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Classical solutions and stability results for Stokesian Hele-Shaw flows

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Abstract. In this paper we study a mathematical model for the motion of a Stokesian fluid in a Hele-Shaw cell surrounded by a gas at uniform pressure. The model is based on a non-Newtonian version of Darcy's law for the bulk fluid, as suggested in [9, 12].

Besides a general existence and uniqueness result for classical solutions, it is also shown that classical solutions exist globally and tend to circles exponentially fast, provided the initial data is sufficiently close to a circle. Finally, our analysis discloses the influence of surface tension and the effective viscosity on the rate of convergence.

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1. Introduction and main results

Despite their importance in applications, the mathematical understanding of moving boundary problems for non-Newtonian fluids is far from being complete. In this paper we offer an analytic framework in which two-dimensional Stokesian fluids[1] with a free interface may be studied. We investigate the dynamic behaviour of such a fluid located between two parallel and transparent plates in a horizontal Hele-Shaw cell. Relative to some typical lateral dimension on the plate, the distance between these plates is assumed to be small so that we shall consider planar flows in the following. This setting is widely used both in experimental and theoretical work, *cf.* [9] and the references therein.

The motivation of our research is twofold. One the one hand we provide an analytic framework which guarantees well-possedness of the full flow problem for general data. This result may serve as the theoretical justification of numerical studies or formal expansions. On the other hand we give some insight in the dynamic behaviour of the flow near equilibria. As a main result we show that circles are exponential stable under small perturbations. However, in contrast to the periodic

[1] In a *Stokesian* fluid the stress tensor is a continuous function of the deformation. The *Newtonian* fluid is a linear Stokesian fluid. Particularly, the viscosity μ is constant in this case.

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strip-like geometry considered in [6], steady states are no longer isolated, but form a three-dimensional submanifold $\mathcal{M}^c_{\text{loc}}$ of the phase space. Nevertheless, a centre manifold analysis allows us to prove that $\mathcal{M}^c_{\text{loc}}$ attracts at an exponential rate any solution which is sufficiently close nearby. This means that by perturbing a circle S_0 initially sufficiently small, the corresponding solution exists globally and converges to a circle S_∞, which is uniquely determined by S_0, since the centre of mass and the fluid volume are preserved by the flow.

To describe our main result precisely, let is introduce the following notation. The initial shape of the fluid's body is assumed to be a small deformation of the unitary disc. Let $\mathbb{S}^1$ denote the unit circle. Further, let $a \in (0, 1/4)$ be fixed and let $\rho \in C([0, T], C^2(\mathbb{S}^1)) \cap C^1([0, T], C^1(\mathbb{S}^1))$, $T > 0$, with $\max_{t\in[0,T]} \|\rho(t)\|_{C(\mathbb{S}^1)} < a$ be a mapping with the property that at each time $t \in [0, T]$ the fluid occupies the domain $\Omega_{\rho(t)}$, where for $\rho \in C^2(\mathbb{S}^1)$ with $\|\rho\|_{C(\mathbb{S}^1)} < a$ we define

$$\Omega_\rho := \left\{x \in \mathbb{R}^2 \,:\, |x| < 1 + \rho\left(\frac{x}{|x|}\right)\right\} \cup \{0\}.$$

The boundary Γ_ρ of the domain Ω_ρ is given by

$$\Gamma_\rho = \left\{x \in \mathbb{R}^2 \,:\, |x| = 1 + \rho\left(\frac{x}{|x|}\right)\right\} = \left\{x(1 + \rho(x)) \,:\, x \in \mathbb{S}^1\right\}.$$

It is suitable to represent Γ_ρ as the 0-level set of an appropriate function. For this, let $N_\rho : R(3/4, 5/4) \to \mathbb{R}$ be the function defined by

$$N_\rho(y) = |y| - 1 - \rho(y/|y|),$$

where $R(3/4, 5/4)$ is the circular ring centred in 0 with radii 3/4 and 5/4, *i.e.*

$$R(3/4, 5/4) := \{x \in \mathbb{R}^2 \,:\, 3/4 < |x| < 5/4\}.$$

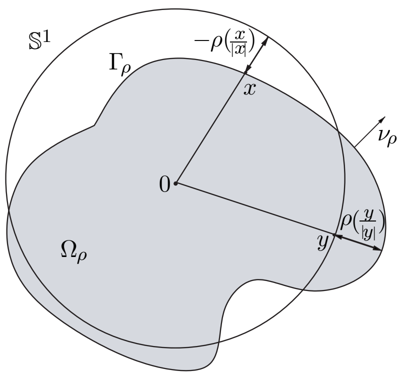

Figure 1.1. The fluid domain.

Since $\Gamma_\rho = N_\rho^{-1}(0)$ it follows that the outward normal ν_ρ at Γ_ρ is $\nu_\rho = DN_\rho/|DN_\rho|$, with DN_ρ denoting the gradient of N_ρ.

The viscous behaviour of the fluid is modeled by a function $\mu \in C^\infty(\mathbb{R}_{\geq 0})$ satisfying the relation

$$\begin{aligned} &0 < \mu(r) \quad \text{for all } r \geq 0, \\ &0 < \mu(r) + 2r\mu'(r) \quad \text{for all } r \geq 0, \\ &r\mu^2(r) \to_{r\to\infty} \infty. \end{aligned} \tag{1.1}$$

We point out here that the first two conditions on the viscosity function were also found in [3], where the full Navier-Stokes problem, but on a fixed domain, is studied. Note that the second and third condition guarantee the invertibility of the mapping

$$h : [0, \infty) \to [0, \infty), \quad h(r) := r\mu^2(r) \quad \text{for } r \geq 0. \tag{1.2}$$

If μ is increasing the fluid is called *shear thickening*. Relation (1.1) holds for all shear thickening fluids having positive viscosities. In addition to the examples mentioned in [5], in particular Oldroyd-B fluids and shear thinning power law fluids, we include also the following cases

$$\mu(r) = \nu_0(1+r)^{s/2} \quad \text{and} \quad \mu(r) = \nu_\infty + \nu_0 r^{s/2},$$

where ν_0 and ν_∞ are positive constants. These are also power-law fluids. For the first example the parameter s belongs to $(-1, \infty)$. The second example was introduced in the mathematical literature by Ladyzhenskaya in [13]. Here (1.1) holds iff $s \in \{0\} \cup [2, \infty)$.

For later purpose we formulate the following stronger version of relation (1.1). Assume there are positive constants m_μ and M_μ such that

$$m_\mu \leq \mu(r) \leq M_\mu \quad \text{and} \quad m_\mu \leq \mu(r) + 2r\mu'(r) \leq M_\mu \quad \text{for all } r \geq 0. \tag{1.3}$$

A further interesting example is the Johnson-Segalman-Oldroyd model with relaxation times λ_k and k-th mode viscosities η_k, $k = 1, ..., N$. The viscosity function is given by

$$\mu(r) = \alpha_0 + \sum_{k=1}^{N} \frac{\alpha_k}{1 + \beta_k^2 \cdot r},$$

where the positive constants occurring in the relation are assumed to satisfy

$$\alpha_0 = \mu_s/\mu_0, \ \alpha_k = \eta_k/\mu_0, \ \beta_k = \lambda_k/\lambda_1 \text{ and } \mu_0 = \mu_s + \sum_{k=1}^{N} \eta_k.$$

We refer to [9] for details. In this case the relation (1.3) holds for any choice of the parameters.

The dynamic of the Stokesian fluid, assumed to be incompressible, is governed by a non-Newtonian Darcy law

$$v = -\frac{Du}{\overline{\mu}(|Du|^2)} \quad \text{in} \quad \Omega_{\rho(t)},$$

cf. [12], where u is the pressure and v is the velocity of the fluid. The *effective viscosity* $\overline{\mu}$ is defined, see [12], by

$$\frac{1}{\overline{\mu}(r)} := \overline{c} \int_{-1}^{1} \frac{s^2}{\widetilde{\mu}(rs^2)}\, ds, \qquad \forall\, r \geq 0,$$

where $\overline{c}$ is a positive constant, $\widetilde{\mu} := \mu \circ h^{-1}$ and h is the function defined by (1.2).

The well-known Laplace-Young condition states that at each point on the boundary $\Gamma_{\rho(t)}$ the pressure has a jump according to the equation $u_{\text{int}} - u_{\text{ext}} = \gamma\kappa_{\rho(t)}$, where $\kappa_{\rho(t)}$ is the curvature of $\Gamma_{\rho(t)}$ (taken to be positive if $\Gamma_{\rho(t)}$ is convex) and γ is the surface tension coefficient, *cf.* [17]. The pressure u_{ext} of the gas surrounding the fluid is assumed to be zero. Thus we get

$$u = \gamma\kappa_{\rho(t)} \quad \text{on} \quad \Gamma_{\rho(t)}.$$

Assuming that a particle located on the boundary $\Gamma_{\rho(0)}$ remains on the boundary of the fluid domain we obtain, using once again Darcy's law, the following kinematic boundary condition

$$\partial_t N_{\rho(t)} - \frac{1}{\overline{\mu}(|Du|^2)} \langle Du, DN_{\rho(t)} \rangle = 0 \quad \text{on} \quad \Gamma_{\rho(t)}.$$

Summarizing, we come to the following moving boundary problem

$$\begin{aligned} \operatorname{div}\left(\tfrac{Du}{\overline{\mu}(|Du|^2)}\right) &= 0 && \text{in } \Omega_{\rho(t)},\ t \in [0,T], \\ u &= \gamma\kappa_{\rho(t)} && \text{on } \Gamma_{\rho(t)},\ t \in [0,T], \\ \partial_t N_{\rho(t)} - \tfrac{1}{\overline{\mu}(|Du|^2)} \langle Du, DN_{\rho(t)} \rangle &= 0 && \text{on } \Gamma_{\rho(t)},\ t \in [0,T], \\ \rho(0) &= \rho_0 && \text{on } \mathbb{S}^1, \end{aligned} \tag{1.4}$$

where ρ_0 is the initial data.

Notice that we have to handle a fully nonlinear problem: the first three equations of (1.4) are all containing nonlinearities in the highest spatial derivative. A further nonlinearity arises from the fact that the a priori unknown domains $\Omega_{\rho(t)}$ are evolving in time.

In order to define the notion of classical solutions to (1.4), we introduce first the Banach spaces which we shall use frequently in this paper. Given $r \geq 0$, the *small Hölder space* $h^r(\mathbb{S}^1)$ denotes the closure of $C^\infty(\mathbb{S}^1)$ in $C^r(\mathbb{S}^1)$. The small Hölder spaces have the nice property that

$$h^r(\mathbb{S}^1) \stackrel{d}{\hookrightarrow} h^s(\mathbb{S}^1) \quad \text{for} \quad 0 \leq s < r,$$

with compact embedding.

