Holomorphic line bundles and divisors
on a domain of a Stein manifold

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Dedicated to Professor Yoshihiro Mizuta on his sixtieth birthday

Abstract. Let $D$ be an open set of a Stein manifold $X$ of dimension $n$ such that
$H^k(D, O) = 0$ for $2 \leq k \leq n - 1$. We prove that $D$ is Stein if and only if
every topologically trivial holomorphic line bundle $L$ on $D$ is associated to some
Cartier divisor $d$ on $D$.

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(secondary).

1. Introduction

For every holomorphic line bundle $L$ on a reduced Stein space $X$ there exists a
global holomorphic section $\sigma \in \Gamma(X, O(L))$ such that the zero set $\{\sigma = 0\}$ is
nowhere dense in $X$. Therefore $L$ is associated to the positive Cartier divisor $\text{div}(\sigma)$
on $X$ (see Gunning [9, pages 122–125]).

Conversely the author [1, Theorem 3] proved that an open set $D$ of a Stein
manifold $X$ of dimension two is Stein if every holomorphic line bundle $L$ on $D$
is associated to some (not necessarily positive) Cartier divisor $d$ on $D$. Moreover
Ballico [4, Theorem 1] proved that an open set $D$ of a Stein manifold $X$ of dimen-
sion more than two of the form $D = \{\varphi < c\}$, where $\varphi : X \to \mathbb{R}$ is a weakly
2-convex function of class $C^2$ in the sense of Andreotti-Grauert, is Stein if every
holomorphic line bundle $L$ on $D$ is associated to some Cartier divisor $d$ on $D$.

In this paper we prove that an open set $D$ of a Stein manifold $X$ of dimen-
sion $n$ such that $H^k(D, O) = 0$ for $2 \leq k \leq n - 1$ is Stein if every topologi-
cally trivial holomorphic line bundle $L$ on $D$ is associated to some Cartier divisor
$d$ on $D$ (see Theorem 4.3). This generalizes both results above (see Corollaries 4.4
and 4.5).

The proof is by induction on $n = \dim X$ and the induction hypothesis is
applied to the complex subspace $(Y, (\mathcal{O}_X/f\mathcal{O}_X)|_Y)$, where $f \in \mathcal{O}_X(X)$ and

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Y := \{[f] = 0\}. Therefore it is inevitable to consider complex spaces which are not necessarily reduced (see Theorem 4.1).

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2. Preliminaries

Throughout this paper complex spaces are always assumed to be second countable. We always denote by $\mathcal{O}$ without subscript the reduced complex structure sheaf of an arbitrary complex space. In other words we always set $\mathcal{O} := \mathcal{O}_X/\mathcal{N}_X$ for a complex space $(X, \mathcal{O}_X)$, where $\mathcal{N}_X$ is the nilradical of the complex structure sheaf $\mathcal{O}_X$.

Let $(X, \mathcal{O}_X)$ be a (not necessarily reduced) complex space and $(\text{red}, \tilde{\text{red}}) : (X, \mathcal{O}) \to (X, \mathcal{O}_X)$ the reduction map. We denote by $[f]$ the valuation $x \mapsto f_x \in \mathcal{O}_{X,x}/\mathcal{m}_x = \mathbb{C}$, $x \in U$, for every $f \in \mathcal{O}_X(U)$, where $U$ is an open set of $X$. Then the assignment $f \mapsto [f]$ is identified with $\tilde{\text{red}} : \mathcal{O}_X(U) \to \mathcal{O}(U)$ (see Grauert-Remmert [8, page 87]).

Let $X$ be a reduced complex space and $e : \mathcal{O} \to \mathcal{O}^*$ the homomorphism of sheaves on $X$ defined by $e_x(f_x) := \exp \left(2\pi \sqrt{-1} f_x\right)$ for every $f_x \in \mathcal{O}_x$ and $x \in X$, where $\mathcal{O}^*$ denotes the multiplicative sheaf of invertible germs of holomorphic functions. Then $e$ induces the homomorphism $e^* : H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*)$. As usual we identify the cohomology group $H^1(X, \mathcal{O}^*)$ with the set of holomorphic line bundles on $X$.

