$

Let G be a connected Ξ_n-graph such that A is G-non-zero at p. Then for each edge (μ, ν) of G, we have $\operatorname{ord}_p A_{\mu\nu} < \infty$. Then (2.1) and (2.2) imply

$$\operatorname*{ord}_p A_{\mu\nu} = \operatorname*{ord}_p B_{\mu\nu} < \infty. \tag{2.3}$$

Hence A and B are G-compatible at p.

Conversely, suppose (i), that A and B are G-compatible at p for some connected Ξ_n-graph G. Then (2.3) holds for each edge (μ, ν) of G. Hence A is G-non-zero at p. By (2.2),

$$\operatorname*{ord}_p X_{\mu\mu} = \operatorname*{ord}_p X_{\nu\nu} < \infty.$$

The connectedness of G implies (2.1). □

Lemma 2.4. *Let $\mathcal{C}$ be an analytic curve, $p \in \mathcal{C}$, and suppose $X \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$ is infinitesimally invertible at p. Then*

(i) $\det X$ *has a local analytic n'th root in some neighborhood $\mathcal{U} \subset \mathcal{C}$ of p.*
(ii) $(\det X)^{-1/n} X \in \mathcal{H}_{\mathcal{U}}\mathrm{SL}_n\mathbb{C}$.
(iii) *If X is Hermitian positive definite on a punctured neighborhood of p, then the n'th root can be chosen so that* $(\det X)^{-1/n} X$ *is Hermitian positive definite in a neighborhood of p.*

Proof. Let t be a local coordinate on $\mathcal{C}$ near p with $t(p) = 0$. Since X is infinitesimally invertible at p, we have

$$X = X_m t^m + \mathrm{O}(t^{m+1}), \quad \det X_m \neq 0. \tag{2.4}$$

Taking the determinant of (2.4),

$$\det X = (\det X_m)t^{nm} + \mathrm{O}(t^{nm+1}). \tag{2.5}$$

Since $\det X_m \neq 0$, then $\operatorname{ord}_p(\det X) = nm$. Since nm is divisible by n, then $\det X$ has a local analytic n'th root in a neighborhood $\mathcal{U} \subset \mathcal{C}$ of p, proving (i).

Taking an n'th root of (2.5),

$$(\det X)^{1/n} = (\det X_m)^{1/n} t^m + \mathrm{O}(t^{m+1}), \tag{2.6}$$

so

$$Y := (\det X)^{-1/n} X = (\det X_m)^{-1/n} X_m + \mathrm{O}(t^1),$$

and hence $Y \in \mathcal{H}_{\mathcal{U}}\mathrm{SL}_n\mathbb{C}$, proving (ii).

To show (iii), suppose $\mathcal{U}$ is a neighborhood of p such that X is positive definite on $\mathcal{U} \setminus \{p\}$. Then $\det X$ is positive on $\mathcal{U} \setminus \{p\}$, and hence by its continuity is nonnegative on $\mathcal{U}$. By (2.5), $\det X_m > 0$, so we can choose $(\det X_m)^{1/n} > 0$ in (2.6). Equation (2.4) and a limit argument imply that m is even. Equation (2.6) then implies that $(\det X)^{1/n}$ is non-negative on $\mathcal{U}$. Then $Y = (\det X)^{1/n} X$ is positive definite in a punctured neighborhood of p. Hence $Y(p)$ is positive semidefinite, and so $Y(p)$ is positive definite since $\det Y(p) = 1$. □

Lemma 2.5. *Let $\mathcal{C}$ be an analytic curve, let $p \in \mathcal{C}$ and let $M_1,\ M_2 \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$. Suppose M_1 and M_2 are irreducible on $\mathcal{C} \setminus \{p\}$. Suppose M_1 is diagonal and no two local analytic eigenvalues of M_1 are identically equal. Then the following are equivalent:*

(i) *M_1 and M_2 are locally simultaneously unitarizable at p.*

(ii) *M_1 and M_2 are pointwise simultaneously unitarizable at each point of a neighborhood of p, and M_2 and M_2^{*-1} are G-compatible at p for some connected Ξ_n-graph G.*

Proof. To show (i) ⇒ (ii), let $V \in \mathcal{H}_{\mathcal{U}}\mathrm{SL}_n\mathbb{C}$ be a local simultaneous unitarizer of M_1 and M_2 at p and let $X = V^*V$. Then

$$XM_k = M_k^{*-1}X, \quad k \in \{1,\ 2\}.$$

Since M_1 is unitarizable, it has unimodular eigenvalues. Since M_1 is diagonal, then $M_1 \in \mathcal{H}_{\mathcal{C}}\mathrm{SU}_n$. Hence $M_1 = M_1^{*-1}$, and so $[X,\ M_1] = 0$. Since no two eigenvalues of M_1 are identically equal, X is diagonal away from the set of isolated points where two eigenvalues of M_1 coincide, so by its continuity, X is diagonal.

Since M_1 is diagonal and M_1 and M_2 are irreducible in a punctured neighborhood of p, then there is no non-zero proper subspace $W \subset \mathbb{C}^n$ generated by standard basis elements of $\mathbb{C}^n$ such that $M_2 W \subset W$ in a neighborhood of p. Hence M_2 is not block-diagonalizable at p via a permutation at p. Since $X(p) \in \mathrm{SL}_n\mathbb{C}$, then X has full rank at p. By Lemma 2.3, M_2 and M_2^{*-1} are G-compatible at p for some connected Ξ_n-graph G. This proves (ii).

To show (ii) ⇒ (i), since M_1 and M_2 are irreducible on $\mathcal{C}\setminus\{p\}$ and are pointwise simultaneous unitarizable at each point in a neighborhood of p, by Lemma 1.7 there exists a neighborhood $\mathcal{U}$ of p and a map $X \in \mathcal{H}_{\mathcal{U}}\mathrm{M}_{n\times n}\mathbb{C} \cap \ker L$ for which X is Hermitian positive definite on $\mathcal{U} \setminus \{p\}$. By condition (ii) and Lemma 2.3, X is infinitesimally invertible at p. By Lemma 2.4, there exists a neighborhood $\mathcal{V}$ of

p and a choice of n'th root of $\det X$ such that $Y = (\det X)^{-1/n} X \in \mathcal{H}_{\mathcal{V}}\mathrm{SL}_n\mathbb{C}$ is Hermitian positive definite.

By the Cholesky Decomposition Proposition 1.8, there exists $V \in \mathcal{H}_{\mathcal{V}}\mathrm{SL}_n\mathbb{C}$ such that $Y = V^*V$. Then by Lemma 1.4(i), $VM_kV^{-1} \in \mathcal{H}_{\mathcal{V}}\mathrm{SU}_n$, $k \in \{1, 2\}$, so V is a local simultaneous unitarizer of M_1 and M_2 at p. $\square$

Lemma 2.6. *For $q \geq 2$, let $M_1, \dots, M_q \in \mathrm{SL}_n\mathbb{C}$ with M_1, M_2 irreducible and M_1, $M_2 \in \mathrm{SU}_n$. If $M_1, \dots, M_q$ are simultaneously unitarizable by an element of $\mathrm{SL}_n\mathbb{C}$, then $M_1, \dots, M_n \in \mathrm{SU}_n$.*

Proof. Let $C \in \mathrm{SL}_n\mathbb{C}$ be a simultaneous unitarizer. Then C^*C commutes with each of M_1 and M_2. By Lemma 1.1, $C^*C \in \mathbb{C}\,\mathrm{I}$. Since C^*C is positive definite, by Lemma 1.9, $C^*C = \mathrm{I}$. Hence $C \in \mathrm{SU}_n$. Since $CM_kC^{-1} \in \mathrm{SU}_n$ for $k \in \{1, \dots, q\}$, then $M_k \in \mathrm{SU}_n$ for $k \in \{1, \dots, q\}$. $\square$

Lemma 2.5, Lemma 2.6 and Theorem 1.10 give the following result.

Theorem 2.7 (Unitarization theorem). *Let $\mathcal{C}$ be an analytic curve, let $p \in \mathcal{C}$ and let $M_1, \dots, M_q \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_n\mathbb{C}$, $q \geq 2$. Suppose M_1 and M_2 are irreducible on $\mathcal{C}$ except at a subset of isolated points. Suppose M_1 is locally diagonalizable at each point $p \in \mathcal{C}$, and that no two local analytic eigenvalues of M_1 are identically equal. Then $M_1, \dots, M_q$ are globally simultaneously unitarizable if and only if the following conditions hold:*

(i) *$M_1, \dots, M_q$ are pointwise simultaneously unitarizable at each point of $\mathcal{C}$.*
(ii) *For each $p \in \mathcal{C}$, let C be a local diagonalizer of M_1 at p, and let $P_2 = CM_2C^{-1}$. Then P_2 and P_2^{*-1} are G-compatible at p for some connected Ξ_n-graph G.*

In this case, any simultaneous unitarizer is unique up to left multiplication by an element of $\mathcal{H}_{\mathcal{C}}\mathrm{SU}_n$.

