

Addendum to: On volumes of arithmetic quotients of $\mathrm{SO}(1, n)$

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Abstract. There are errors in the proof of uniqueness of arithmetic subgroups of the smallest covolume. In this note we correct the proof, obtain certain results which were stated as a conjecture, and we give several remarks on further developments.

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1.1. Let us recall some notation and basic notions. Following [1] we will assume that n is even and $n \geq 4$. The group of orientation preserving isometries of hyperbolic n -space is isomorphic to $\mathrm{SO}(1, n)^o$, the connected component of the identity of the special orthogonal group of signature $(1, n)$, which can be identified with $\mathrm{SO}_0(1, n)$, the subgroup of $\mathrm{SO}(1, n)$ preserving the upper half space. This group is not Zariski closed in SL_{n+1} thus in order to construct arithmetically defined subgroups of $\mathrm{SO}(1, n)^o$ we consider arithmetic subgroups of the orthogonal group $\mathrm{SO}(1, n)$ or, more precisely, of groups $G = \mathrm{SO}(f)$ where f is an admissible quadratic form defined over a totally real number field k (see [1, Section 2.1]).

We have an exact sequence of k -isogenies:

$$1 \rightarrow C \rightarrow \tilde{G} \xrightarrow{\phi} G \rightarrow 1, \quad (1.1)$$

where $\tilde{G}(k) \simeq \mathrm{Spin}(f)$ is the simply connected cover of G and $C \simeq \mu_2$ is the center of \tilde{G} . This induces an exact sequence in Galois cohomology (see [5, Section 2.2.3])

$$\tilde{G}(k) \xrightarrow{\phi} G(k) \xrightarrow{\delta} H^1(k, C) \rightarrow H^1(k, \tilde{G}). \quad (1.2)$$

The main idea of this note is that by using (1.2) certain questions about arithmetic subgroups of G can be reduced to questions about the Galois cohomology group $H^1(k, C)$.

A coherent collection of parahoric subgroups $P = (P_v)_{v \in V_f}$ of \tilde{G} ($V_f = V_f(k)$ denotes the set of finite places of the field k) defines a principal arithmetic subgroup

$\Lambda = \tilde{G}(k) \cap \prod_{v \in V_f} P_v \subset \tilde{G}(k)$ (see [2]). We fix an infinite place v of k for which $G(k_v) \simeq \text{SO}(1, n)$ and denote it by Id . The image of Λ under the central k -isogeny ϕ is an arithmetic subgroup of G and every maximal arithmetic subgroup of $G(k_{Id})$ can be obtained as a normalizer of some $\phi(\Lambda)$ [2, Proposition 1.4]. We will also consider the local stabilizers of P in the adjoint group $G(=\tilde{G})$, defining \bar{P}_v to be the stabilizer of P_v in $G(k_v)$ and $\bar{P} = (\bar{P}_v)_{v \in V_f}$. Clearly, $\bar{P}_v \supset \phi(P_v)$. In the notation of [1] the subgroups $\phi(P_v)$ are called parahoric subgroups of G , however this terminology is non-standard and we will avoid using it here.

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1.2. Given a totally real number field k with the group of units U , let

$$k_\infty^* = \{a \in k^* \mid a_v > 0 \text{ for } v \in V_\infty \setminus Id\}, \quad U_\infty = U \cap k_\infty^*.$$

Lemma 1.1. $\text{Im}(\delta) \simeq k_\infty^*/(k^*)^2$.