Let $\mathcal{d}$ be a Cartier divisor on a reduced complex space $X$ defined by the meromorphic Cousin-II distribution $\{(U_i, m_i)\}_{i \in I}$ on $X$ (see Gunning [9, page 121]). We denote by $[\mathcal{d}]$ the holomorphic line bundle on $X$ defined by the cocycle $\{m_i/m_j\} \in Z^1((U_i)_{i \in I}, \mathcal{O}^*)$ and we say that $[\mathcal{d}]$ is the holomorphic line bundle associated to $\mathcal{d}$. We say that $\mathcal{d}$ is positive (or effective) if $\mathcal{d}$ can be defined by a holomorphic Cousin-II distribution.

Let $(X, \mathcal{O}_X)$ be a (not necessarily reduced) complex space and $D$ an open set of $X$. Then $D$ is said to be locally Stein at a point $x \in \partial D$ if there exists a neighborhood $U$ of $x$ in $X$ such that the open subspace $(D \cap U, \mathcal{O}_X|_{D \cap U})$ is Stein.

Throughout this paper we use the following notation:

$$
\Delta(r) := \{ t \in \mathbb{C} \mid |t| < r \} \text{ for } r > 0, \quad \Delta := \Delta(1),
$$

$$
P(n, \varepsilon) := \Delta(1 + \varepsilon)^n, \quad \text{and}
$$

$$
H(n, \varepsilon) := \Delta^n \cup \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid 1 - \varepsilon < |z_1| < 1 + \varepsilon, \right.
$$

$$
\left. |z_2| < 1 + \varepsilon, |z_3| < 1 + \varepsilon, \ldots, |z_n| < 1 + \varepsilon \right\}
$$

for $n \geq 2$ and $0 < \varepsilon < 1$. 
The pair \((P(n, \varepsilon), H(n, \varepsilon))\) is said to be a Hartogs figure. We have the following lemma which characterizes a Stein open set of \(\mathbb{C}^n\).

**Lemma 2.1 (Kajiwara-Kazama [10, Lemmas 1 and 2]).** Let \(D\) be an open set of \(\mathbb{C}^n\). Then the following two conditions are equivalent.

1. \(D\) is Stein.
2. There do not exist a biholomorphic map \(\varphi: \mathbb{C}^n \to \mathbb{C}^n\), \(\varepsilon \in (0, 1)\) and \(b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n\) such that \(\varphi(H(n, \varepsilon)) \subset D\), \(|b_1| \leq 1 - \varepsilon\), \(\max_{2 \leq v \leq n} |b_v| = 1\) and \(\varphi(b) \in \partial D\).

**3. Lemmas**

In this section we denote by \(\Phi(X, \mathcal{O}_X)\) the composition of the induced homomorphisms

\[ H^1(X, \mathcal{O}_X) \xrightarrow{\text{red}^*} H^1(X, \mathcal{O}) \xrightarrow{e^*} H^1(X, \mathcal{O}^*) \]

for every complex space \((X, \mathcal{O}_X)\).

**Lemma 3.1.** Let \((X, \mathcal{O}_X)\) be a Stein space of pure dimension 2 and \(D\) an open set of \(X\). Let \((\theta, \tilde{\theta}) : (X, \mathcal{O}_X) \to \mathbb{C}^2\) be a holomorphic map, \(R\) and \(W\) open sets of \(X, \varepsilon \in (0, 1)\) and \(b = (b_1, b_2) \in \mathbb{C}^2\) such that \(R \subseteq W \subseteq X \setminus \text{Sing}(X, \mathcal{O}_X)\), \(\theta(W)\) is an open set of \(\mathbb{C}^2\), the restriction \(\theta|_W : W \to \theta(W)\) is biholomorphic, \(\theta(R) = P(2, \varepsilon), (\theta|_W)^{-1}(H(2, \varepsilon)) \subset D\), \(|b_1| \leq 1 - \varepsilon\), \(|b_2| = 1\) and \((\theta|_W)^{-1}(b) \in \partial D\).

Then there exists a cohomology class \(\alpha \in H^1(D, \mathcal{O}_X|_D)\) such that the holomorphic line bundle \(\Phi(D, \mathcal{O}_X|_D)(\alpha)|_{D \cap R}\) on \(D \cap R\) is not associated to any Cartier divisor on \(D \cap R\).