3. Simultaneous unitarizability for $\mathrm{SL}_2\mathbb{C}$

For the case $\mathrm{SL}_2\mathbb{C}$, the Ξ_n-graph condition in Theorem 1.10 is particularly simple and can be recast in terms of commutators.

Note that for A, $B \in \mathrm{M}_{2\times 2}\mathbb{C}$, A and B are irreducible if and only if $[A, B]$ has rank 2.

Definition 3.1. We say that A, $B \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_2\mathbb{C}$ are *infinitesimally irreducible* at $p \in \mathcal{C}$ if $[A, B] \not\equiv 0$ and the leading order term in the series expansion of $[A, B]$ at p has full rank.

The property infinitesimally irreducible (respectively reducible) is preserved under conjugation by an element of $\mathcal{H}_{\mathcal{C}}\mathrm{SL}_2\mathbb{C}$.

Lemma 3.2. *If A is diagonal, then A and B are infinitesimally irreducible if and only if the two off-diagonal terms of B have the same finite order.*

We give a sufficient condition for local diagonalizability:

Lemma 3.3. *Let $\mathcal{C}$ be an analytic curve, $p \in \mathcal{C}$, and $M \in \mathcal{H}_{\mathcal{C}}\mathrm{SU}_2$ be an analytic map. Then M is locally diagonalizable in a neighborhood $\mathcal{U} \subset \mathcal{C}$ of p by a map $C \in \mathcal{H}_{\mathcal{U}}\mathrm{SU}_2$.*

Proof. It follows from the characteristic equation of M that its eigenvalues $\mu_1, \mu_2 = \mu_1^{-1}$ are analytic in a neighborhood $\mathcal{U} \subset \mathcal{C}$ of p. It can be shown, for example by an analytic version of the QR-decomposition, that there exist corresponding analytic eigenvector functions $v_1, v_2 \in \mathcal{H}_{\mathcal{U}}\mathbb{C}^2$ such that $V = (v_1, v_2) \in \mathcal{H}_{\mathcal{U}}\mathrm{SU}_2$. Then $V^{-1}MV = \mathrm{diag}(\mu_1, \mu_2)$, so $C = V^{-1}$ is the required diagonalizer. □

Theorem 3.4 (Unitarization theorem for $\mathbf{SL_2}\,\mathbb{C}$). *Let $\mathcal{C}$ be an analytic curve, $M_1, \dots, M_q \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_2\mathbb{C}$ ($q \geq 2$), and suppose that $[M_r, M_s] \not\equiv 0$ for some fixed choice $r, s \in \{1, \dots, q\}$. Then $M_1, \dots, M_q$ are globally simultaneously unitarizable if and only if the following conditions hold at each $p \in \mathcal{C}$:*

(i) *$M_1, \dots, M_q$ are pointwise simultaneously unitarizable at p.*
(ii) *M_r or M_s is locally diagonalizable at p.*
(iii) *M_r and M_s are infinitesimally irreducible at p.*

In this case, any simultaneous unitarizer V is unique up to left multiplication by an element of $\mathcal{H}_{\mathcal{C}}\mathrm{SU}_2$.

Remark 3.5. Examples exist which show the independence of the three conditions (i)–(iii) of Theorem 3.4.

Proof. Renumber so that $r = 1$ and $s = 2$. Suppose $M_1, \dots, M_q$ are globally simultaneously unitarizable on $\mathcal{C}$. Then (i) clearly holds, and condition (ii) holds by Lemma 3.3. To show condition (iii), let V be a local unitarizer of M_1 and M_2, and let $P_k = VM_kV^{-1}$, $k \in \{1, 2\}$. By Lemma 3.3, there exists a local unitary diagonalizer C of P_1. Let $Q_k = CP_kC^{-1}$, $k \in \{1, 2\}$. Then Q_1 is diagonal. Since Q_2 is unitary, its off-diagonal terms have the same order. Since $[M_1, M_2] \not\equiv 0$, then $[Q_1, Q_2] \not\equiv 0$, so Q_2 is not identically diagonal. Hence the off-diagonal entries of Q_2 are not both identically zero. By Lemma 3.2, Q_1 and Q_2 are infinitesimally irreducible, and hence M_1 and M_2 are infinitesimally irreducible.

Conversely, assume conditions (i)–(iii) and assume M_1 is locally diagonalizable at p. We show that the conditions of Theorem 2.7 are satisfied. Since M_1 is locally diagonalizable, and $[M_1, M_2] \not\equiv 0$, then M_1 does not have identically equal eigenvalues. Let C be a local diagonalizer of M_1 at p, and let $P_k = CM_kC^{-1}$, $k \in \{1, 2\}$. By (iii), the two off-diagonal terms of P_2 and those of P_2^{*-1} all have the same finite order. Hence P_1 and P_2 are irreducible in a punctured neighborhood of p, so M_1 and M_2 are irreducible in a punctured neighborhood of p. Let G be the connected Ξ_2-graph with single edge $(1, 2)$. Then P_2 and ${P_2}^{*-1}$ are G-compatible. The existence and uniqueness of the global simultaneous unitarizer follows by Theorem 2.7. □

4. Simultaneous r-unitarization

The r-Unitarization Theorem 4.4 for $\mathrm{SL}_n\mathbb{C}$-valued loops is a variant of the Unitarization Theorem 1.10 on the standard unit circle $\mathbb{S}^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. This variant has been proven for the case of $\mathrm{SL}_2\mathbb{C}$ [22], where it finds application to the construction of non-simply-connected CMC surfaces.

In the r-Unitarization Theorem, a holomorphic map V is constructed on an annulus $\mathcal{A}_{r,1}$ which simultaneously unitarizes the given set of loops $M_1, \dots, M_q$ in the sense that the VM_kV^{-1} extend holomorphically to $\mathbb{S}^1$ and are unitary there. In this case, the unitarizing loop V does not in general extend holomorphically to $\mathbb{S}^1$, but has zeros and poles there.

We note that a unitary map cannot have poles. This follows from the fact that SU_n and U_n are compact:

Proposition 4.1. *Let $\mathcal{C}$ be an analytic curve. Then $\mathcal{M}_\mathcal{C}\mathrm{U}_n = \mathcal{H}_\mathcal{C}\mathrm{U}_n$ and $\mathcal{M}_\mathcal{C}\mathrm{SU}_n = \mathcal{H}_\mathcal{C}\mathrm{SU}_n$.*

We now extend the Cholesky decomposition theorem to the case of an analytic map on $\mathbb{S}^1$ which is Hermitian positive definite except at a finite subset.

Proposition 4.2 (Meromorphic Cholesky decomposition). *Let $\mathcal{C} = \mathbb{S}^1 \subset \mathbb{C}$ and let $X \in \mathcal{M}_\mathcal{C}\mathrm{M}_{n\times n}\mathbb{C}$ be Hermitian positive definite except at a finite subset of points $S \subset \mathcal{C}$. Then*

(i) *There exists $V \in \mathcal{M}_\mathcal{C}\mathrm{M}_{n\times n}\mathbb{C}$ such that $X = V^*V$.*
(ii) *V is unique up to left multiplication by elements of $\mathcal{H}_\mathcal{C}\mathrm{U}_n$.*

Proof. We will apply the LDU-decomposition stated at the beginning of the proof of Proposition 1.8, with holomorphicity replaced by meromorphicity.

Let $\rho : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a double cover and let $\tau : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ be the deck transformation induced by a single counterclockwise traversal of $\mathcal{C}$. Let ρ^* and τ^* denote the respective pullbacks. Write $D = \mathrm{diag}(d_1, \dots, d_n)$ and for $k \in \{1, \dots, n\}$, define b_k to be either of the global square roots of d_k on $\widetilde{\mathcal{C}}$. Define $\widetilde{B} = \mathrm{diag}(b_1, \dots, b_n)$ and $\widetilde{V} = \widetilde{B}(\rho^*R)$, so $\rho^*X = \widetilde{V}^*\widetilde{V}$. With λ the standard unimodular parameter on $\mathbb{S}^1$, define

$$c_k = 1 \text{ if } \tau^*b_k = b_k, \quad \text{and} \quad c_k = \sqrt{\lambda} \text{ if } \tau^*b_k = -b_k,$$

and define $C = \mathrm{diag}(c_1, \dots, c_n) \in \mathcal{H}_{\widetilde{\mathcal{C}}}\mathrm{U}_n$. Then $\tau^*(C\widetilde{V}) = C\widetilde{V}$. Let $V \in \mathcal{M}_\mathcal{C}\mathrm{M}_{n\times n}\mathbb{C}$ be the unique map satisfying $\rho^*V = U\widetilde{V}$. Then $X = V^*V$, proving (i).