Assume that U is an open subset of $\mathbb{R}^2$. Given $k \in \mathbb{N} \cup \{\infty\}$, the set $BUC^k(U)$ denotes the space of all maps from U to $\mathbb{R}$ which have bounded and uniformly continuous derivatives up to order k. Given $\alpha \in (0, 1)$, the space $BUC^{k+\alpha}(U)$ consists of all $f \in BUC^k(U)$ having uniformly α-Hölder continuous derivatives of order k. Finally, we set $buc^{k+\alpha}(U)$ to be the closure of $BUC^\infty(U)$ in $BUC^{k+\alpha}(U)$.

For our analysis we fix $\alpha \in (0, 1)$ and set

$$\mathcal{V}_\alpha := \begin{cases} \{\rho \in h^{4+\alpha}(\mathbb{S}^1) : \|\rho\|_{C(\mathbb{S}^1)} < a\}, & \text{if (1.3) hold,} \\ \{\rho \in h^{4+\alpha}(\mathbb{S}^1) : \|\rho\|_{C^2(\mathbb{S}^1)} < a\}, & \text{else.} \end{cases}$$

We have defined in this way an open neighbourhood $\mathcal{V}_\alpha$ of the origin in $h^{4+\alpha}(\mathbb{S}^1)$. In addition, if one of the conditions (1.3) is not satisfied, we choose $a < 1/8$, which implies that Ω_ρ is convex for $\rho \in \mathcal{V}_\alpha$ and Γ_ρ has positive curvature. Indeed, for such ρ we have:

$$\begin{aligned} |\kappa_\rho - 1| &= \left| \frac{(1+\rho)^2 + 2\rho'^2 - (1+\rho)\rho''}{((1+\rho)^2 + \rho'^2)^{3/2}} - 1 \right| \\ &\le \left| \frac{(1+\rho)^2}{((1+\rho)^2+\rho'^2)^{3/2}} - 1 \right| + \frac{2\rho'^2}{((1+\rho)^2+\rho'^2)^{3/2}} + \frac{(1+\rho)|\rho''|}{((1+\rho)^2+\rho'^2)^{3/2}} \\ &\le \frac{2a^2}{(1-a)^3} + \frac{(1+a)a}{(1-a)^3} + \frac{|(1+\rho)^2 - ((1+\rho)^2 + \rho'^2)^{3/2}|}{((1+\rho)^2 + \rho'^2)^{3/2}} \\ &\le \frac{|(1+\rho)^4 - ((1+\rho)^2 + \rho'^2)^3|}{(1-a)^3((1-a)^2 + (1-a)^3)} + \frac{3a^2 + a}{(1-a)^3} \\ &\le \frac{a(2+a)(1+a)^4 + 3a^2(1+a)^4 + 3a^4(1+a)^2 + a^6}{(1-a)^3((1-a)^2 + (1-a)^3)} + \frac{a + 3a^2}{(1-a)^3} \\ &\le 0.8. \end{aligned}$$

Consequently $\kappa_\rho \ge 0.2$ and the domain Ω_ρ is strictly convex. The convexity of Ω_ρ is needed to guarantee the solvability of the elliptic problem (2.4) in Section 2.

A pair (u, ρ) is called a *classical Hölder solution* of (1.4) on interval $[0, T]$, $T > 0$, if

$$\begin{aligned} &\rho \in C([0, T], \mathcal{V}_\alpha) \cap C^1([0, T], h^{1+\alpha}(\mathbb{S}^1)), \\ &u(\cdot\,, t) \in buc^{2+\alpha}(\Omega_{\rho(t)}),\ t \in [0, T], \end{aligned}$$

and if (u, ρ) satisfies the equations in (1.4) pointwise.

If (1.3) does not hold, the interface $\rho(t)$ must correspond to a convex domain for $t \in [0, T]$. This is no longer the case if (1.3) is satisfied. The set $\mathcal{V}_\alpha$ is large enough to contain functions, which parametrize non-convex domains too. For example, the mapping ρ given by $\rho(x) = -(1/5)x_1^{10}$ for $x \in \mathbb{S}^1$ belongs to $\mathcal{V}_\alpha$ and the curvature of Γ_ρ in point $(1 - 1/5)$ is negative.

Let $Qu := \operatorname{div}\left(Du/\overline{\mu}(|Du|^2)\right)$ for all $u \in buc^{2+\alpha}(\Omega_\rho)$ and all $\rho \in \mathcal{V}_\alpha$. We use the sum convention $a_{ij}(Du)u_{ij} = \sum_{ij} a_{ij}(Du)u_{ij}$ to compute that $Qu = a_{ij}(Du)u_{ij}$, where

$$a_{ij}(p) = \frac{\delta_{ij}}{\overline{\mu}(|p|^2)} - \frac{2p_i p_j \overline{\mu}'(|p|^2)}{\overline{\mu}^2(|p|^2)}, \quad 1 \le i,\ j \le 2,$$

for all $p = (p_1, p_2) \in \mathbb{R}^2$. The eigenvalues of the matrix $[a_{ij}(p)]_{1\le i,\ j\le 2}$, are

$$\lambda_1(p) = \frac{1}{\overline{\mu}(|p|^2)}, \qquad \lambda_2(p) = \frac{1}{\overline{\mu}(|p|^2)} - \frac{2|p|^2\overline{\mu}'(|p|^2)}{\overline{\mu}^2(|p|^2)}$$

and the quasilinear operator Q is elliptic because at each point $p \in \mathbb{R}^2$ we have

$$\begin{aligned} 0 < \min\{\lambda_1(p), \lambda_2(p)\}|\xi|^2 &\le a_{ij}(p)\xi_i\xi_j \\ &\le \max\{\lambda_1(p), \lambda_2(p)\}|\xi|^2, \quad \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}. \end{aligned}$$

The inequality $0 < \min\{\lambda_1(p), \lambda_2(p)\}$ for $p \in \mathbb{R}^2$ is obtained directly from the properties of the mapping μ. If in addition to (1.1) the relation (1.3) holds, then Q is uniformly elliptic *cf.* [5]. Given the geometric setting considered in this paper we can weaken slightly the assumptions on the viscosity function μ, because sufficiently small deformations of the unitary disc remains convex, whereas in the strip-like geometry of [5] and [6] the cylinder may lose convexity property by arbitrarily small deformations.

The first main result of this paper is proved at the end of Section 3 and guarantees local existence and uniqueness of classical solutions.

Theorem 1.1 (Existence and uniqueness). *Assume that* (1.1) *holds true.*

There exists an open neighbourhood $\mathcal{O}$ *of* 0 *in* $\mathcal{V}_\alpha$ *such that, for any initial data* $\rho_0 \in \mathcal{O}$, *there exists a maximal existence time* $T := T(\rho_0) > 0$ *and a unique classical solution* (u, ρ) *to problem* (1.4) *defined on* $[0, T(\rho_0))$ *which satisfies* $\rho([0, T(\rho_0))) \subset \mathcal{O}$.

Note that Theorem 1.1 implies that

$$\Gamma := \{(t, x) \in (0, T) \times \mathbb{R}^2\,;\ x \in \Gamma_{\rho(t)}\}$$

is a C^1-hypersurface of $\mathbb{R}^3$. Using techniques as in [7], it is in fact possible to show that Γ is a real analytic manifold, *i.e.* $\Gamma \in C^\omega$.

We now turn our attention to the dynamic behaviour of solutions near circles. For this we first note that the volume of fluid enclosed by the moving interface ρ is a constant of motion, *cf.* Lemma 3.6. Moreover, in Section 4 we prove that the circles near the unit circle are the only steady-states of system (1.4) and we show that the set containing all small steady-state solutions is a three dimensional manifold, a so-called *local centre manifold* for the flow. The solutions to (1.4) corresponding to initial data close to an element $\rho_{(c,R)}$ of this centre manifold exist in the large and are attracted exponentially fast by a circle with centre c_0 and radius R, parametrized over $\mathbb{S}^1$ by the mapping $\rho_{(c_0,R)}$ (see Theorem 4.3).

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2. The transformed problem

A fundamental difficulty in treating problem (1.4) is the fact that one has to work with unknown, variable domains Ω_ρ. We overcome this difficulty by transforming problem (1.4) on the unitary disc $\Omega := \mathbb{D}(0, 1)$. Therefore we define for $\rho \in \mathcal{V}_\alpha$ the Hanzawa diffeomorphism $\phi_\rho : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\phi_\rho(x) = \begin{cases} \dfrac{x}{|x|}\left(|x| + \varphi(|x|-1)\rho\left(\dfrac{x}{|x|}\right)\right) & , \ 0 < |x| < 2, \\ x & , \ \text{else}, \end{cases}$$

where $\varphi \in C^\infty(\mathbb{R}, [0, 1])$ satisfies

$$\varphi(r) = \begin{cases} 1 \ , \ |r| \le a, \\ 0 \ , \ |r| \ge 3a, \end{cases}$$

and additionally $\max |\varphi'(r)| < 1/a$. In fact for $|x| \le 1-3a$ and for $|x| \ge 1+3a$ we have $\phi_\rho(x) = x$. Given $y \in \mathbb{S}^1$, the mapping $[0, \infty) \ni r \mapsto r + \varphi(r-1)\rho(y/|y|) \in [0, \infty)$ is strictly increasing and therefore bijective. Taking also in account that for $x \neq 0$ we have

$$D\left(\rho\left(\frac{x}{|x|}\right)\right) = \rho'\left(\frac{x}{|x|}\right)\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right)$$

we can easily verify that $\phi_\rho \in \mathrm{Diff}^{4+\alpha}(\Omega, \Omega_\rho) \cap \mathrm{Diff}^{4+\alpha}(\mathbb{R}^2, \mathbb{R}^2)$. Additionally, we have that $\phi_\rho(\mathbb{S}^1) = \Gamma_\rho$. The push-forward operator induced by ϕ_ρ is defined by

$$\phi_\rho^* : BUC(\Omega_\rho) \to BUC(\Omega), \ u \longmapsto u \circ \phi_\rho.$$

These operators allow us to transform the problem into an abstract Cauchy problem over $\mathbb{S}^1$. General results of the theory of maximal regularity, due to Sinestrari [20], can be used to prove existence of a unique classical solution, corresponding to small initial data. Let $\psi_\rho := \phi_\rho^{-1}$. The solution to (1.4) is then obtained (see Lemma 2.1 below) using the pull-back operators defined by

$$\phi_*^\rho : BUC(\Omega) \to BUC(\Omega_\rho),\ v \longmapsto v \circ \psi_\rho.$$

The transformed differential operators $\mathcal{A}(\rho)$ and $\mathcal{B}$, acting on $buc^{2+\alpha}(\Omega)$, respectively on $\mathcal{V}_\alpha \times buc^{2+\alpha}(\Omega)$ are set to be:

$$\begin{aligned}
&\mathcal{A}(\rho) := \phi_\rho^* \circ Q \circ \phi_*^\rho,\\
&\mathcal{B}(\rho, v)(x) := \tfrac{1}{\overline{\mu}(|D(\phi_*^\rho v)|^2)} \langle D(\phi_*^\rho v), DN_\rho\rangle(\phi_\rho(x)), \quad x \in \mathbb{S}^1.
\end{aligned}$$