**Proof.** Let \(\theta_v := \tilde{\theta}z_v\) for \(v = 1, 2\), where \(z_1\) and \(z_2\) are the coordinates of \(\mathbb{C}^2\). Let \(E_v := \{[\theta_v] \neq b_v\}\) for \(v = 1, 2\). Since \((E_v, \mathcal{O}_X|_{E_v})\) is Stein and \(1/([\theta_v] - b_v) \in \mathcal{O}_X(E_v)\), there exists \(u_v \in \mathcal{O}_X(E_v)\) such that \([u_v] = 1/([\theta_v] - b_v)\) on \(E_v\) for \(v = 1, 2\). Let \(T := \{||[\theta_1]| < 1 + \varepsilon\}\) and \(T_1 := \{||[\theta_1]| < 1 + \varepsilon, ||[\theta_2]| > 1 + \varepsilon/2\} \cup (T \setminus \tilde{\mathcal{R}})\). Then \((T, \mathcal{O}_X|_T)\) is Stein and \([R, T_1]\) is an open covering of \(T\). Since \(H^1([R, T_1], \mathcal{O}_X|_T) = 0\) and \(R \cap T_1 \subseteq E_2\), there exist \(v_0 \in \mathcal{O}_X(R) = \mathcal{O}(R)\) and \(v_1 \in \mathcal{O}_X(T_1)\) such that \(v_2 = v_1 - v_0\) on \(R \cap T_1\). Let \(F := (E_2 \cap R) \cup T_1\). Let \(u \in \mathcal{O}_X(F)\) be defined by \(u = v_0 + u_2\) on \(E_2 \cap R\) and \(v = v_1\) on \(T_1\). Let \(D_1 := D \cap E_1\) and \(D_2 := D \cap ((E_2 \cap T) \cup (T \setminus \tilde{\mathcal{R}}))\). Then \([D_1, D_2]\) is an open covering of \(D\) and \(D_1 \cap D_2 \subseteq E_1 \cap F\). Let \(\alpha \in H^1([D_1, D_2], \mathcal{O}_X|_D)\) be the cohomology class defined by \((u_1v)|_{D_1 \cap D_2} \in \mathcal{O}_X(D_1 \cap D_2) = Z^1([D_1, D_2], \mathcal{O}_X|_D)\). Then by

\(^1\) An open set \(D\) of \(\mathbb{C}^n\) satisfies condition (2) in Lemma 2.1 if and only if \(D\) is \(p\)-convex in the sense of Kajiwara-Kazama [10, page 2]. Note that the sentence “\(\varphi(D)\) is a subset of \(\Omega\) . . .” should be “\(\varphi(D)\) is a subset of \(\Omega\) . . .” in the definition of a boundary mapping in Kajiwara-Kazama [10, page 2].
the argument in Abe [1, page 271] the holomorphic line bundle \( \Phi(D, \mathcal{O}_X|_D)(\alpha)|_{D \cap R} \) is not associated to any Cartier divisor on \( D \cap R \).

A zero set \( N(l) \) of a linear function \( l(z_1, z_2, \ldots, z_n) = \sum_{k=1}^n a_k z_k + b \) on \( \mathbb{C}^n \), where \( a_1, a_2, \ldots, a_n, b \in \mathbb{C} \) and \((a_1, a_2, \ldots, a_n) \neq (0, 0, \ldots, 0)\), is said to be a hyperplane of \( \mathbb{C}^n \).

**Lemma 3.2.** Let \( D \) be an open set of \( \mathbb{C}^n \) and \( H \) a hyperplane of \( \mathbb{C}^n \). Let \( Z := D \cap H \). Then for every Cartier divisor \( \mathfrak{d} \) on \( D \) there exists a Cartier divisor \( \mathfrak{d}' \) on \( D \) such that the support \( |\mathfrak{d}'| \) of \( \mathfrak{d}' \) is nowhere dense in \( Z \) and \( |\mathfrak{d}|_Z = |\mathfrak{d}'|_Z \).