To show uniqueness (ii), suppose $V,\ W \in \mathcal{M}_\mathcal{C}\mathrm{M}_{n\times n}\mathbb{C}$ with $V^*V = W^*W$. Let $U = WV^{-1} \in \mathcal{M}_\mathcal{C}\mathrm{M}_{n\times n}\mathbb{C}$. Then $U^* = U^{-1}$, so U takes values in U_n on $\mathbb{S}^1$ away from its poles. By Proposition 4.1, $U \in \mathcal{H}_\mathcal{C}\mathrm{U}_n$. □

Definition 4.3. Let $M_1, \dots, M_q \in \mathcal{H}_{\mathbb{S}^1}\mathrm{SL}_n\mathbb{C}$. Let $r \in (0,\ 1)$ and suppose $M_1, \dots$ $\dots, M_q$ extend holomorphically to respective maps $\widetilde{M}_1, \dots \widetilde{M}_q \in \mathcal{H}_{\mathcal{A}_{r,1}}\mathrm{SL}_n\mathbb{C}$. Then $M_1, \dots, M_q$ are *simultaneously r-unitarizable* if there exists $V \in \mathcal{H}_{\mathcal{A}_{r,1}}\mathrm{SL}_n\mathbb{C}$ which extends holomorphically to $\mathbb{S}^1$ minus a finite subset such that $V\widetilde{M}_1V^{-1}, \dots$ $\dots, V\widetilde{M}_qV^{-1}$ extend holomorphically to $\mathbb{S}^1$ and their respective restrictions to $\mathbb{S}^1$ are in $\mathcal{H}_{\mathbb{S}^1}\mathrm{SU}_n$.

Theorem 4.4 (r-unitarization theorem). *Let $M_1, \dots, M_q \in \mathcal{H}_{\mathbb{S}^1}\mathrm{SL}_n\mathbb{C}$ $(q \geq 2)$ and suppose for some $i,\ j \in \{1, \dots, r\}$ that M_i and M_j are irreducible except at a finite subset of $\mathbb{S}^1$. Then the following are equivalent:*

(i) *$M_1, \dots, M_q$ are simultaneously r-unitarizable for some $r \in (0,\ 1)$.*

(ii) *$M_1, \dots, M_q$ are pointwise simultaneously unitarizable on $\mathbb{S}^1$ minus a finite subset.*

Moreover, r-unitarizers are unique in the following sense. If $V_1 \in \mathcal{H}_{\mathcal{A}_{r_1,1}}\mathrm{SL}_n\mathbb{C}$ and $V_2 \in \mathcal{H}_{\mathcal{A}_{r_2,1}}\mathrm{SL}_n\mathbb{C}$ are respective r_1- and r_2- unitarizers, then $V_2V_1^{-1}$ extends holomorphically to $\mathbb{S}^1$ and its restriction to $\mathbb{S}^1$ is in $\mathcal{H}_{\mathbb{S}^1}\mathrm{SU}_n$.

Proof. First suppose a simultaneous r-unitarizer V exists as in (i) and let $S \subset \mathbb{S}^1$ be the finite singular set of the extension of V to $\mathbb{S}^1$. Then by the definition of r-unitarizer, for all $p \in \mathbb{S}^1 \setminus S$, $VM_kV^{-1}\big|_p \in \mathrm{SU}_n$, proving (ii).

Conversely, suppose (ii) holds. A simultaneous r-unitarizer V is constructed as follows.

Let L be as in Definition 1.3. By Lemma 1.7, there exists $X_1 \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$ such that $X_1 \in \ker L$, $X_1^* = X_1$, and away from a finite subset of $\mathbb{S}^1$, X_1 is positive definite.

By the Cholesky Decomposition Proposition 4.2 there exists $V_1 \in \mathcal{M}_{\mathcal{C}}\mathrm{M}_{n\times n}\mathbb{C}$ such that $X_1 = V_1^*V_1$.

By the holomorphicity of $M_1, \dots, M_q$ and the meromorphicity of V_1, there exists $r \in (0,\ 1)$ such that $M_1, \dots, M_q$ and V_1 extend holomorphically to $\mathcal{A}_{r,1}$ and $\det V_1$ is non-zero in $\mathcal{A}_{r,1}$.

Let $\widetilde{\mathcal{A}} \to \mathcal{A}_{r,1}$ be an n-fold cover and let $\tau : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ be the deck transformation induced by a single counterclockwise traversal of $\mathbb{S}^1$. Define $V_2 = (\det V)^{-1/n}V_1$ on $\widetilde{\mathcal{A}}$. Then for some $k \in \mathbb{Z}^{\geq 0}$, $\tau^*V_2 = \epsilon^kV_2$, where $\epsilon = e^{2\pi i/n}$. Let λ be the standard unimodular parameter on $\mathbb{S}^1$. Define $U \in \mathcal{H}_{\widetilde{\mathcal{A}}}\mathrm{SU}_n$ by

$$U = \lambda^{-k/n}\,\mathrm{diag}(1, \dots, 1, \lambda^k),$$

so $\tau^*U = \epsilon^{-k}U$. Let $V = UV_2$. Then $\tau^*V = V$, so V on $\widetilde{\mathcal{A}}$ descends to a single valued holomorphic map on $\mathcal{A}_{r,1}$. This gives us $V \in \mathcal{H}_{\mathcal{A}_{r,1}}\mathrm{SL}_2\mathbb{C}$.

Let $S \subset \mathbb{S}^1$ be the singular set of V. By Lemma 1.4(i), on $\mathbb{S}^1 \setminus S$ for each $k \in \{1, \dots, q\}$, $P_k := VM_kV^{-1}$ takes values in SU_n. Since V_1 is meromorphic on S, then $P_k = V_1M_kV_1^{-1}$ is meromorphic on S. By Proposition 4.1, $P_k \in \mathcal{H}_{\mathbb{S}^1}\mathrm{SU}_n$. Hence V is the required simultaneous r-unitarizer.

The uniqueness result follows as in the proof of Theorem 1.10. □

5. The extended Weierstrass representation

We now construct trinoids and symmetric n-noids, conformal CMC immersions of the n-punctured Riemann sphere into each of the space forms Euclidean 3-space $\mathbb{R}^3$, spherical 3-space $\mathbb{S}^3$ and hyperbolic 3-space $\mathbb{H}^3$. We first describe the extended Weierstrass representation used in the construction. As it is thoroughly described in a number of places, such as [4, 3, 22], we give only a brief outline here.

5.1. The Iwasawa decomposition

Given an analytic Lie group G, we denote by ΛG the group $C^\omega(\mathbb{S}^1, G)$ of analytic maps $\mathbb{S}^1 \to G$. Let $\mathcal{D}_1 \subset \mathbb{C}$ be the open disk bounded by $\mathbb{S}^1$. The subgroup $\Lambda^+\mathrm{SL}_2(\mathbb{C}) \subset \Lambda\mathrm{SL}_2\mathbb{C}$ of *positive loops* is the subgroup of loops $B \in \Lambda\mathrm{SL}_2\mathbb{C}$ such that B extends holomorphically to $\mathcal{D}_1$ and $B(0)$ is upper triangular with real positive diagonal entries. The subgroup $\Lambda^*\mathrm{SL}_2\mathbb{C} \subset \Lambda\mathrm{SL}_2\mathbb{C}$ of *unitary loops* is the subgroup of loops $F \in \Lambda\mathrm{SL}_2\mathbb{C}$ which satisfy the condition $F^* = F^{-1}$, where for any $F \in \Lambda\mathrm{SL}_2\mathbb{C}$, $F^* \in \Lambda\mathrm{SL}_2\mathbb{C}$ is defined by

$$F^*(\lambda) = \overline{F(1/\overline{\lambda})}^t. \tag{5.1}$$

Note that $F \in \Lambda^*\mathrm{SL}_2\mathbb{C}$ implies $F(p) \in \mathrm{SU}_2$ at each point $p \in \mathbb{S}^1$.

Multiplication $\Lambda^*\mathrm{SL}_2\mathbb{C} \times \Lambda^+\mathrm{SL}_2(\mathbb{C}) \to \Lambda\mathrm{SL}_2\mathbb{C}$ is a real-analytic diffeomorphism onto [18, 4]. For $\Phi \in \Lambda\mathrm{SL}_2\mathbb{C}$,

$$\Phi = FB$$

with $F \in \Lambda^*\mathrm{SL}_2\mathbb{C}$ and $B \in \Lambda^+\mathrm{SL}_2(\mathbb{C})$, is called the 1-Iwasawa (or just Iwasawa) decomposition of Φ. The chosen normalization of $B(0)$ gives uniqueness of this decomposition. We call F the unitary factor of Φ.

5.2. The extended Weierstrass construction

Every conformal CMC H immersion into one of the 3-dimensional space forms $\mathbb{R}^3$ or $\mathbb{S}^3$ or $\mathbb{H}^3$ can be locally constructed by the extended Weierstrass representation [22, 4] (with $H \neq 0$ for $\mathbb{R}^3$ and $|H| > 1$ for $\mathbb{H}^3$) as follows:

1. Let Σ be a domain in the z-plane $\mathbb{C}$, and choose a holomorphic $C^\omega(\mathbb{S}^1, \mathrm{sl}_2\mathbb{C})$-valued differential form $\xi = A(z,\lambda)dz$ which extends meromorphically to $\mathcal{D}_1$ with a pole only at $\lambda = 0$, which is simple and appears only in the upper-right entry of ξ.
2. Solve the ordinary differential equation $d\Phi = \Phi\xi$.
3. Iwasawa split Φ into $\Phi = FB$. Then F is an extended frame for some CMC immersion.
4. Apply one of three Sym-Bobenko formulas described below to obtain a CMC immersion into $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$.

