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ZBIGNIEW S. KOWALSKI

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ERGODIC PROPERTIES OF PIECEWISE
 MONOTONIC TRANSFORMATIONS

Zbigniew S. Kowalski

Let $(L_1, \|\cdot\|)$ be the space of all Lebesgue integrable functions defined on the interval $[0, 1]$. Lebesgue measure on $[0, 1]$ will be denoted by m .

For a real number $M \geq 1 + \frac{1}{M}$ and an integer $q \geq 2$ we define a set $F(M, q)$ of transformations $\tau : [0, 1] \rightarrow [0, 1]$ for which there exists a partition $0 = a_0 < a_1 < \dots < a_{q_\tau} = 1$ of the unit interval such that the restriction τ_i of τ to the open interval (a_{i-1}, a_i) can be extended to a C^1 + Lipschitz function $\bar{\tau}_i$ on $[a_{i-1}, a_i]$, $1 + \frac{1}{M} \leq |\bar{\tau}_i'| \leq M$, $L_{\bar{\tau}_i} \leq M$, $i = 1, \dots, q_\tau$, $q_\tau \leq q$.

Here $L_{\bar{\tau}_i}$ denotes the Lipschitz constant for $\bar{\tau}_i$ and the number q_τ is chosen minimal one.

Denote by Q_τ the operator acting in L_1 such that for $f \in L_1$

$$Q_\tau f = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f$$

Here P_τ denotes the Frobenius - Perron operator for $\tau \in F(M, q)$. The function $f^* = Q_\tau f$ has the following properties :

i) $f^* \geq 0$ for $f \geq 0$

- ii) $\int_0^1 f^* dm = \int_0^1 f dm$
- iii) $P_\tau f^* = f^*$ and consequently the measure $dm^* = f^* dm$ is τ -invariant
- iv) the function f^* is of bounded variation
- v) $f^*(x^-) = f^*(x)$ for $x \in [0, 1]$ and $f^*(0) = f^*(0^+)$

Let $F = \bigcup_{M \geq 1 + \frac{1}{M}} \bigcup_{q \geq 2} F(M, q)$

The proof of existence of the operator Q_τ with properties i - iv for τ piecewise C^2 is due to Lasota and Yorke [4]. By a simple modification we generalize the above result to the class F.

Properties of $Q_\tau f$ for $f \in L_1$, especially iv, allow us to study ergodic properties of τ and τ -invariant measures.

Theorem 1. [1,2]

Let τ be an element of F. Then the interval $[0, 1]$ can be decomposed into a finite number of open sets which are τ -invariant and τ -ergodic. The number p_τ of these is not greater than

$$\left[\frac{q_\tau - 1}{\inf |\dot{\tau}| - 1} \right] \quad \text{if} \quad \inf |\dot{\tau}| \quad \text{is an integer}$$

and $p_\tau \leq \left[\frac{q_\tau - 1}{[\inf |\dot{\tau}|]} \right]$ otherwise.

Here $[x]$ denotes the integer part of x .

The related result was announced by J.A. Yorke and T Li on Dynamical Systems Conference - Providence 1974.

We put

$$\beta(M, q) = \{ \tau : \tau \in F(M, q) \quad , \quad q_\tau = q \quad , \quad \beta_\tau \cap \tau^{-k}(\bar{\beta}_\tau) = \emptyset \quad \text{for every } k, \quad 0 < k \leq k(\tau) - 1 \}$$

where $\beta_\tau = \{a_0, a_1, \dots, a_{q_\tau}\}$, $\tau^{-1}(x) = \bigcup_{i=1}^{q_\tau} \tau_i^{-1}(x)$,

$\bar{\beta}_\tau = \beta_\tau - \{0, 1\}$ and $k(\tau)$ is the smallest integer such that $\inf |\bar{\beta}_\tau|^{k(\tau)} > 2$.