**Proof.** As usual we identify a Cartier divisor on a complex manifold with a Weil divisor. Let \( \mathfrak{d} = \sum_{\lambda \in \Lambda} \alpha_\lambda A_\lambda \), where \( A_\lambda \) is an irreducible analytic set of dimension \( n - 1 \) and \( \alpha_\lambda \in \mathbb{Z} \) for every \( \lambda \in \Lambda \), be the expression of \( \mathfrak{d} \) as a Weil divisor. Let \( \Lambda'' \) be the set of \( \lambda \in \Lambda \) such that \( A_\lambda \) is a connected component of \( Z \). Let \( \Lambda' := \Lambda \backslash \Lambda'' \), \( \mathfrak{d}' := \sum_{\lambda \in \Lambda'} \alpha_\lambda A_\lambda \) and \( \mathfrak{d}'' := \sum_{\lambda \in \Lambda''} \alpha_\lambda A_\lambda \). Then the support \( |\mathfrak{d}'| = \bigcup_{\lambda \in \Lambda'} A_\lambda \) of \( \mathfrak{d}' \) is nowhere dense in \( Z \). Let \( \{Z_\mu\}_{\mu \in M} \), where \( M \subset \mathbb{N} \), be the set of connected components of \( Z \). There exists a system \( \{U_\mu\}_{\mu \in M} \) of mutually disjoint open sets of \( D \) such that \( U_\mu \) is a neighborhood of \( Z_\mu \) for every \( \mu \in M \). Let \( U_0 := D \backslash Z \). We choose a non-constant linear function \( l \) on \( \mathbb{C}^n \) such that \( \{l = 0\} = H \). If there exists \( \lambda \in \Lambda'' \) such that \( Z_\mu = A_\lambda \), then let \( \beta_\mu := \alpha_\lambda \). Otherwise let \( \beta_\mu := 0 \). Then \( \mathfrak{d}'' \) as a Cartier divisor is defined by the system \( \{(U_0, 1)\} \cup \{(U_\mu, l^{\beta_\mu})\}_{\mu \in M} \). It follows that \( |\mathfrak{d}''| \) is holomorphically trivial on \( U := \bigcup_{\mu \in M} U_\mu \). Since \( U \) is a neighborhood of \( Z \) in \( D \), the restriction \( |\mathfrak{d}''|_Z \) is also holomorphically trivial. Then we have that

\[
|\mathfrak{d}|_Z = |\mathfrak{d}' + \mathfrak{d}''|_Z = (|\mathfrak{d}'| \otimes |\mathfrak{d}''|)_Z = |\mathfrak{d}'|_Z \otimes |\mathfrak{d}''|_Z = |\mathfrak{d}'|_Z.
\]

A complex space \((X, \mathcal{O}_X)\) is said to be *Cohen-Macaulay* if the local \( \mathbb{C}\)-algebra \( \mathcal{O}_{X,x} \) is Cohen-Macaulay for every \( x \in X \) (see Raimondo-Silva [12]).

**Lemma 3.3.** Let \((X, \mathcal{O}_X)\) be a Cohen-Macaulay Stein space of pure dimension \( n \geq 2 \). Let \( D \) be an open set of \( X \) such that \( H^k(D, \mathcal{O}_X|_D) = 0 \) for \( 2 \leq k \leq n - 1 \). Let \((\theta, \tilde{\theta}) : (X, \mathcal{O}_X) \to \mathbb{C}^n \) be a holomorphic map, \( R \) and \( W \) open sets of \( X \), \( \varepsilon \in (0, 1) \) and \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n \) such that \( R \subset W \subset X \backslash \text{Sing}(X, \mathcal{O}_X) \), \( \theta(W) \) is an open set of \( \mathbb{C}^n \), the restriction \( \theta|_W : W \to \theta(W) \) is biholomorphic, \( \theta(R) = P(n, \varepsilon), (\theta|_W)^{-1}(H(n, \varepsilon)) \subset D \), \( |b_1| \leq 1 - \varepsilon \), \( \max_{2 \leq v \leq n} |b_v| = 1 \) and \( (\theta|_W)^{-1}(b) \in \partial D \). Then there exists a cohomology class \( \alpha \in H^1(D, \mathcal{O}_X|_D) \) such that the holomorphic line bundle \( \Phi(D, \mathcal{O}_X|_D)(\alpha)|_{D \cap R} \) on \( D \cap R \) is not associated to any Cartier divisor on \( D \cap R \).