5.3. The Sym-Bobenko formulas

The final step in the extended Weierstrass representation is a Sym-Bobenko formula, which computes the immersion into $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$ from its extended frame.

1. CMC immersions into $\mathbb{R}^3$: The Sym-Bobenko formula [2]
$$-2i\lambda H^{-1}\left(\frac{d}{d\lambda}F\right)F^{-1} \tag{5.2}$$
gives a conformal CMC $H \neq 0$ immersion into $\mathbb{R}^3$ for each fixed $\lambda_0 \in \mathbb{S}^1$. Formula (5.2) gives an immersion into the Lie algebra su_2, which being a real 3-dimensional vector space can be identified with $\mathbb{R}^3$.
2. CMC immersions into $\mathbb{S}^3$: For $\mu \in \mathbb{S}^1 \setminus \{1\}$ and each $\lambda_0 \in \mathbb{S}^1$, the Sym-Bobenko formula [2]
$$F_{\mu\lambda_0}F_{\lambda_0}^{-1} \tag{5.3}$$
gives a conformal CMC $H = i(1+\mu)/(1-\mu)$ immersion into $\mathbb{S}^3$. Here H can take any real value, including 0. Formula (5.3) gives an immersion into the Lie group SU_2, which we are identifying with the unit sphere $\mathbb{S}^3 \in \mathbb{R}^4$.
3. CMC immersions into $\mathbb{H}^3$: For $s \in (0, 1)$ and any $\lambda \in \mathbb{S}^1$, set $\lambda_0 = s\lambda$. Then the Sym-Bobenko formula [2]
$$F_{\lambda_0}\overline{F_{\lambda_0}}^{t} \tag{5.4}$$
gives a conformal CMC $H = (1+s^2)/(1-s^2) > 1$ immersion into $\mathbb{H}^3$ for each fixed $\lambda \in \mathbb{S}^1$. Formula (5.4) gives an immersion into the determinant 1 Hermitian matrices, which we are identifying with $\mathbb{H}^3$.

We choose the following normalizations for the Sym-Bobenko formulas:
$$\begin{aligned}
&\text{for } \mathbb{R}^3\text{: } \lambda_0 = 1\ ,\\
&\text{for } \mathbb{S}^3\text{: } \lambda_0 \in \mathbb{S}^1 \setminus \{\pm 1\}, \text{ and } \mu = \lambda_0^{-2}\ ,\\
&\text{for } \mathbb{H}^3\text{: } \lambda_0 \in (-1, 0) \cup (0, 1).
\end{aligned} \tag{5.5}$$

5.4. Monodromy

The primary result in [4] is that every CMC immersion into $\mathbb{R}^3$ can be obtained via the extended Weierstrass representation, and this is true for the cases of $\mathbb{S}^3$ and $\mathbb{H}^3$ as well [22] (with the restrictions $H \neq 0$ for $\mathbb{R}^3$ and $|H| > 1$ for $\mathbb{H}^3$). This method can also be applied to constructing non-simply-connected CMC immersions, and it is shown in [4] that even when Σ is a non-simply-connected open non-compact Riemann surface, one can still always choose ξ to be well-defined on Σ, as long as the resulting CMC immersion is well-defined on Σ.

However, in this case, closing conditions must be satisfied in order for the resulting CMC immersion to be well-defined on Σ. Considering a deck transformation τ of Σ associated to some loop γ in Σ, let us suppose that we can choose the solution Φ so that $\Phi \circ \tau = M_\gamma \Phi$ with $M_\gamma \in \Lambda^*\mathrm{SL}_2\mathbb{C}$. Then M_γ is the monodromy of Φ about γ, and M_γ is independent of z. Because $M_\gamma \in \Lambda^*\mathrm{SL}_2\mathbb{C}$, we also have $F \circ \tau = M_\gamma F$. Then the immersion obtained from the Sym-Bobenko formula at λ_0 for $\mathbb{R}^3$ will be invariant about γ if $M_\gamma|_{\lambda_0} = \pm\mathrm{I}$ and $\frac{d}{d\lambda} M_\gamma|_{\lambda_0} = 0$. There are similar conditions for the cases of $\mathbb{S}^3$ and $\mathbb{H}^3$. This gives the following sufficient conditions for the resulting CMC immersion to be well-defined on Σ:

$$\text{for } \mathbb{R}^3\text{: } M_\gamma \in \Lambda^*\mathrm{SL}_2\mathbb{C},\ M_\gamma|_{\lambda_0} = \pm\mathrm{I},\ \frac{d}{d\lambda} M_\gamma|_{\lambda_0} = 0\,, \tag{5.6}$$

$$\text{for } \mathbb{S}^3\text{: } M_\gamma \in \Lambda^*\mathrm{SL}_2\mathbb{C},\ M_\gamma|_{\mu\lambda_0} = M_\gamma|_{\lambda_0} = \pm\mathrm{I}\,, \tag{5.7}$$

$$\text{for } \mathbb{H}^3\text{: } M_\gamma \in \Lambda^*\mathrm{SL}_2\mathbb{C},\ M_\gamma|_{\lambda_0} = \pm\mathrm{I}, \tag{5.8}$$

for all loops γ in Σ. These are the conditions we will show are satisfied, to prove the existence of CMC trinoids and symmetric n-noids, by making an appropriate choice of solution Φ of $d\Phi = \Phi\xi$.

In the remainder of the paper we construct families of CMC trinoids and symmetric n-noids. For each family, the hypotheses of the Unitarization Theorem 3.4 are shown to hold under a suitable set of constraints on the end weights. The unitarization theorem then produces a dressing which closes the ends.

6. Constructing *n*-noids

6.1. The *n*-noid potential

We define a class of potentials whose local monodromies have the same eigenvalues as those of a Delaunay surface.

Definition 6.1. Let $\Sigma = \mathbb{P}^1$ be the Riemann sphere with the standard coordinate $z \in \mathbb{C} \cup \{\infty\}$. Let $\lambda_0 \in (\mathbb{R} \cup \mathbb{S}^1) \setminus \{0, -1\}$ be as in (5.5), and let

$$h(\lambda) = \frac{1}{4}\lambda^{-1}(\lambda - \lambda_0)(\lambda - \lambda_0^{-1}). \tag{6.1}$$

Let Q be a meromorphic quadratic differential on Σ all of whose poles are double poles with real quadratic residues. Assume that for each pole of Q, with quadratic residue $w/4$, the function $1 + wh$ is non-negative on $\mathbb{S}^1$. An *n-noid potential* is an extended Weierstrass potential of the form

$$\xi = \begin{pmatrix} 0 & \lambda^{-1}dz \\ \lambda h(\lambda) Q/dz & 0 \end{pmatrix}.$$

Let p be a double pole of Q with quadratic residue $w/4 \in \mathbb{R} \setminus \{0\}$. Choosing a basepoint $z_0 \in \Sigma$, let γ_p be a curve based at z_0 which winds once around p and does not wind around any other poles of Q. Let M_p be the monodromy about γ_p of the solution $\Phi = \Phi(z, \lambda)$ to the equation $d\Phi = \Phi\xi$, $\Phi(z_0, \lambda) = \mathrm{I}$ along γ_p.

Remark 6.2. By [12], the ends of a surface constructed via the extended Weierstrass representation from an n-noid potential are complete and asymptotic to Delaunay surfaces.

Multiplying Φ on the right by an analytic matrix $g = g(\lambda, z)$ does not change the resulting CMC immersion if $g = g_1 \cdot g_2$, where g_1 is a λ-independent diagonal matrix and $g_2 \in \Lambda^+\mathrm{SL}_2(\mathbb{C})$. We then call Φg a *gauge* of Φ. This gauge will change ξ to

$$\xi.g = g^{-1}\xi g + g^{-1}dg. \tag{6.2}$$

If Q is holomorphic at $z = \infty$, then ξ has a pole there. The following lemma shows that this is an artifact of our choice of potential, not a feature of the monodromy representation or induced CMC immersion. This lemma will be used in Section 8.

Lemma 6.3. *Let ξ be an n-noid potential as in Definition 6.1 with $\Sigma = \mathbb{P}^1$. Suppose Q is holomorphic at $z = \infty \in \mathbb{P}^1$, and let M_∞ be a local monodromy at ∞. Then $M_\infty = \mathrm{I}$, and ∞ is a smooth finite point of the CMC immersions induced by the extended Weierstrass representation obtained from ξ.*

Proof. Applying the gauge

$$g = \begin{pmatrix} z & 0 \\ -\lambda & z^{-1} \end{pmatrix},$$

the result follows from the fact that $\xi.g$ is holomorphic at ∞. □

6.2. Delaunay monodromy

We will need several facts about the n-noid monodromy defined in Section 6.1. The first lemma computes the eigenvalues of the monodromy and proves the latter half of the closing conditions 5.6–5.8.