We compute that the curvature κ_ρ of Γ_ρ, $\rho \in \mathcal{V}_\alpha$ is given by

$$\kappa_\rho((1+\rho(x))x) = \frac{(1+\rho)^2 + 2\rho'^2 - (1+\rho)\rho''}{((1+\rho)^2+\rho'^2)^{3/2}}(x), \quad x \in \mathbb{S}^1.$$

It is not difficult to see that if (ρ, u) is a solution of (1.4), then $(\rho, \phi_\rho^* u)$ is a solution of the following problem

$$\begin{aligned}
\mathcal{A}(\rho)v &= 0 && \text{in } \Omega \times [0, T],\\
v &= \gamma \frac{(1+\rho)^2 + 2\rho'^2 - (1+\rho)\rho''}{((1+\rho)^2+\rho'^2)^{3/2}} && \text{on } \mathbb{S}^1 \times [0, T],\\
\partial_t \rho + \mathcal{B}(\rho, v) &= 0 && \text{on } \mathbb{S}^1 \times [0, T],\\
\rho(0) &= \rho_0.
\end{aligned} \tag{2.1}$$

In fact the problems (1.4) and (2.1) are equivalent in the following sense:

Lemma 2.1. *Given $\rho_0 \in \mathcal{V}_\alpha$ we have:*

(a) *If (ρ, u) is a classical Hölder solution for* (1.4)*, then $(\rho, \phi_\rho^* u)$ is a classical Hölder solution for* (2.1).
(b) *If (ρ, v) is a classical Hölder solution for* (2.1)*, then $(\rho, \phi_*^\rho v)$ is a classical Hölder solution for* (1.4).

Proof. Given $\rho \in \mathcal{V}_\alpha$ there exists positive constants K and δ depending only on ρ such that

$$\|\phi_\rho - \phi_{\widetilde{\rho}}\|_{BUC^{4+\alpha}(\mathbb{R}^2)} \le K\|\rho - \widetilde{\rho}\|_{C^{4+\alpha}(\mathbb{S}^1)} \tag{2.2}$$

for all $\widetilde{\rho} \in \mathcal{V}_\alpha$ with $\|\rho - \widetilde{\rho}\|_{C^{4+\alpha}(\mathbb{S}^1)} \le \delta$. In fact, we can choose K large enough such that the relation

$$\|\psi_\rho - \psi_{\widetilde{\rho}}\|_{BUC^{4+\alpha}(\mathbb{R}^2)} \le K\|\rho - \widetilde{\rho}\|_{C^{4+\alpha}(\mathbb{S}^1)} \tag{2.3}$$

holds for $\|\rho - \widetilde{\rho}\|_{C^{4+\alpha}(\mathbb{S}^1)} \le \delta$. Indeed, given $\rho \in \mathcal{V}_\alpha$, let $\psi_\rho =: (\psi_\rho^1, \psi_\rho^2)$.

Using the chain rule, we compute

$$\psi^1_{\rho,1}(\phi_\rho(x))=\begin{cases}\dfrac{1}{1+\varphi'\rho}+\dfrac{x_2}{|x|^3\left(1+\frac{1}{|x|}\varphi\rho\right)(1+\varphi'\rho)}(-x_2\varphi\rho+x_2|x|\varphi'\rho+x_1\varphi\rho'), & 0<|x|<2,\\ 1, & \text{else},\end{cases}$$

$$\psi^1_{\rho,2}(\phi_\rho(x))=\begin{cases}-\dfrac{x_1}{|x|^3\left(1+\dfrac{1}{|x|}\varphi\rho\right)(1+\varphi'\rho)}(-x_2\varphi\rho+x_2|x|\varphi'\rho+x_1\varphi\rho'), & 0<|x|<2,\\ 0, & \text{else},\end{cases}$$

$$\psi^2_{\rho,1}(\phi_\rho(x))=\begin{cases}-\dfrac{x_2}{|x|^3\left(1+\dfrac{1}{|x|}\varphi\rho\right)(1+\varphi'\rho)}(-x_1\varphi\rho+x_1|x|\varphi'\rho-x_2\varphi\rho'), & 0<|x|<2,\\ 0, & \text{else},\end{cases}$$

$$\psi^2_{\rho,2}(\phi_\rho(x))=\begin{cases}\dfrac{1}{1+\varphi'\rho}+\dfrac{x_1}{|x|^3\left(1+\frac{1}{|x|}\varphi\rho\right)(1+\varphi'\rho)}(-x_1\varphi\rho+x_1|x|\varphi'\rho-x_2\varphi\rho'), & 0<|x|<2,\\ 1, & \text{else},\end{cases}$$

where $\varphi=\varphi(|x|-1)$ and $\rho=\rho(x/|x|)$. Relation (2.3) is now immediate. The assertion follows now due to relations (2.2) and (2.3), using arguments similar to the ones in Lemma 1.2 in [5]. □

Given $\rho\in\mathcal{V}_\alpha$, the operator $\mathcal{A}(\rho)$ is elliptic (uniformly elliptic if (1.3) holds) and it carries also a quasilinear structure, in the sense that

$$\mathcal{A}(\rho)v=b_{ij}(x,\rho,Dv)v_{ij}+b_i(x,\rho,Dv)v_i,\quad \forall v\in buc^{2+\alpha}(\Omega),$$

where

$$b_{ij}(x,\rho,Dv)=\psi^i_{\rho,k}(\phi_\rho(x))\psi^j_{\rho,l}(\phi_\rho(x))a_{kl}(D(\phi^\rho_* v)(\phi_\rho(x)))\text{ for }1\le i,j\le 2,$$

$$b_i(x,\rho,Dv)=\psi^i_{\rho,kl}(\phi_\rho(x))a_{kl}(D(\phi^\rho_* v)(\phi_\rho(x)))\text{ for }1\le i\le 2,$$

and

$$D(\phi^\rho_* v)(\phi_\rho(x))=(v_k(x)\psi^k_{\rho,1}(\phi_\rho(x)),v_k(x)\psi^k_{\rho,2}(\phi_\rho(x)))$$

for $v \in buc^{2+\alpha}(\Omega)$ and $x \in \Omega$. Moreover, given $\rho \in \mathcal{V}_\alpha$, $x \in \Omega$ and $p = (p_1, p_2) \in \mathbb{R}^2$, we have

$$b_{ij}(x,\rho,p)\xi_i\xi_j = a_{ij}(p_k\psi^k_{\rho,1}(\phi_\rho(x)),\, p_k\psi^k_{\rho,2}(\phi_\rho(x)))(\xi_k\psi^k_{\rho,i}(\phi_\rho(x)))(\xi_k\psi^k_{\rho,j}(\phi_\rho(x)))$$

for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. The ellipticity, respectively the uniform ellipticity of $\mathcal{A}(\rho)$ is a immediate consequence of relation (1.1), respectively (1.3). We study now the right-hand side of the second equation of (2.1).

Lemma 2.2. *The mapping*

$$\mathcal{V}_\alpha \ni \rho \longmapsto \kappa(\rho) := \frac{(1+\rho)^2 + 2\rho'^2 - (1+\rho)\rho''}{((1+\rho)^2 + \rho'^2)^{3/2}} \in buc^{2+\alpha}(\mathbb{S}^1)$$

is smooth. Given $\rho \in h^{4+\alpha}(\mathbb{S}^1)$, we have $\partial\kappa(0)[\rho] = -\rho - \rho''$.

Proof. We can express κ as $\kappa := \Upsilon \circ \Theta$ where Υ and Θ are the functions defined by

$$\Upsilon : \mathcal{V}_\alpha \to h^{2+\alpha}(\mathbb{S}^1) \times h^{2+\alpha}(\mathbb{S}^1) \times h^{2+\alpha}(\mathbb{S}^1),$$

$$\Upsilon(\rho) = (\rho, \rho', \rho''),$$

$$\Theta : h^{2+\alpha}(\mathbb{S}^1) \times h^{2+\alpha}(\mathbb{S}^1) \times h^{2+\alpha}(\mathbb{S}^1) \to h^{2+\alpha}(\mathbb{S}^1),$$

$$\Theta(\rho_1, \rho_2, \rho_3) = \frac{(1+\rho_1)^2 + 2\rho_2^2 - (1+\rho_1)\rho_3}{((1+\rho_1)^2 + \rho_2^2)^{3/2}}.$$

Since Υ and Θ are smooth, then so is also κ. The second assertion is now obvious. □

Assume first that relation (1.3) is not satisfied. Then, given $\rho \in \mathcal{V}_\alpha$, the boundary curve

$$(\partial\Omega_\rho, \gamma\kappa_\rho) = \{(z, \gamma\kappa_\rho(z)) \,:\, z \in \partial\Omega_\rho\} \subset \mathbb{R}^3$$

satisfies a bounded slope condition since the curvature of $\partial\Omega_\rho$ is everywhere positive and the boundary $\partial\Omega_\rho$, respectively the boundary value $\gamma\kappa_\rho$, belong to the class C^2. This means that for each point $P \in (\partial\Omega_\rho, \gamma\kappa_\rho)$ the curve $(\partial\Omega_\rho, \gamma\kappa_\rho)$ is bounded from above and below on the cylinder $\partial\Omega_\rho \times \mathbb{R}$ by two planes and coincides with them at P. The slopes of the planes are uniformly bounded by a constant which does not depend on the point P. In view of [14, Theorem 6.4.2] we obtain that for each $\rho \in \mathcal{V}_\alpha$, the Dirichlet problem

$$\begin{aligned} Qu &= 0 \quad \text{in } \Omega_\rho, \\ u &= \gamma\kappa_\rho \text{ on } \Gamma_\rho \end{aligned} \tag{2.4}$$

possesses a solution $u \in BUC^{2+\alpha}(\Omega_\rho)$. If (1.3) holds, then Q is uniformly elliptic and the same result for (2.4) is obtained by applying [14, Theorem 4.8.2].

Moreover, the mean value theorem (see [10, Theorem 10.2] for more details) yields that this solution is unique. Notice also that if $\rho \in C^\infty(\mathbb{S}^1)$, then the solution u to (2.4) is smooth, that is $u \in BUC^\infty(\Omega_\rho)$.

Together with Lemma 2.2 we obtain the following existence, uniqueness and regularity result.

Theorem 2.3. *Given $\rho \in \mathcal{V}_\alpha$, there exists a unique solution $\mathcal{T}(\rho) \in buc^{2+\alpha}(\Omega)$ of the quasilinear Dirichlet problem*

$$\begin{aligned} \mathcal{A}(\rho)v &= 0 \qquad in \ \ \Omega, \\ v &= \gamma\kappa(\rho) \ \ on \ \mathbb{S}^1. \end{aligned} \tag{2.5}$$

Moreover, the mapping $[\mathcal{V}_\alpha \ni \rho \longmapsto \mathcal{T}(\rho) \in buc^{2+\alpha}(\Omega)]$ is smooth.