**Proof.** The proof proceed by induction on \( n = \dim X \). By Lemma 3.1 the assertion is true if \( n = 2 \). We consider the case when \( n \geq 3 \). Let \( \theta_v := \tilde{\theta} z_v \) for \( v = 1, 2, \ldots, n \), where \( z_1, z_2, \ldots, z_n \) are the coordinates of \( \mathbb{C}^n \). We replace \( W \) by the connected component of \( W \) which contains \( \tilde{R} \). Let \( X_0 \) be the irreducible component of \( X \) which contains \( W \). Since \((X, \mathcal{O}_X)\) is Stein, there exists \( f \in \mathcal{O}_X(X) \) such that
$[f] = [\theta_n] - b_n$ on $X_0$ and $[f] \neq 0$ on any irreducible component of $X$. Let

$$Y := \{[f] = 0\} = \text{Supp} \left( \frac{\mathcal{O}_X}{\mathcal{O}_X} \right)$$

and $\mathcal{O}_Y := \left( \frac{\mathcal{O}_X}{\mathcal{O}_X} \right)|_Y$. By the active lemma (see Grauert-Remmert [8, page 100]) we have that

$$\dim \mathcal{O}_{X,x}/f_{X,x} = \dim \mathcal{O}_{X,x} = n - 1 = \dim \mathcal{O}_{X,x} - 1$$

for every $x \in Y$. Therefore $f_x$ is not a zero divisor of $\mathcal{O}_{X,x}$ for every $x \in Y$ and $(Y, \mathcal{O}_Y)$ is a Cohen-Macaulay Stein space of pure dimension $n - 1$ (see Grauert-Remmert [6, page 141] or Serre [15, page 85]). Let $m : \mathcal{O}_X \to \mathcal{O}_X$ be the homomorphism defined by $m_x(h_x) := f_x h_x$ for every $h_x \in \mathcal{O}_{X,x}$ and $x \in X$. Since the sequence

$$0 \to \mathcal{O}_X \xrightarrow{m} \mathcal{O}_X \xrightarrow{\iota} \mathcal{O}_{X}/f_{X} \xrightarrow{0}$$

is exact, we have the long exact sequence of cohomology groups

$$\cdots \to H^k(D, \mathcal{O}_{X}|_D) \to H^k(D \cap Y, \mathcal{O}_{Y}|_D \cap Y) \to H^{k+1}(D, \mathcal{O}_{X}|_D) \to \cdots$$

Since by assumption $H^k(D, \mathcal{O}_{X}|_D) = 0$ for $2 \leq k \leq n - 1$, we have that $H^k(D \cap Y, \mathcal{O}_{Y}|_D \cap Y) = 0$ for $2 \leq k \leq n - 2$ and that the homomorphism $\tilde{\iota}^* : H^1(D, \mathcal{O}_{X}|_D) \to H^1(D \cap Y, \mathcal{O}_{Y}|_D \cap Y)$ is surjective. Let $(\theta', \tilde{\theta}') : (Y, \mathcal{O}_Y) \to \mathbb{C}^{n-1}$ be the holomorphic map such that $\tilde{\theta}' \circ z_v = (\tilde{\theta}_v)\mid Y$ for $v = 1, 2, \ldots, n - 1$ (see Grauert-Remmert [8, page 22]). Let $R' := R \cap Y$, $W' := W \cap Y$ and $b' := (b_1, b_2, \ldots, b_{n-1})$. Then $\theta(x) = (\theta'(x), b_n)$ for every $x \in W'$, $R' \subset W' \subset Y \setminus \text{Sing}(Y, \mathcal{O}_Y)$, $\theta'(W')$ is an open set of $\mathbb{C}^{n-1}$ and the restriction $\theta'|_{W'} : W' \to \theta'(W')$ is biholomorphic. We have that

$$\theta'(R') \times \{b_n\} = \theta(R') = P(n, \varepsilon) \cap \{z_n = b_n\} = P(n - 1, \varepsilon) \times \{b_n\},$$

$$(\theta'|_{W'})^{-1}(H(n - 1, \varepsilon)) = (\theta|_{W'})^{-1}(H(n - 1, \varepsilon) \times \{z_n = b_n\})$$

$$= (\theta|_{W'})^{-1}(H(n, \varepsilon)) \cap W' \subset D \cap Y, \quad \text{and}$$