Proof. From (1.2) we have $\text{Im}(\delta : G(k) \rightarrow H^1(k, \mu_2)) = \text{Ker}(H^1(k, \mu_2) \rightarrow H^1(k, \tilde{G}))$. The Hasse principle for simply connected k -groups implies that $H^1(k, \tilde{G})$ is isomorphic to $\prod_{v \in V_\infty} H^1(k_v, \tilde{G})$ [5, Theorem 6.6], and hence

$$\text{Im}(\delta) = \text{Ker}(H^1(k, \mu_2) \rightarrow \prod_{v \in V_\infty} H^1(k_v, \tilde{G})).$$

The group $H^1(k, \mu_2)$ is canonically isomorphic to $k^*/(k^*)^2$ [5, Lemma 2.6]. It is well known that for all $v \in V_\infty$ such that the group $G(k_v)$ is anisotropic, the map ϕ in (1.2) is surjective and hence for all such v , $\text{Im}(\delta_v) = \text{Ker}(H^1(k_v, \mu_2) \rightarrow H^1(k_v, \tilde{G}))$ is trivial. For the remaining one infinite place $v(= Id) \in V_\infty$, $\phi(\tilde{G}(k_v))$ is a subgroup of index 2 in $G(k_v)$ which consists of the orthogonal transformations with the trivial spinor norm. Collecting this information together we obtain the required isomorphism. □

1.3 The proof of the uniqueness part in [1, Theorem 4.1] contains errors but the result is correct. We will now give another argument for it. In order to do so we first establish a more general fact and then apply it to the cases considered in [1].

Let $P = (P_v)_{v \in V_f}$ and $P' = (P'_v)_{v \in V_f}$ be two coherent collections of parahoric subgroups of \tilde{G} such that for all $v \in V_f$, P'_v is conjugate to P_v under an element of $G(k_v)$. For all but finitely many v , $P_v = P'_v$ hence there is an element $g \in G(\mathbb{A}_f)$ (\mathbb{A}_f denotes the ring of finite adèles of k) such that P' is the conjugate of P under g . We have $\bar{P} = \prod_{v \in V_f} \bar{P}_v$ is the stabilizer of P in $G(\mathbb{A}_f)$. The number of distinct

$G(k)$ -conjugacy classes of coherent collections P' as above is the cardinality $c(\overline{P})$ of $\mathcal{C}(\overline{P}) = G(k) \backslash G(\mathbb{A}_f) / \overline{P}$, which is called the class group of G relative to \overline{P} . The class number $c(\overline{P})$ is known to be finite (see e.g. [2, Proposition 3.9]). The following result can be used for obtaining further information about its value.

Proposition 1.2. *Let $G = SO(f)$, $\tilde{G} = Spin(f)$ for an admissible quadratic form f defined over k and let $P = (P_v)_{v \in V_f}$ a coherent collection of parahoric subgroups of \tilde{G} . The class number $c(P)$ divides the order $h_{\infty, 2}$ of a restricted 2-class group of k given by*

$$h_{\infty, 2} = \frac{2^{[k:\mathbb{Q}]-1} h_2}{[U : U_{\infty}]},$$

where h_2 is the order of the 2-class group of k .

Proof. Recall two isomorphisms (see [5, Proposition 8.8], a minor modification is needed in order to adjust the statement to our setting but the argument remains the same):

$$G(k) \backslash G(\mathbb{A}_f) / \overline{P} \simeq G(\mathbb{A}_f) / \overline{P} G(k);$$

$$G(\mathbb{A}_f) / \overline{P} G(k) \simeq \delta_{\mathbb{A}_f}(G(\mathbb{A}_f)) / \delta_{\mathbb{A}_f}(\overline{P} G(k)),$$

where $\delta_{\mathbb{A}_f}$ is the restriction of the product map $\prod_v G(k_v) \rightarrow \prod_v H^1(k_v, \mathbb{C})$ to $G(\mathbb{A}_f)$.