Proof. In view of the facts discussed above it suffices to prove that $\mathcal{T}$ is smooth. Indeed, knowing that $\mathcal{T}$ is smooth and that (2.4) has a unique solution for $\rho \in \mathcal{V}_\alpha$, we obtain in view of $\mathcal{T}(C^\infty(\mathbb{S}^1)) \subset BUC^\infty(\Omega)$, that $\rho \in \mathcal{V}_\alpha$ implies $\mathcal{T}(\rho) \in buc^{2+\alpha}(\Omega)$.

We proceed now and prove that $\mathcal{T}$ is a smooth operator. Let $\mathcal{S} : \mathcal{V}_\alpha \times BUC^{2+\alpha}(\Omega) \to \mathcal{L}(BUC^{2+\alpha}(\Omega), BUC^\alpha(\Omega))$ be the operator defined by

$$\mathcal{S}(\rho, v)[u] := b_{ij}(x, \rho, Dv)u_{ij} + b_i(x, \rho, Dv)u_i,$$

where we use the standard sum convention. Given $(\rho, v) \in \mathcal{V}_\alpha \times BUC^{2+\alpha}(\Omega)$, $\mathcal{S}(\rho, v)$ is a linear, uniformly elliptic differential operator of second order satisfying

$$\mathcal{S} \in C^\infty(\mathcal{V}_\alpha \times BUC^{2+\alpha}(\Omega), \mathcal{L}(BUC^{2+\alpha}(\Omega), BUC^\alpha(\Omega))).$$

It follows that $\mathcal{I} : \mathcal{V}_\alpha \times BUC^{2+\alpha}(\Omega) \to BUC^{2+\alpha}(\Omega)$, with

$$\mathcal{I}(\rho, v) := (\mathcal{S}(\rho, v), \mathrm{tr})^{-1}(0, \gamma\kappa(\rho)),$$

is a smooth operator. We have denoted by tr the trace operator on $\mathbb{S}^1$. The smoothness is obtained in view of Lemma 2.2, taking also into account that the function mapping a linear operator to its inverse is analytical.

The function $[\rho \mapsto (\rho, \mathcal{T}(\rho))]$ is a parametrization of the 0-level set of the smooth mapping

$$\mathcal{F} : \mathcal{V}_\alpha \times BUC^{2+\alpha}(\Omega) \to BUC^{2+\alpha}(\Omega), \qquad \mathcal{F}(\rho, v) := v - \mathcal{I}(\rho, v).$$

The derivative of $\mathcal{F}$ with respect to v is given by the relation

$$\partial_v\mathcal{F}(\rho, v) = \mathrm{id}_{BUC^{2+\alpha}(\Omega)} - \partial_v\mathcal{I}(\rho, v),$$

where $\partial_v\mathcal{I}(\rho, v)$ is a compact operator. This is due to the fact that $\mathcal{S}$ and $\mathcal{I}$ have natural extensions to operators on $\mathcal{V}_\alpha \times BUC^{1+\alpha}(\Omega)$ and the embedding

$BUC^{2+\alpha}(\Omega) \hookrightarrow BUC^{1+\alpha}(\Omega)$ is compact. Consequently, $\partial_v \mathcal{F}(\rho, v)$ is a Fredholm operator of index 0.

Let us compute the Fréchet derivative $\partial_v \mathcal{I}(\rho, \mathcal{T}(\rho))$ for $\rho \in \mathcal{V}_\alpha$. Let $\rho \in \mathcal{V}_\alpha$ be given and set $v := \mathcal{T}(\rho)$. Given $w \in BUC^{2+\alpha}(\Omega)$, the mapping $\partial_v \mathcal{I}(\rho, v)[w]$ is the unique solution of the following Dirichlet problem:

$$\begin{aligned} b_{ij}(x,\rho,Dv)z_{ij}+b_i(x,\rho,Dv)z_i &= -\left[\overline{\partial} b_{ij}(x,\rho,Dv)v_{ij} + \overline{\partial} b_i(x,\rho,Dv)v_i\right] \\ &\qquad \cdot (D(\phi_*^\rho w)(\phi_\rho)) \text{ in } \Omega \\ z &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{L}$$

where

$$\overline{\partial} b_{ij}(x,\rho,Dv) = \psi^i_{\rho,k}(\phi_\rho(x))\psi^j_{\rho,l}(\phi_\rho(x))\partial a_{kl}(D(\phi_*^\rho v)(\phi_\rho(x))) \text{ for } 1 \le i, j \le 2,$$

$$\overline{\partial} b_i(x,\rho,Dv) = \psi^i_{\rho,kl}(\phi_\rho(x))\partial a_{kl}(D(\phi_*^\rho v)(\phi_\rho(x))) \text{ for } 1 \le i \le 2,$$

and ∂a_{ij} are the usual Fréchet derivatives of the smooth mappings $a_{ij} : \mathbb{R}^2 \to \mathbb{R}$, $1 \le i, j \le 2$.

We state that the linear operator $\partial_v \mathcal{F}(\rho, \mathcal{T}(\rho))$ is an isomorphism for all $\rho \in \mathcal{V}_\alpha$. Taking into consideration that $\partial_v \mathcal{F}(\rho, \mathcal{T}(\rho))$ is a Fredholm operator of index 0, it suffices to prove that it is one-to-one. Indeed, let $w \in BUC^{2+\alpha}(\Omega)$ be a function with the property that $\partial_v \mathcal{F}(\rho, \mathcal{T}(\rho))[w] = 0$, that is $w = \partial_v \mathcal{I}(\rho, \mathcal{T}(\rho))[w]$. Then, w is the solution of (L), hence $w = 0$.

The implicit function theorem yields that $\mathcal{T}$ is smooth in a neighbourhood of ρ for all $\rho \in \mathcal{V}_\alpha$. This completes the proof. □

3. The nonlinear Cauchy problem

Replacing in the third equation of (2.1) v by $\mathcal{T}(\rho)$, the solution to (2.5), we reduce problem (1.4) to a nonlinear Cauchy problem over the unit circle:

$$\partial_t \rho + \Phi(\rho) = 0, \qquad \rho(0) = \rho_0, \tag{3.1}$$

where $\Phi(\cdot) := \mathcal{B}(\cdot, \mathcal{T}(\cdot))$ is a nonlinear and nonlocal operator of third order.

In order to prove Theorem 1.1 we observe first that Φ is a smooth mapping and that $-\partial\Phi(0)$ generates a strongly continuous analytic semigroup in $\mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))$ with domain of definition $h^{4+\alpha}(\mathbb{S}^1)$, that is $\partial\Phi(0) \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$.

We study now the regularity properties of the operator $\mathcal{B}$. First we prove the following result:

Lemma 3.1. *The nonlinear operator $\mathcal{B}$ belongs to $C^\infty(\mathcal{V}_\alpha \times buc^{2+\alpha}(\Omega), h^{1+\alpha}(\mathbb{S}^1))$. Moreover, given $(\rho_0, v_0) \in (-a, a) \times \mathbb{R} \subset \mathcal{V}_\alpha \times buc^{2+\alpha}(\Omega)$, we have*

$$\partial\mathcal{B}(\rho_0, v_0)[\rho, v] = \frac{1}{\overline{\mu}(0)} \operatorname{tr} \partial_\nu v$$

for all $[\rho, v] \in h^{4+\alpha}(\mathbb{S}^1) \times buc^{2+\alpha}(\Omega)$, *where* ν *is the unit exterior normal vector field to* $\partial\Omega$.

Proof. Given $(\rho, v) \in \mathcal{V}_\alpha \times buc^{2+\alpha}(\Omega)$ we compute:

$$\langle D(\phi_*^\rho v), DN_\rho\rangle(\phi_\rho) = \left[x_i \psi^j_{\rho,i}(\phi_\rho) + \frac{\rho'}{1+\rho}\left(x_2\psi^j_{\rho,1}(\phi_\rho) - x_1\psi^j_{\rho,2}(\phi_\rho)\right)\right] v_j$$

and the regularity assumption is now obvious. We used here again the usual sum convention.

Let $(\rho_0, v_0) \in (-a, a) \times \mathbb{R}$ be fixed and $\theta > 0$, with the property any $\rho \in h^{4+\alpha}(\mathbb{S}^1)$ satisfying $\|\rho - \rho_0\|_{C^{4+\alpha}(\mathbb{S}^1)} \leq \theta$ belongs to $\mathcal{V}_\alpha$. We prove that

$$\partial\mathcal{B}(\rho_0, v_0)[\rho, v](x) = \frac{1}{\overline{\mu}(0)}\langle D(\phi_*^{\rho_0} v), DN_{\rho_0}\rangle(\phi_{\rho_0}(x)), \quad x \in \mathbb{S}^1$$

for $[\rho,v]\in h^{4+\alpha}(\mathbb{S}^1)\times buc^{2+\alpha}(\Omega)$ satisfying $\|[\rho,v]-[\rho_0,v_0]\|_{C^{4+\alpha}(\mathbb{S}^1)\times BUC^{2+\alpha}(\Omega)} \leq \theta$. We notice first that $\mathcal{B}(\rho_0, v_0) = 0$. Further on, we write

$$\begin{aligned}
&\mathcal{B}(\rho_0+\rho, v_0+v) - \frac{1}{\overline{\mu}(0)}\langle D(\phi_*^{\rho_0} v), DN_{\rho_0}\rangle(\phi_{\rho_0}) \\
&= \left(\frac{1}{\overline{\mu}(|D(\phi_*^{\rho_0+\rho}(v_0+v))|^2)} - \frac{1}{\overline{\mu}(0)}\right)\langle D(\phi_*^{\rho_0+\rho}(v_0+v)), DN_{\rho_0+\rho}\rangle(\phi_{\rho_0+\rho}) \\
&\quad + \frac{1}{\overline{\mu}(0)}\Big(\langle D(\phi_*^{\rho_0+\rho}(v_0+v)), DN_{\rho_0+\rho}\rangle(\phi_{\rho_0+\rho}) - \langle D(\phi_*^{\rho_0} v), DN_{\rho_0}\rangle(\phi_{\rho_0})\Big) \\
&=: E_1 + E_2.
\end{aligned}$$

Using standard arguments we find a positive constant χ depending only on (ρ_0, v_0) such that $\|E_i\|_{C^{1+\alpha}(\mathbb{S}^1)} \leq \chi\|[\rho, v]\|^2_{C^{4+\alpha}(\mathbb{S}^1)\times BUC^{2+\alpha}(\Omega)}$. The relation $\langle D(\phi_*^{\rho_0} v), DN_{\rho_0}\rangle(\phi_{\rho_0}) = \partial_\nu v$ leads then to the conclusion.