Proposition 6.4. *Let M_p be a monodromy arising from an n-noid potential as in Section* 6.1. *Then*

(i) *The eigenvalues of M_p are* $\exp(\pm 2\pi i \rho_w)$, *where*

$$\rho_w(\lambda) = \frac{1}{2} - \frac{1}{2}\sqrt{1 + wh(\lambda)}. \tag{6.3}$$

(ii) *With λ_0 as in* (5.5),

$$M_p(\lambda_0^{\pm 1}) = \mathrm{I} \quad \textit{and} \quad \textit{if } \lambda_0 = 1, \textit{ then } \left.\frac{d}{d\lambda}\right|_{\lambda_0} M_p = 0. \tag{6.4}$$

Proof. The eigenvalues of M_p can be computed using the theory of regular singularities [22, 5].

The first part of (6.4), $M_p(\lambda_0^{\pm 1}) = \mathrm{I}$, can be computed directly as the monodromy associated to $\xi(\lambda_0^{\pm 1})$. To show the second part of (6.4), assume $\lambda_0 = 1$.

Define the parameter θ by $\lambda = e^{i\theta}$ and let $L = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. Then ξ has the symmetry $\xi(-\theta) = L(\theta)\xi(\theta)L^{-1}(\theta)$, from which it follows that $M_p(-\theta) = L(\theta)M_p(\theta)L^{-1}(\theta)$. Then

$$0 = \frac{d}{d\theta}\bigg|_{\theta=0} \big(M_p(-\theta)L(\theta) - L(\theta)M_p(\theta)\big) = -2\left(\frac{d}{d\theta}M_p\right)\bigg|_{\theta=0} .$$

The result $\frac{d}{d\lambda}\big|_{\lambda_0} M_p = 0$ follows. □

6.3. Local diagonalizability

We show that subject to a bound on the end weights, the n-noid monodromies satisfy the local diagonalizability condition (ii) of the Unitarization Theorem 3.4.

Lemma 6.5. *Let $\mathcal{C}$ be an analytic curve and $M \in \mathcal{H}_{\mathcal{C}}\mathrm{M}_{2\times 2}\mathbb{C}$ with local analytic eigenvalues $\mu_1, \mu_2 \in \mathcal{H}_{\mathcal{C}}\mathbb{C}$ at $p \in \mathcal{C}$. Then*

(i) $\mathrm{ord}_p(\mu_1 - \mu_2) \geq \mathrm{ord}_p(M - \mu_1\,\mathrm{I}) = \mathrm{ord}_p(M - \mu_2\,\mathrm{I})$.
(ii) *Assume M is not identically a scalar multiple of* I. *Then the eigenlines of M are non-coincident at p if and only if* $\mathrm{ord}_p(\mu_1 - \mu_2) = \mathrm{ord}_p(M - \mu_1\,\mathrm{I})$.

Proof. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define adjoint $(M) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. To prove (i), since $M +$ adjoint $(M) = (\mu_1 + \mu_2)\,\mathrm{I}$, then

$$M - \mu_1\,\mathrm{I} = -(\mathrm{adjoint}\,(M) - \mu_2\,\mathrm{I}) = -\,\mathrm{adjoint}\,(M - \mu_2\,\mathrm{I}).$$

Hence

$$\mathop{\mathrm{ord}}_p(M - \mu_1\,\mathrm{I}) = \mathop{\mathrm{ord}}_p(-\,\mathrm{adjoint}\,(M - \mu_2\,\mathrm{I})) = \mathop{\mathrm{ord}}_p(M - \mu_2\,\mathrm{I}).$$

Then, using $(\mu_1 - \mu_2)\,\mathrm{I} = (M - \mu_2\,\mathrm{I}) - (M - \mu_1\,\mathrm{I})$, we have

$$\mathop{\mathrm{ord}}_p(\mu_1 - \mu_2) \geq \min(\mathop{\mathrm{ord}}_p(M - \mu_1\,\mathrm{I}),\ \mathop{\mathrm{ord}}_p(M - \mu_2\,\mathrm{I})) = \mathop{\mathrm{ord}}_p(M - \mu_1\,\mathrm{I}).$$

To prove (ii), let t be a local coordinate at p on $\mathcal{C}$ such that $t = 0$ at p. By (i), we can define $n = \mathrm{ord}_p(M - \mu_1\,\mathrm{I}) = \mathrm{ord}_p(M - \mu_2\,\mathrm{I})$. Write

$$M - \mu_k\,\mathrm{I} = A_k t^n + \mathrm{O}(t^{n+1}), \quad k \in \{1, 2\},$$

with $A_1 \neq 0$ and $A_2 \neq 0$. For $k \in \{1, 2\}$, the eigenline map $\mathcal{C} \to \mathbb{P}^1$ corresponding to μ_k can be written locally as $[v_k]$, where $v_k = a_k + \mathrm{O}(t)$ for some $a_k \in \mathbb{C}^2 \setminus \{0\}$. Then $(M - \mu_k\,\mathrm{I})v_k = 0$ implies $a_k \in \ker A_k$. Then

$$(\mu_1 - \mu_2)\,\mathrm{I} = (M - \mu_2\,\mathrm{I}) - (M - \mu_1\,\mathrm{I}) = (A_2 - A_1)t^n + \mathrm{O}(t^{n+1}),$$

so $\mathrm{ord}_p(\mu_1 - \mu_2) > n$ if and only if $A_1 = A_2$. If $A_1 = A_2$, then $a_1 \in \ker A_1$ and $a_2 \in \ker A_1$, and $a_1 \neq 0$, $a_2 \neq 0$, $A_1 \neq 0$ imply $[a_1] = [a_2]$. Conversely, if $[a_1] = [a_2]$, then $a_1 \in \ker A_1$ and $a_1 \in \ker A_2$, so $a_1 \in \ker(A_1 - A_2)$. Since $A_1 - A_2$ is a scalar multiple of I, then $A_1 - A_2 = 0$. □

Lemma 6.6. *Let $M = M_p \in \Lambda\mathrm{SL}_2\mathbb{C}$ be an n-noid monodromy at p as above. Let $\rho = \rho_w$ be as in* (6.3) *and assume $|\rho| < \frac{1}{2}$ on $\mathbb{S}^1$. Then M is locally diagonalizable at each point of $\mathbb{S}^1$.*

Proof. Let μ be an eigenvalue of M. Then μ is locally analytic on $\mathbb{S}^1$ because $\frac{1}{2}\operatorname{tr} M \in [-1, 1]$ on $\mathbb{S}^1$. Let λ_0 be as in (5.5). Because $|\rho| < \frac{1}{2}$, μ is never -1 on $\mathbb{S}^1$, and μ is $+1$ on $\mathbb{S}^1$ only at $\lambda_0^{\pm 1}$. Define $n = n_{\lambda_0} : \mathbb{S}^1 \to \{0, 1, 2\}$ by

$$\begin{aligned} n_{\lambda_0}(p) &= 0 \text{ if } p \in \mathbb{S}^1 \setminus \{\lambda_0^{\pm 1}\}\,, \\ n_{\lambda_0}(\lambda_0^{\pm 1}) &= 1 \text{ if } \lambda_0 \in \mathbb{S}^1 \setminus \{1\}\,, \\ n_{\lambda_0}(\lambda_0) &= 2 \text{ if } \lambda_0 = 1. \end{aligned}$$

Then for all $p \in \mathbb{S}^1$, we have $\operatorname{ord}_p(\mu - 1) = n(p) = \operatorname{ord}_p(\mu - \mu^{-1})$, and by (6.4), $\operatorname{ord}_p(M - \mathrm{I}) \geq n(p)$, Then using $M - \mu\,\mathrm{I} = (M - \mathrm{I}) - (\mu - 1)\,\mathrm{I}$, we have

$$\operatorname*{ord}_p(M - \mu\,\mathrm{I}) \geq \min(\operatorname*{ord}_p(M - \mathrm{I}),\ \operatorname*{ord}_p(\mu - 1)) = n(p) = \operatorname*{ord}_p(\mu - \mu^{-1}),$$

and the result follows by Lemma 6.5. □

6.4. Unitarizing three loops whose product is I

The following well-known proposition [7] (see also [1, 23]) gives a condition for simultaneous unitarizability of three matrices whose product is I in terms of their traces.

Proposition 6.7 ([7]). *Let $M_1, M_2, M_3 \in \mathrm{SL}_2\mathbb{C}$ and suppose $M_1M_2M_3 = \mathrm{I}$. For $k \in \{1, 2, 3\}$, let $t_k = \frac{1}{2}\operatorname{tr} M_k$ and suppose $t_k \in [-1, 1]$. Define*

$$T = 1 - t_1^2 - t_2^2 - t_3^2 + 2t_1t_2t_3. \tag{6.5}$$

Then the M_1, M_2, M_3 are reducible if and only if $T = 0$, and are irreducible and simultaneously unitarizable if and only if $T > 0$.