For every finite place v , $H^1(k_v, \tilde{G})$ is trivial (see [5, Theorem 6.4]) which implies $\delta_v : G(k_v) \rightarrow H^1(k_v, \mathbb{C})$ is surjective. Thus the image of $\delta_{\mathbb{A}_f}(G(\mathbb{A}_f))$ can be identified with the restricted direct product $\prod' H^1(k_v, \mathbb{C})$ with respect to the subgroups $\delta_v(\overline{P}_v)$. Also $\delta_{\mathbb{A}_f}(G(k))$ naturally identifies with the image of $\delta(G(k))$ in $H^1(k, \mathbb{C})$ under the embedding $\psi : H^1(k, \mathbb{C}) \rightarrow \prod' H^1(k_v, \mathbb{C})$. Hence we have an isomorphism

$$\delta_{\mathbb{A}_f}(G(\mathbb{A}_f)) / \delta_{\mathbb{A}_f}(\overline{P} G(k)) \simeq \prod' H^1(k_v, \mathbb{C}) / (\prod_v \delta_v(\overline{P}_v) \cdot \psi(\text{Im } \delta(G(k)))).$$

The group $H^1(k_v, \mu_2)$ is canonically isomorphic to $k_v^* / (k_v^*)^2$, by Lemma 1.1 $\text{Im } \delta(G(k)) \simeq k_{\infty}^* / (k^*)^2$, so we obtain

$$\frac{\prod' H^1(k_v, \mathbb{C})}{\prod_v \delta_v(\overline{P}_v) \cdot \psi(\text{Im } \delta(G(k)))} \simeq \frac{\prod' k_v^* / (k_v^*)^2}{\delta_P \cdot k_{\infty}^* / (k^*)^2} \simeq \frac{J_f}{\delta_P \cdot J_f^2 k^*} \cdot \frac{k^*}{k_{\infty}^*},$$

where J_f is the ring of finite idèles of k and δ_P denotes $\prod_v \delta_v(\overline{P}_v)$.

Now, $\#(J_f / J_f^2 k^*) = h_2$, the group k^* / k_{∞}^* splits as a product of local factors and $\#(k^* / k_{\infty}^*) = 2^{[k:\mathbb{Q}]-1} / [U : U_{\infty}]$ (see [4, Chapter 6]). This implies the proposition. \square

In order to give a precise formula for the class number $c(P)$ one has to analyze the image of $\prod_v \delta_v(\overline{P}_v)$ in $\prod' H^1(k_v, \mathbb{C})$. Still in many practical cases this appears to be unnecessary. Thus in order to prove the uniqueness of the minimal hyperbolic orbifolds we need to consider $k = \mathbb{Q}[\sqrt{5}]$ (in the compact case) and $k = \mathbb{Q}$ (for the non-compact orbifolds). In both cases $h_2 = h = 1$. For $k = \mathbb{Q}[\sqrt{5}]$, $U_\infty = \{1, \frac{1-\sqrt{5}}{2}\}$ and thus $[U : U_\infty] = 2$ which implies $h_{\infty,2} = 1$. For $k = \mathbb{Q}$, clearly, $h_{\infty,2} = 1$ as well. So in all the cases $c(P) = 1$ which implies that the corresponding arithmetic subgroups are defined uniquely up to a conjugation by $g \in \text{SO}(1, n)$. It is clear that we can always chose $g \in \text{SO}_0(1, n)$ and therefore the smallest orbifolds constructed in [1] are unique up to an (orientation preserving) isometry.

1.4. We now turn to Conjecture 4.1 and its analogue for the non-cocompact orbifolds in [1, Section 4.4]. Recall that in [1] the numbers $N(r)$, $N'(r)$ were defined for every $r \geq 2$ and estimated from above. These numbers are related to the index of the principal arithmetic subgroups in their normalizers. We now prove

Proposition 1.3. *For every $r \geq 2$, $N(r) = N'(r) = 1$.*

Proof. Let Λ be a principal arithmetic subgroup of \tilde{G} which corresponds to a compact or non-compact hyperbolic n -orbifold of the minimal volume, $\Lambda' = \phi(\Lambda)$ and $\Gamma = N_G(\Lambda')$.