Indeed, the estimate for E_1 follows easily from the mean value theorem and the fact that v_0 is constant. Moreover, we have that

$$\begin{aligned}
\overline{\mu}(0)E_2 &= \langle D(\phi_*^{\rho_0+\rho} v), DN_{\rho_0+\rho}\rangle(\phi_{\rho_0+\rho}) - \langle D(\phi_*^{\rho_0} v), DN_{\rho_0}\rangle(\phi_{\rho_0}) \\
&= \Bigg\{x_i(\psi^j_{\rho_0+\rho,i}(\phi_{\rho_0+\rho}) - \psi^j_{\rho_0,i}(\phi_{\rho_0})) \\
&\quad + \Bigg[\frac{\rho'}{1+\rho_0+\rho}\left(x_2\psi^j_{\rho_0+\rho,1}(\phi_{\rho_0+\rho}) - x_1\psi^j_{\rho_0+\rho,2}(\phi_{\rho_0+\rho})\right) \\
&\quad - \frac{\rho'}{1+\rho}\left(x_2\psi^j_{\rho,1}(\phi_\rho) - x_1\psi^j_{\rho,2}(\phi_\rho)\right)\Bigg]\Bigg\} v_j
\end{aligned}$$

which, in view of (2.2) and (2.3) completes the proof. □

Summarizing, we obtain from Theorem 2.3 and Lemma 3.1 that $\Phi \in C^\infty(\mathcal{V}_\alpha, h^{1+\alpha}(\mathbb{S}^1))$. Using the chain rule we have

$$\partial\Phi(0) = \partial\mathcal{B}(0, \mathcal{T}(0)) \circ (\mathrm{id}_{h^{4+\alpha}(\mathbb{S}^1)}, \partial\mathcal{T}(0)).$$

Let us first notice that $\mathcal{T}(0) = \gamma$. We are left to determine the derivative $\partial\mathcal{T}(0)$. Using Lemma 2.1 we obtain:

Lemma 3.2. *Given $\rho \in \mathcal{V}_\alpha$, the mapping $\partial\mathcal{T}(0)[\rho] \in buc^{2+\alpha}(\Omega)$ is the unique solution of the linear Dirichlet problem*

$$\begin{aligned} \Delta\omega &= 0 \qquad\qquad\quad , \ in \ \Omega, \\ \omega &= -\gamma(\rho + \rho'') \ , \ on \ \mathbb{S}^1. \end{aligned} \tag{3.2}$$

Proof. The proof is standard and is left to the reader. □

Thus, given $\rho \in \mathcal{V}_\alpha$, the derivative $\partial\Phi(0)$ satisfies $\partial\Phi(0)[\rho] = (1/\overline{\mu}(0))\partial_\nu\omega$, where ω is the unique solution to (3.2). If we consider the Fourier expansion of $\rho = \sum_{k\in\mathbb{Z}} \widehat{\rho}(k)x^k$, we obtain from the well-known Poisson integral formula that

$$\omega(rx) = \sum_{k\in\mathbb{Z}} \gamma(k^2 - 1)r^{|k|}\widehat{\rho}(k)x^k$$

for all $r \le 1$ and $x \in \mathbb{S}^1$. Particularly, $\partial_\nu\omega(x) = \sum_{k\in\mathbb{Z}} \gamma|k|(k^2 - 1)\widehat{\rho}(k)x^k$ for $x \in \mathbb{S}^1$, and we conclude

$$-\partial\Phi(0)\left[\sum_{k\in\mathbb{Z}} \widehat{\rho}(k)x^k\right] = \sum_{k\in\mathbb{Z}} \zeta|k|(1 - k^2)\widehat{\rho}(k)x^k \tag{3.3}$$

for all $\sum_{k\in\mathbb{Z}} \widehat{\rho}(k)x^k \in h^{4+\alpha}(\mathbb{S}^1)$, where $\zeta := \gamma/\overline{\mu}(0)$.

We will use the Fourier representation (3.3) to prove that $\partial\Phi(0) \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$. The coefficients $\lambda_k := \zeta|k|(1 - k^2)$, $k \in \mathbb{Z}$ will play an important role in our further analysis. Given $r \ge 0$, the Sobolev space $H^r(\mathbb{S}^1)$ is defined by

$$H^r(\mathbb{S}^1) := \{\rho \in L^2(\mathbb{S}^1) : \sum_{k\in\mathbb{Z}} (1 + k^2)^r|\widehat{\rho}(k)|^2 < \infty\},$$

and is endowed with the scalar product $\langle\rho, \varsigma\rangle := \sum_{k\in\mathbb{Z}} (1 + k^2)^r\widehat{\rho}(k)\overline{\widehat{\varsigma}(k)}$. The smooth functions are dense in $H^r(\mathbb{S}^1)$ and the Sobolev embedding $H^{k+r}(\mathbb{S}^1) \hookrightarrow C^k(\mathbb{S}^1)$ holds for all $k \in \mathbb{N}$, provided $r > 1/2$. Moreover, $H^{k+s}(\mathbb{S}^1) \overset{d}{\hookrightarrow} h^{k+\beta}(\mathbb{S}^1)$ for all $k \in \mathbb{N}$, $\beta \in [0, 1]$, and $s > 3/2$.

Let us now consider the operator $-\partial\Phi(0)$, given by (3.3), as an operator between Sobolev spaces. Given $\mathrm{Re}\,\lambda \ge 1$ the operator $\lambda + \partial\Phi(0)$ is an isomorphisms. More precisely, it belongs to $\mathrm{Isom}(H^{r+3}(\mathbb{S}^1), H^r(\mathbb{S}^1))$ for all $r \ge 0$. This can be seen using the Fourier expansions of the functions belonging to the spaces $H^s(\mathbb{S}^1)$, $s \ge 0$, together with the relation $\lim_{|k|\to\infty}(\lambda_{|k|}/|k|^3) = -\zeta$. Consequently, for any $\mathrm{Re}\,\lambda \ge 1$ and $r \ge 0$, the resolvent $R(\lambda, -\partial\Phi(0))$ is a well-defined element of $\mathcal{L}(H^r(\mathbb{S}^1), H^{r+3}(\mathbb{S}^1))$. Applying this result we obtain:

Proposition 3.3. *Let* $k \in \{1,4\}$ *and suppose* $R(\lambda, -\partial\Phi(0)) \in \mathcal{L}(C^{1+\alpha}(\mathbb{S}^1), C^{k+\alpha}(\mathbb{S}^1))$ *for some* $\operatorname{Re}\lambda \geq 1$. *Then* $R(\lambda, -\partial\Phi(0)) \in \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1), h^{k+\alpha}(\mathbb{S}^1))$.

Proof. From the assumption we deduce that $R(\lambda, -\partial\Phi(0)) \in \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1), C^{4+\alpha}(\mathbb{S}^1))$. Given $\rho \in h^{1+\alpha}(\mathbb{S}^1)$, we find a sequence $(\rho_n)_n \subset H^r(\mathbb{S}^1)$, $r > 3$, such that $\rho_n \to \rho$ in $C^{1+\alpha}(\mathbb{S}^1)$. It follows that

$$R(\lambda, -\partial\Phi(0))\rho_n \longrightarrow R(\lambda, -\partial\Phi(0))\rho \quad \text{in} \quad C^{k+\alpha}(\mathbb{S}^1).$$

Thanks to the above mentioned result we obtain

$$R(\lambda, -\partial\Phi(0))\rho \in \overline{H^{r+3}(\mathbb{S}^1)}^{\|\cdot\|_{C^{k+\alpha}(\mathbb{S}^1)}} = h^{k+\alpha}(\mathbb{S}^1). \qquad \square$$

This proposition is very useful because it allows us to transfer the problem of studying the spectrum of the operator $-\partial\Phi(0)$ considered as an operator between small Hölder spaces to the case when it acts between Hölder spaces, which is more at hand due to the identification $C^s(\mathbb{S}^1) = B^s_{\infty,\infty}(\mathbb{S}^1)$ for $s > 0$, $s \notin \mathbb{N}$. Here we have denoted by $B^s_{\infty,\infty}(\mathbb{S}^1)$ the usual Besov spaces over $\mathbb{S}^1$ (for more details see [16]).

In order to show that $\partial\Phi(0) \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$ it suffices, *cf.* [1], to find a constant $\chi \geq 1$ such that

$$\lambda + \partial\Phi(0) \in \operatorname{Isom}(h^{4+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1)), \tag{3.4}$$

$$|\lambda| \cdot \|R(\lambda, -\partial\Phi(0))\|_{\mathcal{L}(h^{1+\alpha}(\mathbb{S}^1))} \leq \chi, \tag{3.5}$$

for all $\operatorname{Re}\lambda \geq 1$. We shall prove that there exists a constant χ such that relations (3.4) and (3.5) hold. The proof is based on a straightforward generalization of a result appeared in [2].

Theorem 3.4. *Let* r, s *be two positive constants and let* $(M_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ *be a sequence satisfying the following conditions*

$$\text{(i)} \quad \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s}|M_k| < \infty,$$

$$\text{(ii)} \quad \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s+1}|M_{k+1} - M_k| < \infty,$$

$$\text{(iii)} \quad \sup_{k\in\mathbb{Z}\setminus\{0\}} |k|^{r-s+2}|M_{k+2} - 2M_{k+1} + M_k| < \infty.$$

The mapping

$$\sum_{k\in\mathbb{Z}} \widehat{\rho}(k)x^k \longmapsto \sum_{k\in\mathbb{Z}} M_k\widehat{\rho}(k)x^k$$

belongs then to $\mathcal{L}(B^s_{\infty,\infty}(\mathbb{S}^1), B^r_{\infty,\infty}(\mathbb{S}^1))$.

Proof. The proof follows similarly to that of Theorem 4.5 (i) in [2] with the obvious modifications. □

Theorem 3.5.

$$\partial\Phi(0) \in \mathcal{H}(h^{4+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1)).$$

Proof. Let $\operatorname{Re}\lambda \geq 1$ be fixed. Given $\sum_{k\in\mathbb{Z}} \widehat{\rho}(k)x^k \in L^2(\mathbb{S}^1)$, we have

$$R(\lambda, -\partial\Phi(0))\left[\sum_{k\in\mathbb{Z}} \widehat{\rho}(k)x^k\right] = \sum_{k\in\mathbb{Z}} M_k^\lambda \widehat{\rho}(k)x^k,$$

where $M_k^\lambda = 1/(\lambda - \lambda_k),\ k \in \mathbb{Z}$. We prove first that $R(\lambda, -\partial\Phi(0)) \in \mathcal{L}(h^{1+\alpha}(\mathbb{S}^1), h^{4+\alpha}(\mathbb{S}^1))$. Thanks to Proposition 3.3 and Theorem 3.4 it suffices to show that the coefficients (M_k^λ) satisfy conditions (i), (ii) and (iii) of the above theorem (with $s = 1 + \alpha$ and $r = 4 + \alpha$).