$$(\theta'|_{W'})^{-1}(b') = (\theta|_{W'})^{-1}(b) \in \partial (D \cap Y),$$

where $\partial (D \cap Y)$ denotes the boundary of $D \cap Y$ in $Y$. By induction hypothesis there exists $\alpha' \in H^1(D \cap Y, \mathcal{O}_Y|_{D \cap Y})$ such that the holomorphic line bundle $\Phi(\mathcal{O}_{D \cap Y, \mathcal{O}_Y|_{D \cap Y}})(\alpha'|_{D \cap R'})$ on $D \cap R'$ is not associated to any Cartier divisor on $D \cap R'$. Since $\tilde{\iota}^*$ is surjective, there exists $\alpha \in H^1(D, \mathcal{O}_X|_D)$ such that $\tilde{\iota}^*(\alpha) = \alpha'$. Assume that there exists a Cartier divisor $0$ on $D \cap R$ such that $\Phi(\mathcal{O}_{D, \mathcal{O}_X|_D})(\alpha)|_{D \cap R} = [0]$. By Lemma 3.2 there exists a Cartier divisor $c$ on $D \cap R$ such that the support $|c|$ is nowhere dense in $D \cap R'$ and $|c|_{D \cap R'} = |c||_{D \cap R'}$. Then we have that

$$\Phi(\mathcal{O}_{D \cap Y, \mathcal{O}_Y|_{D \cap Y}})(\alpha'|_{D \cap R'}) = \Phi(\mathcal{O}_{D, \mathcal{O}_X|_D})(\alpha)|_{D \cap R'} = [0]|_{D \cap R'} = [c]|_{D \cap R'}$$

and it is a contradiction. It follows that $\Phi(\mathcal{O}_{D, \mathcal{O}_X|_D})(\alpha)|_{D \cap R}$ is not associated to any Cartier divisor on $D \cap R$. \qed
4. Theorems

**Theorem 4.1.** Let \((X, \mathcal{O}_X)\) be a (not necessarily reduced) Cohen-Macaulay Stein space of pure dimension \(n\) and \(D\) an open set of \(X\). Assume that the following two conditions are satisfied.

i) \(H^k(D, \mathcal{O}_X|_D) = 0\) for \(2 \leq k \leq n - 1\).

ii) For every holomorphic line bundle \(L\) on \(D\) which is an element of the image of the composition \(\Phi\) of the homomorphisms

\[
\begin{align*}
H^1(D, \mathcal{O}_X|_D) & \xrightarrow{\text{red}^*} H^1(D, \mathcal{O}) \xrightarrow{e^*} H^1(D, \mathcal{O}^*)
\end{align*}
\]

there exists a Cartier divisor \(\mathfrak{d}\) on \(D\) such that \(L = [\mathfrak{d}]\).

Then \(D\) is locally Stein at every point \(x \in \partial D \setminus \text{Sing}(X, \mathcal{O}_X)\).

**Proof.** We may assume that \(n \geq 2\). Assume that there exists a point \(x_0 \in \partial D \setminus \text{Sing}(X, \mathcal{O}_X)\) such that \(D\) is not locally Stein at \(x_0\). Since \(X\) is Stein, there exist a holomorphic map \((f, \tilde{f}) : (X, \mathcal{O}_X) \rightarrow \mathbb{C}^n\) and an open set \(W\) of \(X\) such that \(x_0 \in W \subset X \setminus \text{Sing}(X, \mathcal{O}_X), f(W)\) is an open set of \(\mathbb{C}^n\) and \(f|_W : W \rightarrow f(W)\) is biholomorphic (see Grauert-Remmert [7, page 151]). Take a Stein open set \(V\) of \(\mathbb{C}^n\) such that \(f(x_0) \in V \subset f(W)\). Let \(U := (f|_W)^{-1}(V)\). Then \(U\) is Stein, \(x_0 \in U \subset W\) and \(f(U) = V\). Since \(D\) is not locally Stein at \(x_0\), the open set \(f(D \cap U)\) of \(\mathbb{C}^n\) is not Stein. By Lemma 2.1 there exist a biholomorphic map \(\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n, \varepsilon \in (0, 1)\) and \(b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n\) such that \(\varphi(H(n, \varepsilon)) \subset f(D \cap U), |b_1| \leq 1 - \varepsilon, \max_{2 \leq v \leq n} |b_v| = 1\) and \(\varphi(b) \in \partial (f(D \cap U)).\) Let \((\theta, \tilde{\theta}) := \varphi^{-1} \circ (f, \tilde{f}) : (X, \mathcal{O}_X) \rightarrow \mathbb{C}^n\). We have that \(\theta(W) = \varphi^{-1}(f(W))\) is an open set of \(\mathbb{C}^n\) and \(\theta|_W : W \rightarrow \partial f(W)\) is biholomorphic. Let \(P := P(n, \varepsilon)\) and \(H := H(n, \varepsilon)\). Since \(V\) is Stein and \(\varphi(H) \subset f(D \cap U) \subset V\), we have that \(\varphi(P) \subset V \subset f(W)\).