If $M_1, M_2, M_3 : \mathcal{C} \to \mathrm{SL}_2\mathbb{C}$ are analytic maps on an analytic curve $\mathcal{C}$, then their infinitesimal irreducibility at a zero of T can in some cases be computed by the following technique.

Lemma 6.8. *Let $\mathcal{C}$ be an analytic curve and $p \in \mathcal{C}$. Let $M_1, M_2, M_3 \in \mathcal{H}_{\mathcal{C}}\mathrm{SL}_2\mathbb{C}$ satisfy $M_1M_2M_3 = \mathrm{I}$. Let T be as in* (6.5) *with $t_k = \frac{1}{2}\operatorname{tr} M_k$, $k \in \{1, 2, 3\}$. Then for any pair j, k in $\{1, 2, 3\}$, if $\operatorname{ord}_p[M_j, M_k] \geq n \geq 0$,*

$$\det([M_j, M_k]^{(n)})(p) = \frac{4}{b_{2n,n}} T^{(2n)}(p),$$

where the superscript (n) denotes differentiation n times with respect to a coordinate at p, and $b_{r,s}$ denotes the binomial coefficient.

Proof. We first show that for any analytic map $X : \mathcal{C} \to \mathrm{M}_{2\times 2}\mathbb{C}$ with $\operatorname{tr} X \equiv 0$, if $\operatorname{ord}_p X \geq n \geq 0$, then

$$\det(X^{(n)}(p)) = \frac{1}{b_{2n,n}} (\det X)^{(2n)}(p). \tag{6.6}$$

Differentiating the Cayley-Hamilton equation $(\det X)\,\mathrm{I} = -X^2$, we have

$$(\det X)^{(2n)} \cdot \mathrm{I} = -\textstyle\sum_{k=0}^{2n} b_{2n,k} X^{(k)} X^{(2n-k)}.$$

As $X^{(k)}(p) = 0$ for $k < n$, all terms in the sum are zero except possibly when $k = n$. This yields

$$(\det X)^{(2n)}(p) \cdot \mathrm{I} = -b_{2n,n} (X^{(n)}(p))^2 = b_{2n,n} \det(X^{(n)}(p)) \cdot \mathrm{I},$$

proving (6.6).

A calculation shows

$$\det([M_j, M_k]) = 4T. \tag{6.7}$$

The result follows directly from (6.7) and (6.6) with $X = [M_j, M_k]$. $\square$

7. Trinoids

In [22] a three-parameter family of constant mean curvature trinoids was constructed for each mean curvature H in each of the space forms $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, using r-Iwasawa decomposition for $r < 1$. Here we show, employing the 1-unitarization Theorem 3.4, that these immersions can be constructed with less machinery, using only the 1-Iwasawa decomposition.

7.1. Trinoid potentials

Definition 7.1 (Trinoid potentials). Let $\Sigma = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the thrice-punctured Riemann sphere. The family of *trinoid potentials* ξ on Σ, parametrized by λ_0 and $w_0, w_1, w_\infty \in \mathbb{R} \setminus \{0\}$, is given by ξ in Definition 6.1 with

$$Q = \frac{w_\infty z^2 + (w_1 - w_0 - w_\infty)z + w_0}{4z^2(z-1)^2} dz^2.$$

Q is the unique quadratic differential with double poles at $\{0, 1, \infty\}$ (the ends of the surface) with respective quadratic residues $w_0/4$, $w_1/4$, $w_\infty/4$, and no other poles.

A set of generators of the monodromy representation of a trinoid potential ξ is defined as follows. Choose a basepoint $z_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. For $k \in \{0, 1, \infty\}$, a set of closed curves γ_k based at z_0 can be chosen which wind respectively around $k \in \mathbb{P}^1$ once and not around any other point in $\{0, 1, \infty\}$, satisfying $\gamma_0\gamma_1\gamma_\infty = \mathrm{I}$. Define $M_k : \mathbb{C}^* \to \mathrm{SL}_2\mathbb{C}$ as the monodromy of the solution $\Phi = \Phi(z, \lambda)$ to the equation $d\Phi = \Phi\xi$, $\Phi(z_0, \lambda) = \mathrm{I}$ along γ_k. Then by the choice of $\gamma_0, \gamma_1, \gamma_\infty$, we have $M_0 M_1 M_\infty = \mathrm{I}$.

7.2. Pointwise unitarizability

A key step from [22] in the trinoid construction is showing, with a suitable set of inequalities, that the monodromy representation is pointwise unitarizable on $\mathbb{S}^1$. The following lemma is a restatement of the required lemma in [22] with inequalities replaced by strict inequalities.

Lemma 7.2 ([22]). *Let ξ be a trinoid potential parametrized by $\lambda_0, w_0, w_1, w_\infty$, and let $\{M_0, M_1, M_\infty\}$ be the generators of the monodromy representation for ξ as described above. For $k \in \{0, 1, \infty\}$ define $\rho_k = \rho_{w_k}$ as in* (6.3) *and $n_k = \rho_{w_k}(-1)$ and $m_k = \rho_{w_k}(1)$. Suppose the following inequalities hold for every permutation (i, j, k) of* $(0, 1, \infty)$*:*

$$|n_0| + |n_1| + |n_\infty| < 1 \text{ and } |n_i| < |n_j| + |n_k| \text{ for all space forms,} \tag{7.1}$$

$$|m_0| + |m_1| + |m_\infty| < 1 \text{ and } |m_i| < |m_j| + |m_k| \text{ for } \mathbb{S}^3 \text{ and } \mathbb{H}^3\,, \tag{7.2}$$

$$|w_i| < |w_j| + |w_k| \text{ for } \mathbb{R}^3. \tag{7.3}$$

Then the monodromy representation for ξ is pointwise unitarizable on $\mathbb{S}^1$, and is irreducible on $\mathbb{S}^1 \setminus \{\lambda_0^{\pm 1}\}$. Moreover, $|\rho_k| < \frac{1}{2}$ on $\mathbb{S}^1$.

7.3. Infinitesimal irreducibility

Lemma 7.3. *Let ξ be a trinoid potential parametrized by λ_0, w_0, w_1, w_∞. Let M_0, M_1, M_∞ be the monodromies for ξ as in Section* 7.2*. Suppose condition* (7.3) *holds. Then M_0, M_1, M_∞ are pairwise infinitesimally irreducible at $\{\lambda_0^{\pm 1}\}$.*

Proof. Let $'$ denote differentiation with respect to λ, and let the superscript (k) denote differentiation k times with respect to λ. Let T be as in (6.5). Choose distinct $j, k \in \{0, 1, \infty\}$.

Let $\chi \in \mathbb{R}$ be defined by

$$\chi = (w_0 + w_1 + w_\infty)(-w_0 + w_1 + w_\infty)(w_0 - w_1 + w_\infty)(w_0 + w_1 - w_\infty). \tag{7.4}$$

Note that $\chi = 0$ if and only if $|w_i| = |w_j| + |w_k|$ for some permutation $(i,\ j,\ k)$ of $(0,\ 1,\ \infty)$. Hence by condition (7.3), $\chi \neq 0$.

First take the case $\lambda_0 \neq 1$. A calculation using $M_r(\lambda_0^{\pm 1}) = \mathrm{I}$, $r \in \{j, k\}$, shows $\operatorname{ord}_{\lambda_0^{\pm 1}}[M_j,\ M_k] \geq 2$. By Lemma 6.8 and a calculation,

$$\det([M_j, M_k]^{(2)}(\lambda_0^{\pm 1})) = \frac{4}{b_{4,2}} T^{(4)}(\lambda_0^{\pm 1}) = \frac{4}{b_{4,2}} 3 \cdot 2^{-11} \pi^4 (1 - \lambda_0^{\mp 2})^4 \chi.$$

Thus since $\chi \neq 0$, then $\det([M_j, M_k]^{(2)}(\lambda_0^{\pm 1})) \neq 0$ and M_j and M_k are infinitesimally irreducible at $\lambda_0^{\pm 1}$ by Definition 3.1.

For the case $\lambda_0 = 1$, a calculation using $M_r(1) = \mathrm{I}$, $M_r'(1) = 0$, $r \in \{j, k\}$, shows $\mathrm{ord}_1[M_j, M_k] \geq 4$. By Lemma 6.8 and a calculation,

$$\det([M_j, M_k]^{(4)}(1)) = \frac{4}{b_{8,4}} T^{(8)}(1) = \frac{4}{b_{8,4}} 315 \cdot 2^{-7} \pi^4 \chi.$$

Thus since $\chi \neq 0$, then $\det([M_j, M_k]^{(4)}(1)) \neq 0$ and M_j and M_k are infinitesimally irreducible at 1 by Definition 3.1. □

7.4. Constructing trinoids

The results of Sections 7.1–7.3 are now brought together to construct a family of trinoids.