The relation $\lim_{k\to\infty} k^3/(\lambda-\lambda_k) = 1/\zeta$ implies (i). Further on, we write for $k \geq 1$

$$k^4|M_{k+1}^\lambda - M_k^\lambda| = \frac{k^3}{|\lambda - \lambda_{k+1}|}\frac{k^3}{|\lambda - \lambda_k|}\frac{|\lambda_{k+1} - \lambda_k|}{k^2} \underset{k\to\infty}{\longrightarrow} \frac{3}{\zeta},$$

because of $(\lambda_k - \lambda_{k+1})/k^2 \to 3\zeta$, and (ii) is proved. We also have

$$\begin{aligned} &k^5|M_{k+2} - 2M_{k+1} + M_k| \\ &= \frac{k^3}{|\lambda - \lambda_{k+2}|}\frac{k^3}{|\lambda - \lambda_{k+1}|}\frac{k^3}{|\lambda - \lambda_k|}\frac{1}{k^4}\Big|\lambda(\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k) + \\ &\qquad\qquad + \lambda_k(\lambda_{k+1} - \lambda_{k+2}) + \lambda_{k+2}(\lambda_{k+1} - \lambda_k)\Big|, \end{aligned}$$

and $(\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k)/k^4 \to 0$, respectively $(\lambda_k(\lambda_{k+1} - \lambda_{k+2}) + \lambda_{k+2}(\lambda_{k+1} - \lambda_k))/k^4 \to 12\zeta^2$. This proves (iii). Thus, $\lambda + \partial\Phi(0) \in \operatorname{Isom}(h^{4+\alpha}(\mathbb{S}^1), h^{1+\alpha}(\mathbb{S}^1))$ for all $\operatorname{Re}\lambda \geq 1$.

We prove now (3.5). Denoting by $N_k^\lambda := \lambda M_k^\lambda$ for $\operatorname{Re}\lambda \geq 1$ and $k \in \mathbb{Z}$, it suffices to show that the conditions (i), (ii) and (iii) of Theorem 3.4 (with $s = r = 1 + \alpha$) hold for (N_k^λ) uniformly in $\lambda \in \{\lambda : \operatorname{Re}\lambda \geq 1\}$. Let us notice first that $|\lambda - \lambda_k| \geq \max\{1, |\lambda|, |\lambda_k|\}$ for all $\operatorname{Re}\lambda \geq 1$ and $k \in \mathbb{Z}$. Moreover, $\lambda_k = 0$ iff $|k| \leq 1$. Therefore, we have

$$\sup_{\operatorname{Re}\lambda\geq 1}\sup_{k\in\mathbb{Z}} |N_k^\lambda| \leq 1.$$

We also get

$$|k||N_{k+1}^\lambda - N_k^\lambda| = \frac{|\lambda|}{|\lambda - \lambda_{k+1}|}\frac{|k|^3}{|\lambda - \lambda_k|}\frac{|\lambda_{k+1} - \lambda_k|}{k^2} \leq \frac{|\lambda_{k+1} - \lambda_k|}{k^2}x_k,$$

where

$$x_k := \begin{cases} 1 & , \ |k| \le 1, \\ \dfrac{|k|^3}{|\lambda_k|} & , \ |k| \ge 2, \end{cases}$$

and we are left to verify that (iii) holds. For $k \neq 0$ we have

$$\begin{aligned} & k^2 |N^\lambda_{k+2} - 2N^\lambda_{k+1} + N^\lambda_k| \\ &= \frac{|\lambda|}{|\lambda - \lambda_{k+2}|} \frac{k^3}{|\lambda - \lambda_{k+1}|} \frac{k^3}{|\lambda - \lambda_k|} \frac{1}{k^4} \Big| \lambda(\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k) + \\ & \qquad\qquad + \lambda_k(\lambda_{k+1} - \lambda_{k+2}) + \lambda_{k+2}(\lambda_{k+1} - \lambda_k) \Big| \\ &\le x_k \frac{|\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k|}{|k|} + x_k x_{k+1} \frac{|\lambda_k(\lambda_{k+1} - \lambda_{k+2}) + \lambda_{k+2}(\lambda_{k+1} - \lambda_k)|}{k^4}. \end{aligned}$$

The relation $(\lambda_{k+2} - 2\lambda_{k+1} + \lambda_k)/k \underset{k\to\infty}{\longrightarrow} -6\zeta$ leads to conclusion. □

We are now able to sketch the proof of Theorem 1.1. We fix $0 < \beta < \alpha$ and set $\theta := (\alpha - \beta)/3$. Since $\alpha \in (0, 1)$ was arbitrary in our analysis, we deduce from Theorem 3.5, using the fact that $\mathcal{H}(h^{4+\beta}(\mathbb{S}^1), h^{1+\beta}(\mathbb{S}^1))$ is open in $\mathcal{L}(h^{4+\beta}(\mathbb{S}^1), h^{1+\beta}(\mathbb{S}^1))$, the existence of an open neighbourhood $\mathcal{O}_\beta$ of 0 in $\mathcal{V}_\beta$ with the property that $\partial\Phi(x) \in \mathcal{H}(h^{4+\beta}(\mathbb{S}^1), h^{1+\beta}(\mathbb{S}^1))$ for all $x \in \mathcal{O}_\beta$. Let $\mathcal{O} := \mathcal{O}_\beta \cap h^{4+\alpha}(\mathbb{S}^1)$. Letting $C > 0$ be the norm of the compact embedding $h^{4+\alpha}(\mathbb{S}^1) \hookrightarrow h^{4+\beta}(\mathbb{S}^1)$, we find that $B_{h^{4+\alpha}(\mathbb{S}^1)}(x, r/C) \subset \mathcal{O}$ for any ball $B_{h^{4+\beta}(\mathbb{S}^1)}(x, r) \subset \mathcal{O}_\beta$. Thus $\mathcal{O}$ is an open neighbourhood of 0 in $\mathcal{V}_\alpha$.

Moreover, given $x \in \mathcal{O}$, the operator $-\partial\Phi(x)$ is the part in $h^{1+\alpha}(\mathbb{S}^1)$ of a sectorial operator $B : h^{4+\beta}(\mathbb{S}^1) \subset h^{1+\beta}(\mathbb{S}^1) \longrightarrow h^{1+\beta}(\mathbb{S}^1)$ with $D_B(\theta) = h^{1+\alpha}(\mathbb{S}^1)$ and $D_B(\theta+1) = h^{4+\alpha}(\mathbb{S}^1)$, where $D_B(\theta)$ and $D_B(\theta+1)$ are suitable interpolation spaces, associated to the operator B. We refer to [15] for details. We have used here the well-known interpolation property of the small Hölder spaces

$$(h^{\sigma_0}(\mathbb{S}^1), h^{\sigma_1}(\mathbb{S}^1))_\theta = h^{(1-\theta)\sigma_0+\theta\sigma_1}(\mathbb{S}^1), \ \text{ if } \theta \in (0,1) \text{ and } (1-\theta)\sigma_0+\theta\sigma_1 \notin \mathbb{N}. \quad (3.6)$$

Thus, we have establish that the assumptions of Theorem 8.4.1 in [15] hold and the proof of Theorem 1.1 is now obvious. Consequently, given $\rho_0 \in \mathcal{O}$, there exists a positive time $T(\rho_0) > 0$ and a unique classical solution (u, ρ) to problem (1.4) on $[0, T(\rho_0)]$ satisfying $\rho([0, T(\rho_0)]) \subset \mathcal{O}$. Moreover, the solution may be extended on a maximal interval $[0, T(\rho_0))$ and if ρ is uniformly continuous with values in $h^{4+\alpha}(\mathbb{S}^1)$, then either

$$\lim_{t \nearrow T(\rho_0)} \rho(t) \in \partial\mathcal{O} \quad \text{or} \quad T(\rho_0) = +\infty.$$

We prove now a conservation law for the fluid volume:

Lemma 3.6 (Conservation of volume). *Given* $\rho\in\mathcal{V}_\alpha$*, we have* $\int_{\mathbb{S}^1}(1+\rho)\Phi(\rho)dx=0$.

Proof. Let $\rho \in \mathcal{V}_\alpha$ be given. Denoting by u the unique solution of the Dirichlet problem

$$\begin{aligned} \operatorname{div}\left(\tfrac{Du}{\overline{\mu}(|Du|^2)}\right) &= 0 \quad \text{in } \Omega_\rho, \\ u &= \gamma\kappa_\rho \text{ on } \Gamma_\rho, \end{aligned}$$

we obtain

$$\begin{aligned} \int_{\mathbb{S}^1}(1+\rho)\Phi(\rho)\,dx &= \int_{\mathbb{S}^1}(1+\rho)\mathcal{B}(\rho,\mathcal{T}(\rho))\,dx \\ &= \int_{\mathbb{S}^1}(1+\rho)\left\langle\frac{Du}{\overline{\mu}(|Du|^2)}, DN_\rho\right\rangle(\phi_\rho)\,dx \\ &= \frac{1}{2\pi}\int_{\partial\Omega_\rho}\left\langle\frac{Du}{\overline{\mu}(|Du|^2)}, \nu_\rho\right\rangle d\sigma \\ &= \frac{1}{2\pi}\int_{\Omega_\rho}\operatorname{div}\left(\frac{Du}{\overline{\mu}(|Du|^2)}\right)dx = 0. \end{aligned}$$

This completes the proof. □

Given $\rho \in \mathcal{V}_\alpha$, the volume of the domain Ω_ρ is $\operatorname{vol}(\Omega_\rho) = \pi\int_{\mathbb{S}^1}(1+\rho)^2dx$. Fixing $\rho_0 \in \mathcal{O}$, we denote by $\rho : [0, T(\rho_0)) \to \mathcal{V}_\alpha$ the solution to (1.4). We compute

$$\frac{d}{dt}\operatorname{vol}(\Omega_{\rho(t)}) = -2\pi\int_{\mathbb{S}^1}(1+\rho(t))\Phi(\rho(t))\,dx = 0,$$

thus the volume of the fluid is preserved. Furthermore, we know [18] that also the centre of mass of the fluid domain is preserved by the flow.

4. Equilibria and stability properties

We determine the equilibria of the problem (1.4) by solving the following free boundary problem

$$\begin{aligned} \operatorname{div}\left(\tfrac{Du}{\overline{\mu}(|Du|^2)}\right) &= 0 \quad \text{in } \Omega_\rho, \\ u &= \gamma\kappa_\rho \text{ on } \Gamma_\rho, \\ \langle Du, DN_\rho\rangle &= 0 \quad \text{on } \Gamma_\rho. \end{aligned} \tag{4.1}$$

Assuming that $\rho \in \mathcal{V}_\alpha$ is known, we obtain from the first and the third equation of (4.1), using the fact that the outward normal at $\partial\Omega_\rho$ is $\nu_\rho = DN_\rho/|DN_\rho|$, that u must be a constant function. Consequently, the curvature of $\partial\Omega_\rho$ is constant and

$\partial\Omega_\rho$ must be a circle. Furthermore, it is obvious that any circle sufficiently near the unit circle determines a steady-state solution for (1.4). There exists also an open neighbourhood $\mathcal{U}$ of the origin in $\mathbb{R}^3$ with the property that

$$\mathcal{E} := \{(\rho_{(c,R)}, \gamma/(R+1)) \in \mathcal{V}_\alpha \times buc^{2+\alpha}(\Omega) \, : \, (c, R) \in \mathcal{U}\}$$

contains all the equilibria of problem (1.4) in $\mathcal{V}_\alpha$. Given $(c, R) \in \mathcal{U}$, the mapping

$$\rho_{(c,R)}(x) = \sqrt{(R+1)^2 - |c|^2 + < c, x >^2} + < c, x > -1, \quad x \in \mathbb{S}^1,$$

defines an element of $\mathcal{V}_\alpha$ and the boundary $\partial\Omega_{\rho_{(c,R)}}$ is exactly the circle with centre c and radius (R+1).