The metric $\rho(\tau_1, \tau_2) = \int_0^1 |\tau_1 - \tau_2| dm$ for $\tau_1, \tau_2 \in F(M, q)$ gives the compact topology on $F(M, q)$. By uniqueness of invariant measure for τ -ergodic and by continuity of the operator P_τ as a function of parameter τ we get ;

Theorem 2. [1]

If $\tau_n \rightarrow \tau$, $\tau_n \in F(M, q)$, $\tau \in B(M, q)$ and τ is ergodic with respect to the measure m then for any $f \in L_1$

$$Q_{\tau_n} f \rightarrow Q_\tau f \text{ in } L_1\text{-norm}.$$

Theorem 3. [1]

Let $\tau_n \rightarrow \tau$ and $\tau_n \in F(M, q)$, $\tau \in B(M, q)$. Then $Q_{\tau_n} f \rightarrow Q_\tau f$ for every $f \in L_1$ iff $\lim_{n \rightarrow \infty} p_{\tau_n} = p_\tau$.

Let $H(M, q)$ be the subset of $F(M, q)$ such that for every sequence $\tau_n \in F(M, q)$ and $\tau \in H(M, q)$ if $\tau_n \rightarrow \tau$ in ρ then $Q_{\tau_n} f \rightarrow Q_\tau f$ in L_1 for every $f \in L_1$. By theorem 2, if τ is ergodic and $\tau \in B(M, q)$ then $\tau \in H(M, q)$.

There exists an ergodic transformation τ such that $\tau \in H(M, q) - B(M, q)$.

For example $\tau(x) = \frac{\sqrt{5} + 1}{2} x \text{ mod } (1)$ belongs to $H(M, 2) - B(M, 2)$.

The above is a consequence of the following theorem ;

Theorem 4. [3]

Transformation $\beta(x) = \beta x \bmod (1)$ for $\beta > 1$ belongs to $H(M, q_\beta)$.

Let $D_\tau = \{x : Q_\tau^{-1}(x) > 0\}$.

Theorem 5. [3]

For every $\tau \in F(M, q)$ there exist disjoint sets C_i , $i = 1, \dots, p_\tau$, such that C_i is finite sum of intervals, $\tau(C_i) \approx C_i$, τ is ergodic on C_i and

$$\bigsqcup_{i=1}^{p_\tau} C_i = D_\tau.$$

Theorem 5 gives a description of the support for τ -invariant measure. The related result in piecewise C^2 -case has been obtained, but by a mistaken way, by A.A. Kosjakin and E.A. Sandler (Izv. Vyss. Uceb. Zav. N.3 118 1972). By construction of C_i in proof of theorem 5 we get ;

Theorem 6. [1,3]

If $\tau \in H(M, q)$ then $q_\tau = q$ in nonergodic case and if $M > 2$ then $q_\tau \geq q - 2$ for τ -ergodic.

This gives examples of ergodic elements of the set $F(M, q) - H(M, q)$ for $q > 3$.

Let λ be a real such that $1 < \lambda < \sqrt[3]{\frac{7}{4}}$ and let δ be any small positive number.

We choose a transformation $\tau \in F(M, 2)$ such that

$$\begin{aligned} \dot{\tau}(x) &= -\lambda \text{ for } x \in (0, \delta) \sqcup (\frac{3}{4} - \delta, \frac{3}{4}) \quad \dot{\tau}(x) = \lambda \text{ for } \\ x \in (1 - \delta, 1) \sqcup (\frac{3}{4}, \frac{3}{4} + \delta), \quad \tau(0) &= 1, \quad \tau(\frac{3}{4}) = 0 \text{ and } \tau(1) = \frac{3}{4}. \end{aligned}$$

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In [3] it has been shown that $\tau \in F(M,2) - H(M,2)$. At last we get the following fixpoint Theorem ;

Theorem 7. [3]

For $\tau \in F$ there exists a positive integer k such that τ^k has a fixpoint.

References

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Zbigniew S. Kowalski
Wrocław University
Wrocław
Poland