Theorem 7.4 (Trinoids). *Let ξ be a trinoid potential on $\Sigma = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ satisfying the inequalities* (7.1)–(7.3). *Then there exists a solution Ψ of the equation $d\Psi = \Psi\xi$ such that Ψ induces a CMC H immersion of Σ into the appropriate space form $\mathbb{R}^3$ or $\mathbb{S}^3$ or $\mathbb{H}^3$ via the extended Weierstrass representation, where the mean curvature H is subject to the restrictions in Section* 5.3.

Proof. Let $z_0 \in \Sigma$ be a basepoint, and let Φ be the solution to $d\Phi = \Phi\xi$, $\Phi(z_0, \lambda) = \mathrm{I}$. We show that the hypotheses of Theorem 3.4 hold for the generators $\{M_0, M_1, M_\infty\}$ of the trinoid monodromy representation.

By Lemma 7.2, M_0, M_1, M_∞ are pairwise irreducible on $\mathbb{S}^1 \setminus \{\lambda_0^{\pm 1}\}$ and hence no two identically commute. By the same lemma, M_0, M_1, M_∞ are pointwise unitarizable on $\mathbb{S}^1$, condition (i) of Theorem 3.4.

Lemma 7.2 provides the bound $|\rho_k| < \frac{1}{2}$, so by Lemma 6.6, M_0, M_1, M_∞ are each locally diagonalizable at each point of $\mathbb{S}^1$, condition (ii) of Theorem 3.4.

Since M_0, M_1, M_∞ are irreducible on $\mathbb{S}^1 \setminus \{\lambda_0^{\pm 1}\}$, they are pairwise infinitesimally irreducible there. By Lemma 7.3, M_0, M_1, M_∞ are pairwise infinitesimally irreducible at $\{\lambda_0^{\pm 1}\}$. Therefore M_0, M_1, M_∞ are pairwise infinitesimally irreducible on $\mathbb{S}^1$, condition (iii) of Theorem 3.4.

Thus all conditions of Theorem 3.4 are satisfied, so by that theorem there exists an analytic loop $C \in \Lambda \mathrm{SL}_2\mathbb{C}$ which unitarizes the monodromy representation of Φ. In the case of the spaceform $\mathbb{H}^3$, C may be singular at λ_0, so let $C = C_u \cdot C_+$ be the 1-Iwasawa decomposition of C. Then C_+ likewise unitarizes the monodromy representation of Φ, and is nonsingular at λ_0 for any spaceform.

Then $\Psi = C_+\Phi$ satisfies the appropriate closing condition (5.6)–(5.8) since condition (6.4) is independent of conjugation by an analytic loop. Hence the immersion induced by Ψ is well-defined on Σ. □

Remark 7.5. In the case in which any of the inequalities (7.1)–(7.3) becomes an equality, the trinoid can be constructed via the r-Unitarization Theorem 4.4 (see [22]).

8. Symmetric n-noids

The above ideas are now applied to the construction of CMC symmetric n-noids, genus-zero surfaces similar to trinoids, but having n ends and full dihedral symmetry of order n.

8.1. Symmetric n-noid potentials

Definition 8.1 (Symmetric n-noid potentials). Let n be an integer with $n \geq 3$. Let Σ be the n-punctured Riemann sphere $\Sigma = \mathbb{P}^1 \setminus \{z^n = 1\}$. Let $w \in \mathbb{R} \setminus \{0\}$. The family of *symmetric n-noid potentials* ξ, parametrized by λ_0 and w, is given by ξ in Definition 6.1 with

$$Q = \frac{n^2 w z^{n-2}}{4(z^n - 1)^2} dz^2 \,.$$

Q is chosen to have double poles at $\{z^n = 1\}$ (the ends of the surface), with quadratic residue $w/4$ at each pole, and no other poles. The choice of ends gives Q a symmetry that will imply the pointwise simultaneously unitarizability of the monodromy group for ξ on $\mathbb{S}^1$.

A set of generators of the monodromy representation of a symmetric n-noid potential ξ is defined as follows. A set of closed curves $\gamma_0, \ldots, \gamma_{n-1}, \gamma_\infty$ based at 0 can be chosen which respectively wind around $e^{2\pi i 0/n}, \ldots, e^{2\pi i (n-1)/n}$, ∞ once and not around any other of these points, satisfying $\gamma_0 \ldots \gamma_{n-1}\gamma_\infty = \mathrm{I}$. Define $M_0, \ldots, M_{n-1}$, $M_\infty : \mathbb{C}^* \to \mathrm{SL}_2\mathbb{C}$ as the monodromies of the solution $\Phi(z, \lambda)$ to the equation $d\Phi = \Phi\xi$, $\Phi(0, \lambda) = \mathrm{I}$ along $\gamma_0, \ldots, \gamma_{n-1}$, γ_∞ respectively. This choice gives $M_0 \cdots M_{n-1} M_\infty = \mathrm{I}$.

Lemma 8.2. *Let ξ be a symmetric n-noid potential. Define the gauge*

$$g = \mathrm{diag}(\alpha^{1/2}, \alpha^{-1/2}), \quad \alpha = e^{2\pi i/n}. \tag{8.1}$$

Let $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ be the automorphism of $\mathbb{P}^1$ defined by $\tau(z) = \alpha z$. Then, referring to (6.2) *for the action of a gauge on a potential,*

(i) *ξ has the symmetry $\tau^*\xi = \xi.(g^{-1}) = \mathrm{Ad}\, g \cdot \xi$.*
(ii) *Let $\Phi = \Phi(z, \lambda)$ solve $d\Phi = \Phi\xi$, $\Phi(0, \lambda) = \mathrm{I}$. Then Φ has the symmetry $\tau^*\Phi = g\Phi g^{-1}$.*
(iii) *The monodromy matrices $M_0, \ldots, M_{n-1}$ of Φ satisfy $M_k = g^k M_0 g^{-k}$ for $k = 0, \ldots, n-1$.*

Proof. Showing (i) is a calculation. By (i), every solution of $d\Phi = \Phi\xi$ has the symmetry $\tau^*\Phi = A\Phi g^{-1}$ for some A. Evaluating at $z = 0$ yields $A = g$, implying (ii). Symmetry (iii) follows. $\square$

8.2. Pointwise unitarizability

The techniques of Section 6.4 do not directly apply to the generators $M_0, \dots, M_{n-1}$ of the symmetric n-noid monodromy representation, but rather to the triple M_0, g, $(M_0 g)^{-1}$, whose product is I and whose traces are computable. The pointwise or loopwise unitarizability of this triple implies the same for the symmetric n-noid monodromy representation.

The value of w in the symmetric n-noid potential determines the weight of the Delaunay potential to which the symmetric n-noid potential is asymptotic [12, 22, 14, 15]. So the condition (8.2) in the lemma below amounts to a restriction on the weight of the ends.

Lemma 8.3. *Let ξ be an n-noid potential with parameters w and λ_0. With h as in* (6.1) *and ρ_w as in* (6.3)*, let M_0 be the symmetric n-noid monodromy defined above and let g be as in* (8.1)*. Suppose*

$$|\rho(1)| < \frac{1}{n} \quad \textit{and} \quad |\rho(-1)| < \frac{1}{n}\,. \tag{8.2}$$

Then g and M_0 are pointwise simultaneously unitarizable on $\mathbb{S}^1$, and are irreducible on $\mathbb{S}^1 \setminus \{\lambda_0^{\pm 1}\}$.

Proof. Since ρ attains it minimum and maximum on $\mathbb{S}^1$ at $\lambda = 1$ and $\lambda = -1$ in some order, condition (8.2) is equivalent to the condition that ρ takes values in $(-1/n,\ 1/n)$ on $\mathbb{S}^1$.

By Lemma 6.3 we have $M_0 \cdots M_{n-1} = \mathrm{I}$. A calculation using Lemma 8.2(iii) implies

$$(M_0 g)^n = -\mathrm{I}\ . \tag{8.3}$$

It follows that the eigenvalues of $M_0 g$ are n'th roots of -1, and are hence constant. With α as in (8.1), using $M_0(\lambda_0) = \mathrm{I}$ we get

$$\text{the eigenvalues of } M_0 g \text{ are } \alpha^{\pm 1/2}. \tag{8.4}$$

Now consider the triple $(M_0,\ g,\ (M_0 g)^{-1})$. Their product is I, and

$$t := \frac{1}{2}\operatorname{tr} M_0 = \cos(2\pi\rho_w) \quad \frac{1}{2}\operatorname{tr} g = \frac{1}{2}\operatorname{tr}((M_0 g)^{-1}) = (\alpha^{1/2} + \alpha^{-1/2})/2.$$