In order to study the stability of the 0 solution for the problem (3.1) we shall proceed like in [15] and [4] and construct a three dimensional invariant submanifold of the phase space with the property that the eigenspace corresponding to the eigenvalue $\lambda_1 = 0$ is tangential to it in 0.

Let $0 < \beta < \alpha < 1$ and $\theta := (\alpha - \beta)/3$. Letting $G(\rho) := A\rho - \Phi(\rho)$ for $\rho \in \mathcal{V}_\beta$ we write problem (3.1) as follows

$$\begin{aligned} \partial_t \rho &= -A\rho + G(\rho), \; t \geq 0, \\ \rho(0) &= \rho_0, \end{aligned} \tag{4.2}$$

where $A := \partial\Phi(0) \in \mathcal{H}(h^{4+\beta}(\mathbb{S}^1), h^{1+\beta}(\mathbb{S}^1))$. We have $G \in C^\infty(\mathcal{V}_\alpha, D_A(\theta))$, $G(0) = 0$ as well as $\partial G(0) = 0$. These relations follow by reason of $D_A(\theta) = (h^{4+\beta}(\mathbb{S}^1), h^{1+\beta}(\mathbb{S}^1))_\theta = h^{1+\alpha}(\mathbb{S}^1)$ and $D_A(\theta + 1) = \{\rho \in h^{4+\beta}(\mathbb{S}^1) \, : \, A\rho \in D_A(\theta)\} = h^{4+\alpha}(\mathbb{S}^1)$. Notice that the part of A in $D_A(\theta)$ is denoted again by A. We refer also to [15] for details.

The embedding $h^{4+\beta}(\mathbb{S}^1) \hookrightarrow h^{1+\beta}(\mathbb{S}^1)$ is compact, and therefore $-A$ is an operator with a compact resolvent. We infer to Theorem III.8.29 in [11] to conclude that its spectrum consists only of eigenvalues having finite multiplicity. Thus, $\sigma(-A) = \{\lambda_k \, : \, k \geq 1\}$ and the multiplicity of λ_1 is equal to 3, respectively the multiplicity of λ_k is 2 for $k \geq 2$, as can be deduced from (3.3). We find ourself in a critical case of stability: the value $\lambda_1 = 0$ belongs to the spectrum of $-A$.

Let us denote by $P \in \mathcal{L}(h^{1+\beta}(\mathbb{S}^1))$ the spectral projection associated with the nonnegative spectral set $\{0\}$:

$$P = \frac{1}{2\pi i} \int_C R(z, -A) \, dz,$$

where C is the circle centred in 0 with radius ζ. Notice that the closed ball bounded by C contains non of the negative eigenvalues of the operator $-A$. Using the Fourier expansions of the functions $\rho \in h^{1+\beta}(\mathbb{S}^1)$ we see that P is a Fourier multiplier.

More precisely, given $\rho = \sum_{k\in\mathbb{Z}} \widehat{\rho}(k)x^k \in h^{1+\beta}(\mathbb{S}^1)$ we compute

$$\begin{aligned} P\left[\sum_{k\in\mathbb{Z}} \widehat{\rho}(k)x^k\right] &= \frac{1}{2\pi i}\int_C R(z,-A)\rho\, dz = \frac{1}{2\pi i}\int_C \sum_{k\in\mathbb{Z}} \frac{1}{z-\lambda_k}\widehat{\rho}(k)x^k\, dz \\ &= \frac{1}{2\pi i}\sum_{k\in\mathbb{Z}}\int_C \frac{1}{z-\lambda_k}\widehat{\rho}(k)x^k\, dz \\ &= \sum_{k\in\{-1,0,1\}}\left(\frac{1}{2\pi i}\int_C \frac{1}{z-\lambda_k}\, dz\right)\widehat{\rho}(k)x^k \\ &= \sum_{k\in\{-1,0,1\}} \widehat{\rho}(k)x^k. \end{aligned}$$

Note that $\rho \in h^{1+\beta}(\mathbb{S}^1)$ ensures the uniform convergence of the series $\sum_{k\in\mathbb{Z}} \frac{1}{z-\lambda_k}\widehat{\rho}(k)x^k$. Particularly, Theorem 3.4 implies $P \in \mathcal{L}(h^{k+\sigma}(\mathbb{S}^1))$ for all $k \in \mathbb{N}$ and $\sigma \in (0,1)$.

We set $X_1 = P(h^{1+\beta}(\mathbb{S}^1))$, $X_2 = (I-P)(h^{1+\beta}(\mathbb{S}^1))$, and

$$A_1 : X_1 \to X_1,\ A_1x = Ax$$

$$A_2 : X_2 \cap h^{4+\beta}(\mathbb{S}^1) \to X_2,\ A_2x = Ax.$$

Then $A_1 \in L(X_1)$ is the 0_{X_1} operator, $\sigma(-A_2) = \{\lambda_k : k \geq 2\}$ and $D_{A_2}(\theta) = D_A(\theta)\cap X_2$, $D_{A_2}(\theta+1) = D_A(\theta+1)\cap X_2$. Setting $\widetilde{h}^{k+\sigma}(\mathbb{S}^1) := \{\rho \in h^{k+\sigma}(\mathbb{S}^1) : \widehat{\rho}(m) = 0 \text{ for } |m| \leq 1\}$ for $k \in \mathbb{N}$ and $\sigma \in (0,1)$, we have

$$D_{A_2}(\theta) = \widetilde{h}^{1+\alpha}(\mathbb{S}^1) \qquad \text{and} \qquad D_{A_2}(\theta+1) = \widetilde{h}^{4+\alpha}(\mathbb{S}^1).$$

Moreover, A_2 is the part in $\widetilde{h}^{1+\alpha}(\mathbb{S}^1)$ of the operator $A_2 \in \mathcal{H}(\widetilde{h}^{4+\beta}(\mathbb{S}^1), \widetilde{h}^{1+\beta}(\mathbb{S}^1))$. Let us further notice that $Ph^{k+\sigma}(\mathbb{S}^1)$ is a three-dimensional space, thus

$$h^{k+\sigma}(\mathbb{S}^1) = Ph^{k+\sigma}(\mathbb{S}^1) \oplus \widetilde{h}^{k+\sigma}(\mathbb{S}^1),\ k \in \mathbb{N},\ \sigma \in (0,1)$$

is a topological direct sum.

Choose $r_0 > 0$ such that $\overline{B}_{h^{4+\alpha}(\mathbb{S}^1)}(0, r_0(1+K)) \subset \mathcal{O}$ and let $\psi : X_1 \to [0,1]$ be a smooth cutoff function satisfying

$$\psi(x) = 1 \quad \text{for} \quad ||x||_{C^{1+\beta}(\mathbb{S}^1)} \leq \frac{1}{2} \quad \text{and} \quad \psi(x) = 0 \quad \text{for} \quad ||x||_{C^{1+\beta}(\mathbb{S}^1)} \geq 1.$$

We have denoted by K the norm of $(X_1, \|\cdot\|_{C^{4+\alpha}(\mathbb{S}^1)}) \hookrightarrow (X_1, \|\cdot\|_{C^{1+\beta}(\mathbb{S}^1)})$. Given $r \leq r_0$, the mapping

$$G_r : X_1 \times B_{\widetilde{h}^{4+\alpha}(\mathbb{S}^1)}(0, r_0) \subset h^{4+\alpha}(\mathbb{S}^1) \to h^{1+\alpha}(\mathbb{S}^1),$$

$$G_r(\rho) := G\left(\psi\left(\frac{P\rho}{r}\right)P\rho + (I-P)\rho\right), \quad \rho \in X_1 \times B_{\widetilde{h}^{4+\alpha}(\mathbb{S}^1)}(0, r_0),$$

has the same regularity properties as G and $G(0) = 0$, $\partial G(0) = 0$. Further on, for $r \le r_0$ the abstract Cauchy problem:

$$\begin{aligned} \partial_t \rho &= -A\rho + G_r(\rho),\ t \ge 0 \\ \rho(0) &= \rho_0, \end{aligned} \tag{4.3}$$

is equivalent to the problem (4.2) for small solutions. More precisely, the solutions of (4.2) remaining in $B_{X_1}(0, r/2) \times B_{\widetilde{h}^{4+\alpha}(\mathbb{S}^1)}(0, r_0) \subset h^{4+\alpha}(\mathbb{S}^1)$ coincide with the solutions of (4.3). Given $\rho \in X_1 \times B_{\widetilde{h}^{4+\alpha}(\mathbb{S}^1)}(0, r_0)$, we assume that $-A+\partial G_r(\rho)$ is the part in $h^{1+\alpha}(\mathbb{S}^1)$ of a sectorial operator $B : h^{4+\beta}(\mathbb{S}^1) \subset h^{1+\beta}(\mathbb{S}^1) \to h^{1+\beta}(\mathbb{S}^1)$, such that $D_B(\theta) = h^{1+\alpha}(\mathbb{S}^1)$ and $D_B(\theta + 1) = h^{4+\alpha}(\mathbb{S}^1)$. This can be achieved by choosing r_0 small enough.

Under this assumptions we obtain for each initial data $\rho_0 \in X_1 \times B_{\widetilde{h}^{4+\alpha}(\mathbb{S}^1)}(0, r_0)$, applying again Theorem 8.4.1 in [15], existence and uniqueness of a maximal defined solution $\rho \in C([0, T(\rho_0)), X_1 \times B_{\widetilde{h}^{4+\alpha}(\mathbb{S}^1)}(0, r_0)) \cap C^1([0, T(\rho_0)), h^{1+\alpha}(\mathbb{S}^1))$.

Clearly, problem (4.3) is equivalent to the following coupled system

$$\begin{aligned} x' &= -A_1 x + f(x, y), \\ y' &= -A_2 y + g(x, y), \\ x(0) &= x_0, \quad y(0) = y_0, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} &f : X_1 \times B_{\widetilde{h}^{4+\alpha}(\mathbb{S}^1)}(0, r_0) \subset X_1 \times \widetilde{h}^{4+\alpha}(\mathbb{S}^1) \to X_1, \\ &f(x, y) = PG\left(\psi\left(\tfrac{x}{r}\right) x + y\right), \\ &g : X_1 \times B_{\widetilde{h}^{4+\alpha}(\mathbb{S}^1)}(0, r_0) \subset X_1 \times \widetilde{h}^{4+\alpha}(\mathbb{S}^1) \to \widetilde{h}^{1+\alpha}(\mathbb{S}^1), \\ &g(x, y) = (I - P)G\left(\psi\left(\tfrac{x}{r}\right) x + y\right). \end{aligned}$$

Being interested in the stability of the equilibria to problem (3.1) located near the trivial solution 0 it will be sufficient to consider the problem (4.4) for small r.