Let \(R := (\theta|_W)^{-1}(P)\). Then we have that \(\theta(R) = P, (\theta|_W)^{-1}(H) \subset U\) and \((\theta|_W)^{-1}(b) \in \partial D\). By Lemma 3.3 there exists a holomorphic line bundle \(L \in \im \Phi\) such that \(L|_{D \cap R}\) is not associated to any Cartier divisor on \(D \cap R\). On the other hand by assumption there exists a Cartier divisor \(\mathfrak{d}\) on \(D\) such that \(L = [\mathfrak{d}]\) and therefore \(L|_{D \cap R} = [\mathfrak{d}]|_{D \cap R}\), which is a contradiction. It follows that \(D\) is locally Stein at every point \(x \in \partial D \setminus \text{Sing}(X, \mathcal{O}_X)\).

**Remark 4.2.** Condition i) in Theorem 4.1 can be replaced by the following weaker one:

i’) The dimension of \(H^k(D, \mathcal{O}_X|_D)\) is at most countably infinite for every integer \(k\) such that \(2 \leq k \leq n - 1\).

**Proof.** If condition i’) is satisfied, then by Ballico [3, Proposizione 7], which generalizes Siu [16, Theorem A], we have that \(\dim H^k(D, \mathcal{O}_X|_D) < +\infty\) for \(2 \leq k \leq n - 1\). We also have that \(H^k(D, \mathcal{O}_X|_D) = 0\) for \(k \geq n\) by Siu [17] and by Reiffen [13, page 277]. It follows that \(H^k(D, \mathcal{O}_X|_D) = 0\) for \(k \geq 2\) by Raimondo-Silva [12].

Every complex manifold is Cohen-Macaulay (see Grauert-Remmert [6, page 142]). The image of $H^1(D, \mathcal{O}) \to H^1(D, \mathcal{O}^*)$ coincides with the set of topologically trivial holomorphic line bundles on $D$. Therefore by Theorem 4.1 and by Docquier-Grauert [5] we obtain the following theorem.

**Theorem 4.3.** Let $X$ be a Stein manifold of dimension $n$ and $D$ an open set of $X$ such that $H^k(D, \mathcal{O}) = 0$ for $2 \leq k \leq n - 1$. Then the following four conditions are equivalent.

1. $D$ is Stein.
2. For every holomorphic line bundle $L$ on $D$ there exists a positive Cartier divisor $\mathcal{D}$ on $D$ such that $L = [\mathcal{D}]$.
3. For every holomorphic line bundle $L$ on $D$ there exists a Cartier divisor $\mathcal{D}$ on $D$ such that $L = [\mathcal{D}]$.
4. For every topologically trivial holomorphic line bundle $L$ on $D$ there exists a Cartier divisor $\mathcal{D}$ on $D$ such that $L = [\mathcal{D}]$.

**Corollary 4.4 (Abe [1, Theorem 3]).** Let $X$ be a Stein manifold of dimension 2 and $D$ an open set of $X$. Then the four conditions in Theorem 4.3 are equivalent.

Let $X$ be a complex manifold of dimension $n$ and $\varphi : X \to \mathbb{R}$ a function of class $C^2$. Then $\varphi$ is said to be weakly 2-convex if for every $x \in X$ the Levi form of $\varphi$ at $x$ has at most one negative eigenvalue. By the theorem of Andreotti-Grauert [2] we have the following corollary.

**Corollary 4.5 (Ballico [4, Theorem 1]).** Let $X$ be a Stein manifold and $\varphi : X \to \mathbb{R}$ a weakly 2-convex function of class $C^2$. Let $D := \{ \varphi < c \},$ where $c \in \mathbb{R}$ is a constant. Then the four conditions in Theorem 4.3 are equivalent.

We also have the following corollary (see Serre [14, page 65]).

**Corollary 4.6 (Laufer [11, Theorem 4.1]).** Let $X$ be a Stein manifold of dimension $n$ and $D$ an open set of $X$. Then the following two conditions are equivalent.

1. $D$ is Stein.
2. $H^k(D, \mathcal{O}) = 0$ for $1 \leq k \leq n - 1$.

**References**


