NOBUO AOKI

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ERGODIC AUTOMORPHISMS OF COMPACT METRIC GROUPS 
ARE ISOMORPHIC TO BERNOULLI SHIFTS 

Nobuo Aoki

I wish to discuss its title. Let X be a compact, metric group and μ be its normalized Haar measure. Then (X,μ) is a Lebesgue space. Let σ be an automorphism of X, then σ is an invertible measure preserving transformation from X onto X. Our problem is concerned with measure theoretic properties of σ.

Today we will outline a proof of the following

Theorem.
An ergodic automorphism of a compact, metric, abelian group is a Bernoulli shift.

In order to outline the Theorem, we prepare the following Lemma 1. Let X be a compact, metric, abelian group and σ be an ergodic automorphism of X. Then there exists a sequence of subgroup \( X \supset X_1 \supset \ldots \cap_{n=1}^{\infty} X_n = \{e\} \), invariant with respect to σ, such that there are a σ-invariant totally disconnected subgroup \( X_D^{(n)} \), a σ-invariant infinite dimensional connected subgroup \( X_B^{(n)} \) and a σ-invariant finite dimensional connected subgroup \( X_A^{(n)} \), so that \( \sigma|_{X_D^{(n)}} \) is isomorphic to a Bernoulli shift, \( \sigma|_{X_B^{(n)}} \) is a Bernoulli automorphism (see [28]) and \( \sigma|_{X_A^{(n)}} \) is ergodic, and \( (X/X_n^{(n)}, \sigma) \) is an algebraic factor of \( (X_A^{(n)} \times X_B^{(n)} \times X_D^{(n)}, \sigma \times \sigma \times \sigma) \).

To show the ergodic system \( (X_A^{(n)}, \sigma), n \geq 1 \), have the
Bernoulli properties, let $\overline{T}^r$ be a compact, connected, abelian group with the character group $Q^r$, being the $r$-vector of rational numbers and $\sigma$ be an automorphism of $\overline{T}^r$, then it is sufficient to prove that if $(\overline{T}^r, \sigma)$ is ergodic then $(\overline{T}^r, \sigma)$ is Bernoullian.

We write $Q^r = \{g_1, g_2, \ldots\}$. Let $Q_n$ denote a subgroup generated by $\{g_k \sigma^j : j \in \mathbb{Z}, 1 \leq k \leq n\}$ for $n \geq 1$. Since rank $(Q^r) = r$, there exists an integer $m > 0$ such that rank $(Q_n) = \text{rank} (Q^r)$ for $n \geq m$.

We denote by $\overline{T}(Q_n)$ the annihilator of $Q_n$ in $\overline{T}^r$ for $n \geq 1$.

Then we have

**Lemma 2.** $(\overline{T}(Q_n), \sigma)$ is Bernoullian for $n \geq 1$.

The proof is done by ideas of Katznelson together with the following facts.

We may consider $\sigma$ as operating on $R^r$. So we have the decomposition

$$R^r = V_{-p} \oplus \cdots \oplus V_{-q} \oplus \cdots \oplus V_g$$

such that $V_j, -p < j < q$, are $\sigma$-invariant, all the eigenvalues of $\sigma|V_j$ have modulus 1, and all the eigenvalues of $\sigma|V_j, j \neq 0$, have the same modulus $\rho_j$ where

$$\rho_{-p} < \cdots < 1 = \rho_0 < \cdots < \rho_q$$

**Lemma 3.** If $\{0\} \neq Q^r \cap \overline{V}$, where $\overline{V} = V_{-p} \oplus \cdots \oplus V_g$, then $R^r$ is decomposed by subspaces $R^r_{1}(\sigma R^r = R^r)$ and $R^r_{r-r^r_1}$ with the following properties: there exists an automorphism $\sigma'$ of $R^r_{r-r^r_1}$ and a homomorphism $\lambda$ from $R^r_{r-r^r_1}$ into $R^r_{1}$ such that

$$\sigma(\chi_1, \chi_2) = (\sigma' \chi_1, \lambda(\chi_1) + \sigma \chi_2), (\chi_1, \chi_2) \in R^r_{1} \oplus R^r_{r-r^r_1},$$

and $\sigma \mid R^r_{1}$ gives $R^r_{1} = V_{-p} \oplus \cdots \oplus V_g$. 
where \( V_j \subseteq V_i \) for \( j = -p, \ldots, 0 \), and \( \sigma' \) gives
\[
R_{r-1} = V_{-p} \oplus \cdots \oplus V_0 \oplus \cdots \oplus V_q
\]
where \( V_j \subseteq V_i \) for \( j = -p, \ldots, q \), such that
\[
(V_{-p} \oplus \cdots \oplus V_0) \cap Q^r = \emptyset.
\]

Lemma 4. If \( \sigma \) acts ergodically on \( Q^r \), then so is
\[
\sigma|_{Q^r} \text{ and } Q^r \cap V_1 = \emptyset.
\]
If \( Q^r \cap (V_{-p} \oplus \cdots \oplus V_q) \neq \emptyset \), then we again decompose \( R_{r-1} \) as in Lemma 3. However, since the dimension of \( R^r \) is finite, such the argument ends in finite time.

For \( m \geq 0 \), we denote by \( V^{(m)} \) the set of all \( r \)-vectors such that each component of a vector consists of an element of \( \{0, \pm 1, \ldots, \pm m\} \). We define
\[
V(1,M) = \{ \sum_{m=1}^M \sigma^{-m} \alpha_m : \alpha_m \in V^{(m)}, 1 \leq m \leq M \},
\]
\[
V(K,K+M) = \{ \sum_{m=K}^{K+M} \sigma^{-m} \alpha_m : \alpha_m \in V^{(m)}, K \leq m \leq K+M \}
\]
for \( K > 0 \) and \( M > 0 \). Then \( V(1,M) \) and \( V(K,K+M) \) are subsets of \( R^r \).

Lemma 5. Assume that \( R^r \) is decomposed as follows. If
\[
R^r = V_{-p} \oplus \cdots \oplus V_q \quad \text{then } V \cap Q^r = \emptyset,
\]
and if \( R^r = V_{-p} \oplus \cdots \oplus V_{-1} \oplus \cdots \oplus V_q \), then \( (V_{-p} \oplus \cdots \oplus V_q) \cap Q^r = \emptyset \) and \( (V_0 \oplus \cdots \oplus V_q) \cap Q^r = \emptyset \). Then there exists \( K_1 > 0 \) such that
\[
V(1,M) \cap V(K,K+M) = \emptyset
\]
for \( K \geq K_1 \) and an odd integer \( M > 0 \).

In the proofs of Lemmas 3, 4 and 5, we use no results of number theory.

We now conclude that \( (X,\sigma) \) is Bernoullian. Using theorem and structures of skew product transformations, I can prove the result for the case of non-abelian. But, proofs of these results will appear elsewhere.
REFERENCES


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Department of Mathematics
Tokyo Metropolitan University
Tokyo, Japan