From [2, Proposition 2.9], which in turn follows from the work of J. Rohlfs, using the fact that the center of our group G is trivial, we obtain:

$$[\Gamma : \Lambda'] = \#(H^1(k, \mu_2)_\Theta \cap \delta(G(k))) = \#\text{Im}(\delta : G(k) \rightarrow H^1(k, \mu_2)).$$

By Lemma 1.1 we can identify the image of δ . The cases we are interested in are

$$k = \mathbb{Q} : \text{Im}(\delta) = \{k^{*2}, (-1)k^{*2}\};$$

$$k = \mathbb{Q}[\sqrt{5}] : \text{Im}(\delta) = \left\{ k^{*2}, \frac{1-\sqrt{5}}{2}k^{*2} \right\}.$$

In both cases $[\Gamma : \Lambda'] = \#\text{Im}(\delta) = 2$. Now it is easy to see that $\Lambda' = \phi(\Lambda) \subset \text{SO}_0(1, n)$. From the other side there always exists $g \in \text{SO}(1, n) \setminus \text{SO}_0(1, n)$ which normalizes $\phi(\Lambda)$. For example take $g = \text{diag}(-1, -1, 1, \dots, 1)$. As in all the cases under consideration the quadratic form associated to Λ is diagonal [1, Sections 4.3, 4.4], g stabilizes Λ and clearly $g \in \text{SO}(1, n) \setminus \text{SO}_0(1, n)$. From these facts it follows that Λ' is a maximal arithmetic subgroup in $\text{SO}_0(1, n)$ and thus $N(r)$ (or $N'(r)$) = 1. □

This proposition makes *precise* the statements of Theorem 4.1 and 4.4 of [1]. It also implies that Table 2 of *loc. cit.* gives the precise values of the covolumes of the smallest n -dimensional hyperbolic orbifolds in even dimensions up to 18.

One other corollary is that cocompact and non-cocompact arithmetic subgroups of $SO(1, 2r)^\circ$ of the smallest covolumes can be obtained as the stabilizers of certain lattices described in [1, Section 4.3]. We remark that since the fields of definition of the groups have class number 1, the lattices in both cases are free as \mathcal{O}_k -modules.

1.5. Correction: on p. 765, l. 9 one should read “grow super-exponentially” instead of “grow exponentially”. (It follows from [1] that the Euler characteristic is bounded from below by $\text{const} \cdot \left(\prod_{i=1}^r \frac{(2i-1)!}{(2\pi)^{2i}}\right)^{[k:\mathbb{Q}]}$ which for large enough r is $\geq \text{const} \cdot (2r - 1)!$)

We conclude this addendum with a few remarks on related results which appeared after the paper was published.

1.6. In [1, Section 4.5] we observed that for $r > 2$ the minimal covolume among the arithmetic lattices in $SO(1, 2r)$ is attained on a non-uniform lattice. This interesting phenomenon was first discovered by A. Lubotzky for SL_2 over local fields of positive characteristic. Recently, in [6] A. Salehi Golsefidy proved that lattices of minimal covolume in classical Chevalley groups over local fields of characteristic $p > 7$ are all non-uniform. This result gives further support to a **conjecture** that *generically (i.e. for groups of high enough rank or fields of large enough positive characteristic) the minimal covolume is always attained on a non-uniform lattice.*

1.7. In [3] M. Conder and C. Maclachlan constructed a compact orientable hyperbolic 4-manifold which has Euler characteristic 16. The previously known smallest example which was used in order to formulate the main result in [1, Section 5] had $\chi = 26$. The construction of [3] agrees with our Theorem 5.5 and it also allows us to give a more precise formulation of the theorem:

Theorem 5.5'. *If there exists a compact orientable arithmetic hyperbolic 4-manifold M with $\chi(M) \leq 16$, then M is defined over $\mathbb{Q}[\sqrt{5}]$ and has the form $\Gamma_M \backslash \mathcal{H}^4$ with Γ_M being a torsion-free subgroup of index $7200\chi(M)$ of the group Γ_1 of the smallest arithmetic hyperbolic 4-orbifold.*

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