Hence with T as in (6.5),

$$T(\lambda) = (1 - t)(t - (\alpha + \alpha^{-1})/2). \tag{8.5}$$

Condition (8.2) implies that ρ_w takes values in $(-1/n,\ 1/n)$ on $\mathbb{S}^1$, and ρ_w is zero only at $\lambda_0^{\pm 1}$. Hence t takes values in $((\alpha + \alpha^{-1})/2,\ 1]$ and the zero set of T on $\mathbb{S}^1$ is $\{\lambda_0^{\pm 1}\}$. Then by Proposition 6.7, $(M_0,\ g,\ (M_0 g)^{-1})$ are simultaneously unitarizable and irreducible on $\mathbb{S} \setminus \{\lambda_0^{\pm 1}\}$. We have by (6.4) that $M_0 = \mathrm{I}$, so $(M_0,\ g,\ (M_0 g)^{-1})$ are simultaneously unitarizable at $\{\lambda_0^{\pm 1}\}$, and hence on $\mathbb{S}^1$. $\square$

8.3. Infinitesimal irreducibility

Lemma 8.4. *Let ξ be a symmetric n-noid potential. Let M_0 be the symmetric n-noid monodromy defined in Section* 8.2 *and g as in* (8.1). *Then g and M_0 are infinitesimally irreducible at $\{\lambda_0^{\pm 1}\}$.*

Proof. If $\lambda_0 \neq 1$, then $M_0(\lambda_0^{\pm 1}) = \mathrm{I}$, so $\operatorname{ord}_{\lambda_0^{\pm 1}}[g,\ M_0] \geq 1$. By Lemma 6.8 and a calculation using (8.5), taking derivatives with respect to the parameter θ for $\lambda = e^{i\theta}$, with α as in (8.1),

$$\det([g,\ M_0]^{(1)}(\lambda_0^{\pm 1})) = \frac{4}{b_{2,1}} T^{(2)}(\lambda_0^{\pm 1}) = \frac{1}{\alpha b_{2,1}} 2^{-5}\pi^2(1-\alpha)^2(\lambda_0 - \lambda_0^{-1})^2 w^2,$$

and since $\lambda_0^2 \neq 1$, $\alpha \neq 1$ and $w \neq 0$, then $\det([g,\ M_0]^{(1)}(\lambda_0^{\pm 1})) \neq 0$ and g and M_0 are infinitesimally irreducible at $\lambda_0^{\pm 1}$ by the definition of infinitesimal irreducibility in Section 3.

If $\lambda_0 = 1$, then $M_0(1) = \mathrm{I}$ and $M_0^{(1)}(1) = 0$, so $\operatorname{ord}_1[g,\ M_0] \geq 2$. By Lemma 6.8 and a calculation using (8.5),

$$\det([g,\ M_0]^{(2)}(1)) = \frac{4}{b_{4,2}} T^{(4)}(1) = -\frac{1}{\alpha b_{4,2}} 3 \cdot 2^{-3}\pi^2(1-\alpha)^2 w^2,$$

so $\det([g,\ M_0]^{(2)}(1)) \neq 0$ and g and M_0 are infinitesimally irreducible at 1. $\square$

8.4. Constructing symmetric n-noids

The results of Sections 8.1–8.3 are now brought together to construct a family of symmetric n-noids.

Theorem 8.5. *With $n \geq 3$, let ξ be a symmetric n-noid potential on $\Sigma = \mathbb{P}^1 \setminus \{z^n = 1\}$ such that the inequalities* (8.2) *hold. Then there exist solutions Ψ of the equation $d\Psi = \Psi\xi$ such that the Ψ induce a 1-parameter family of CMC H immersions of Σ into each of the space forms $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, where $H \in \mathbb{R}$ is subject to the restrictions of Section* 5.3.

Proof. Let M_0 be the monodromy described in Section 8.1 and let g be as in 8.2. Let Φ solve the equation $d\Phi = \Phi\xi$ with $\Phi(0,\ \lambda) = \mathrm{I}$. We show that the hypotheses of Theorem 3.4 hold for the triple $\{M_0,\ g,\ (M_0 g)^{-1}\}$.

By Lemma 8.3, M_0 and g are irreducible on $\mathbb{S}^1 \setminus \{\lambda_0^{\pm}\}$, and hence $[M_0,\ g] \not\equiv 0$. By the same lemma, M_0 and g are pointwise simultaneously unitarizable at every point of $\mathbb{S}^1$, condition (i) of Theorem 3.4. Condition (ii) of Theorem 3.4 holds because g is diagonal. Lemmas 8.3 and 8.4 show that M_0 and g are infinitesimally irreducible at every point of $\mathbb{S}^1$, condition (iii) of Theorem 3.4.

Thus all conditions of Theorem 3.4 are satisfied, so there exists an analytic loop $C \in \Lambda\mathrm{SL}_2\mathbb{C}$ which simultaneously unitarizes M_0 and g. C unitarizes the monodromy representation described in Section 8.1, because it is contained in the group generated by M_0 and g. In the case of the spaceform $\mathbb{H}^3$, C may be singular at λ_0, so let $C = C_u \cdot C_+$ be the 1-Iwasawa decomposition of C. Then C_+ likewise unitarizes the monodromy representation.

Let $\Psi = C_+\Phi$. Then the solution Ψ to $d\Phi = \Psi\xi$, $\Psi(0, \lambda) = C_+$ has unitary monodromy satisfying the appropriate closing condition (5.6)–(5.8), since condition (6.4) is independent of conjugation by an analytic loop. Hence the CMC immersion induced by Ψ via the extended Weierstrass representation into the appropriate space form is well-defined on Σ. Note that $z = \infty$ is a finite smooth point of the immersion, by Lemma 6.3, so the surface has n ends. For each spaceform and each choice of n, this produces a one-parameter family of surfaces parametrized by w. □

Remark 8.6. The methods in this section can be extended to a broader family of symmetric n-noids whose end axes are not coplanar [21].

References

[1] I. BISWAS, *On the existence of unitary flat connections over the punctured sphere with given local monodromy around the punctures*, Asian J. Math **3** (1999), 333–344.

[2] A. I. BOBENKO, *Constant mean curvature surfaces and integrable equations*, Russian Math. Surveys **46** (1991), 1–45.

[3] J. DORFMEISTER and G. HAAK, *Construction of non-simply connected CMC surfaces via dressing*, J. Math. Soc. Japan **55** (2003), 335–364.

[4] J. DORFMEISTER, F. PEDIT, and H. WU, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom. **6** (1998), 633–668.

[5] J. DORFMEISTER and H. WU, *Construction of constant mean curvature trinoids from holomorphic potentials*, preprint, 2000.

[6] J. DORFMEISTER and H. WU *Unitarization of loop group representations of fundamental groups*, preprint, 2005.

[7] W. M. GOLDMAN, *Topological components of spaces of representations*, Invent. Math. **93** (1988), 557–607.

[8] K. GROSSE-BRAUCKMANN, *New surfaces of constant mean curvature*, Math. Z. **214** (1993), 527–565.

[9] K. GROSSE-BRAUCKMANN, R. KUSNER and J. M. SULLIVAN, *Triunduloids: embedded constant mean curvature surfaces with three ends and genus zero*, J. Reine Angew. Math. **564** (2003), 35–61.

[10] K. GROSSE-BRAUCKMANN, R. KUSNER and J. M. SULLIVAN, *Coplanar constant mean curvature surfaces*, math.DG/0509210, 2005.

[11] N. KAPOULEAS, *Complete constant mean curvature surfaces in Euclidean three space*, Ann. of Math. **131** (1990), 239–330.

[12] M. KILIAN, W. ROSSMAN and N. SCHMITT, *Delaunay ends of constant mean curvature surfaces*, preprint, 2006.

[13] M. KILIAN, N. SCHMITT and I. STERLING, *Dressing CMC n-noids*, Math. Z. **246** (2004), 501–519.

[14] N. KOREVAAR, R. KUSNER, W. MEEKS III and B. SOLOMON, *Constant mean curvature surfaces in hyperbolic space*, Amer. J. Math. **114** (1992), 1–43.

[15] N. KOREVAAR, R. KUSNER and B. SOLOMON, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geom. **30** (1989), no. 2, 465–503.

[16] R. MAZZEO and F. PACARD, *Constant mean curvature surfaces with Delaunay ends*, Comm. Anal. Geom. **9** (2001), 169–237.
[17] I. MCINTOSH, *Global solutions of the elliptic 2d periodic Toda lattice*, Nonlinearity **7** (1994), 85–108.
[18] A. PRESSLEY and G. SEGAL, "Loop Groups", Oxford Science Monographs, Oxford Science Publications, 1988.
[19] J. RATZKIN, *An end to end gluing construction for metrics of constant positive scalar curvature*, Indiana Univ. Math. J. **52** (2003), 703–726.
[20] N. SCHMITT, *CMCLab*, http://www.gang.umass.edu/software.
[21] N. SCHMITT, *Constant mean curvature n-noids with symmetries*, preprint, 2006.
[22] N. SCHMITT, M. KILIAN, S. KOBAYASHI and W. ROSSMAN, *Unitarization of monodromy representations and constant mean curvature trinoids in 3-dimensional space forms*, J. London Math. Society, to appear.
[23] M. UMEHARA and K. YAMADA, *Metrics of constant curvature 1 with three conical singularities on the 2-sphere*, Illinois. J. Math. **44** (2000), 72–94.

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