A pair (x, y) is called solution of (4.4) if there exists a constant $T > 0$ such that

$$\begin{aligned} &x \in C^1([0, T], Ph^{1+\alpha}(\mathbb{S}^1)), \\ &y \in C^1([0, T], \widetilde{h}^{1+\alpha}(\mathbb{S}^1)) \cap C([0, T], \widetilde{h}^{4+\alpha}(\mathbb{S}^1)), \end{aligned}$$

and if (x, y) satisfies the system (4.4) pointwise. In view of [15, Proposition 9.2.1] we can choose r_0 small enough to guarantee, for $r \le r_0$, the existence of a constant positive $C(r)$ with the property that solutions to (4.4) satisfying initially $\|\rho_0\|_{C^{4+\alpha}(\mathbb{S}^1)} \le C(r)$ exist globally.

We state now a theorem on the existence and smoothness of invariant manifolds for the system (4.4), result which can be found in [19, Theorem 4.1] and [15, Theorem 9.2.2] (see also [4] and [8]).

Theorem 4.1 (Existence of centre manifolds). *Given $k \in \mathbb{N}_{>0}$, there exists $r_k \in (0, r_0]$ and for each $r \in (0, r_k]$ there is a unique mapping*

$$\sigma \in BC^k(X_1, \widetilde{h}^{4+\alpha}(\mathbb{S}^1)),$$

satisfying

$$\sigma(0) = 0 \qquad \partial\sigma(0) = 0.$$

Moreover

$$\|\sigma(x) - \sigma(\overline{x})\|_{C^{4+\alpha}(\mathbb{S}^1)} \leq b\|x - \overline{x}\|_{C^{1+\beta}(\mathbb{S}^1)}$$

for a suitable constant b and

$$\|\sigma(x)\|_{C^{4+\alpha}(\mathbb{S}^1)} \leq r, \quad \forall x \in X_1.$$

Let $\mathcal{M} := \mathcal{M}(r, k) = \{(x, \sigma(x)) : x \in X_1\} \subset h^{4+\alpha}(\mathbb{S}^1)$. Then $\mathcal{M}$ is a globally invariant 3*-dimensional C^k-manifold for the problem* (4.4)*, i.e. given $(x_0, y_0) \in \mathcal{M}$, the solution (x, y) to* (4.4) *exists in the large and $(x(t), y(t)) \in \mathcal{M}$ for $t \geq 0$.*

Denote by $z(\,\cdot\,) = z(\,\cdot\,, x, \sigma)$ the global solution of the initial value problem for the reduced ordinary differential equation

$$z'(t) = f(z(t), \sigma(z(t))), \quad t \in \mathbb{R},$$
$$z(0) = x.$$

The function σ is the unique fixed point of the following equation

$$\sigma(x) = \int_{-\infty}^{0} e^{tA_2} g(z(t, x, \sigma), \sigma(z(t, x, \sigma)))\, dt, \tag{4.5}$$

and for $(x_0, y_0) \in \mathcal{M}$ we have that $(x(t), y(t)) = (z(t, x_0, \sigma), \sigma(z(t, x_0, \sigma)))$, $t \geq 0$ is the globally defined solution to (4.4)*.*

Additionally, if $\rho : \mathbb{R} \to h^{4+\alpha}(\mathbb{S}^1)$ is a globally defined solution of (3.1) *with*

$$\rho(t) \in W(r) := B_{X_1}\left(0, \frac{r}{2}\right) \times B_{\widetilde{h}^{4+\alpha}(\mathbb{S}^1)}(0, r),$$

i.e. $\|P\rho(t)\|_{C^{1+\beta}(\mathbb{S}^1)} < r/2$ and $\|(I - P)\rho(t)\|_{C^{4+\alpha}(\mathbb{S}^1)} < r$ for all $t \geq 0$, then $(I - P)\rho(t) = \sigma(P\rho(t))$ and $P\rho$ is the unique solution of the following initial value problem

$$z'(t) = f(z(t), \sigma(z(t))), \quad t \in \mathbb{R},$$
$$z(0) = P\rho_0.$$

Thus, $\mathcal{M}$ contains all small global solutions of (3.1)*. The tangent space to $\mathcal{M}$ in* 0 *is X_1, the eigenspace corresponding to the eigenvalue* 0

$$T_0(\mathcal{M}) = \mathrm{im}(\mathrm{id}_{X_1}, \partial\sigma(0)) = X_1 \times \{0\} \cong X_1.$$

Fix now $k \geq 2$. Given $r \in (0, r_k]$, we construct now a locally invariant C^k-manifold $\mathcal{M}^c_{\text{loc}}$ for the problem (3.1) containing just stationary solutions of (3.1). Let $V \subset \mathcal{U}$ be a small neighbourhood of 0 in $\mathbb{R}^3$ satisfying

$$\rho_{(c,R)} \in W(r), \ \forall (c, R) \in V.$$

By Theorem 4.1 we have then $\rho_{(c,R)} = (P\rho_{(c,R)}, \sigma(P\rho_{(c,R)}))$ for $(c, R) \in V$. The mapping $[\mathcal{U} \ni (c, R) \mapsto \rho(c, R) := \rho_{(c,R)} \in h^{4+\alpha}(\mathbb{S}^1)]$ is smooth and we compute

$$\partial\rho(0, 0)[c, R] = \frac{c}{2}x^{-1} + R + \frac{\overline{c}}{2}x = \langle c, x\rangle + R, \qquad [c, R] \in \mathbb{R}^3. \tag{4.6}$$

Given $(c, R) \in \mathcal{U}$, we can represent the periodic function $P\rho_{(c,R)}$ uniquely by its trigonometric series

$$P\rho_{(c,R)} = \widetilde{R} + \langle \widetilde{c}, x\rangle, \tag{4.7}$$

where $\widetilde{R} = \widehat{\rho}_{(c,R)}(0)$ and $\widetilde{c} = 2\widehat{\rho}_{(c,R)}(-1)$.

Using this relation, we state that the mapping $\mathcal{F} : \mathcal{U} \to \mathbb{R}^3$, defined by $\mathcal{F}(c, R) := (\widetilde{c}, \widetilde{R})$, where $(\widetilde{c}, \widetilde{R})$ are given by (4.7), is smooth, satisfies $F(0) = 0$, and additionally, by (4.7), $\partial\mathcal{F}(0) = \text{id}_{\mathbb{R}^3}$. If V is small enough, then $\mathcal{F} : V \to \mathcal{F}(V)$ is a smooth diffeomorphism. Given $(\widetilde{c}, \widetilde{R}) \in \mathcal{F}(V)$ we have:

$$P\rho\mathcal{F}^{-1}(\widetilde{c}, \widetilde{R}) = \widetilde{R} + \langle \widetilde{c}, x\rangle,$$

thus $P\rho\mathcal{F}^{-1}$ is the restriction to $\mathcal{F}(V)$ of the isomorphism $[\mathbb{R}^3 \ni (\widetilde{c}, \widetilde{R}) \mapsto \langle \widetilde{c}, x\rangle + \widetilde{R} \in X_1]$. We conclude that $P\rho(V)$ is an open neighbourhood of 0 in X_1. Define $\mathcal{M}^c_{\text{loc}}$ as the graph of the restriction of σ to the open set $P\rho(V)$. We have obtain in this way a local invariant manifold for the system (3.1). The example of A. Kelley (Example 13.7) in [21] shows that invariant manifolds are in general not unique. In the context of our problem we know additionally

$$\begin{aligned}\mathcal{M}^c_{\text{loc}} &= \{(x, \sigma(x)) \,:\, x \in P\rho(V)\} = \left\{(P\rho_{(c,R)}, \sigma(P\rho_{(c,R)})) \,:\, (c, R) \in V\right\} \\ &= \left\{\rho_{(c,R)} \,:\, (c, R) \in V\right\}.\end{aligned}$$

This means that the (a priori non-unique) invariant manifold $\mathcal{M}^{\text{loc}}_c$ consists in equilibria only, *i.e.* in circles, and is therefore unique.

The manifold $\mathcal{M}$ attracts the solutions of (4.4) for small initial data. This result is found in [15]. More precisely we have:

Theorem 4.2. *Given $\omega \in (0, -\lambda_2 = 6\zeta)$, there exist positive constants $M = M(\omega)$ and $\overline{r} = \overline{r}(\omega)$ such that for $r \leq \overline{r}$, $(x_0, y_0) \in X_1 \times \widetilde{h}^{4+\alpha}(\mathbb{S}^1)$, with $\|(x_0, y_0)\|_{C^{4+\alpha}(\mathbb{S}^1)} \leq C(r)$, the solution (x, y) to (4.4) exists in the large and satisfies*

$$\|y(t) - \sigma(x(t))\|_{C^{4+\alpha}(\mathbb{S}^1)} \leq Me^{-\omega t}\|y_0 - \sigma(x_0)\|_{C^{4+\alpha}(\mathbb{S}^1)} \quad \textit{for} \quad t \in [0, \infty).$$

The following result on asymptotic stability ensures that for small initial data ρ_0 the solution to (3.1) exists in the large and there exists a steady state belonging to the local centre manifold $\mathcal{M}^c_{\rm loc}$, uniquely determined by the initial data, which attracts the solution exponentially. The proof of this result follows similarly as the proof in [8, Theorem 6.5], which is an adoption of the proof in [19, Proposition 9.2.4].

Theorem 4.3. *Let $\omega \in (0, 6\gamma/\overline{\mu}(0))$ and $r \le \overline{r}$ be given. There exist positive constants $K = K(\omega)$ and a neighbourhood $\mathcal{W}(r)$ of 0 in $h^{4+\alpha}(\mathbb{S}^1)$ such that, for $\rho_0 \in \mathcal{W}(r)$ the solution to (3.1) exists in the large and there exists $z_0 \in P\rho(V)$ with*

$$\|\rho(t)-(z_0,\sigma(z_0))\|_{C^{4+\alpha}(\mathbb{S}^1)} \le Ke^{-\omega t}\|(I-P)\rho_0-\sigma(P\rho_0))\|_{C^{4+\alpha}(\mathbb{S}^1)} \quad \text{for} \quad t\in[0,\infty).$$

Notice that for $z_0 \in P\rho(V)$, the mapping $(z_0, \sigma(z_0))$ belongs to the local centre manifold $\mathcal{M}^c_{\rm loc}$ and is uniquely determined by the centre of mass and volume of the initial data ρ_0. Notice also the regularizing effect of the surface tension and viscosity. Fluids with large surface tension coefficient and small viscosity converge more rapidly to circles.

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