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THE WEIL GROUP AS AUTOMORPHISMS
OF THE LUBIN-TATE GROUP

Jack Morava

Introduction:

Let L be a finite extension of \mathbb{Q}_p , with maximal abelian extension L_{ab} ; then the canonical monomorphism \underline{a} of [8, XIII§3] maps the multiplicative group of L onto an open dense subgroup $W(L_{ab}/L)$ of the Galois group of L_{ab} over L . These modified Galois [or W -] groups can be defined more generally, and behave very much like Galois groups [8, appendix II], but for some purposes they are more convenient.

For example, there is a representation of $W(L_{ab}/L)$ on the \mathbb{Q}_p -vector space L , defined by the obvious multiplication map $L^X \times L \rightarrow L$.

The trace of this representation defines a p -adic character of $W(L_{ab}/L)$ and therefore [via the natural homomorphism from $W(\overline{\mathbb{Q}_p}/L)$ to $W(L_{ab}/L)$] a p -adic character of $W(\overline{\mathbb{Q}_p}/L)$. In this note we construct an extension of this character to $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ when L is normal over \mathbb{Q}_p .

Our construction uses a theorem of Šafarevič: if d is the degree of L over \mathbb{Q}_p , and D is a division algebra with center \mathbb{Q}_p and invariant $d^{-1} \in \mathbb{Q}$, $E = \text{Br}(\mathbb{Q}_p)$, then L can be embedded as a commutative subfield of D ; let $N(L)$ be the normaliser of the multiplicative group L^\times of L in D^\times . The canonical morphism \underline{a} then extends [8, appendix III] to a canonical isomorphism $\underline{w} : N(L) \xrightarrow{\sim} W(L_{\text{ab}}/\mathbb{Q}_p)$, and the composition of \underline{w}^{-1} with the reduced trace from D to \mathbb{Q}_p defines a character of $W(L_{\text{ab}}/\mathbb{Q}_p)$ and therefore of $W(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

In §1 we identify $N(L)$ with a group of "extended automorphisms" of the Lubin-Tate group of L ; this action defines a cocycle (and thus a representation) \mathfrak{w} of $N(L)$, whose trace is the character described above.

The present work was motivated by the construction of a (topological) spectrum which admits $N(L)$ as a group of automorphisms, such that the representation defined on its $2n^{\text{th}}$ homotopy group is the n^{th} tensor power of \mathfrak{w} [6]. However, the result of §1 suggests the hope of a constructive proof of the Weil-Šafarevič theorem [which might shed some light on the interpretation of $W(L_{\text{ab}}/\mathbb{Q}_p)$ as a group of automorphisms [9]] and could therefore be of wider interest.

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§1, proof of the main result

1.1. A continuous homomorphism $\phi : A[[T]] \rightarrow A[[T]]$ of commutative

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formal power series rings will be called an extended endomorphism, if

- i) $\phi(T)$ lies in the ideal generated by T , and
- ii) the image of the composition $A \hookrightarrow A[[T]] \xrightarrow{\phi} A[[T]]$ lies in A .

Consequently $\phi(\sum_{i \geq 0} a_i T^i) = \sum_{i \geq 0} \phi(a_i) \phi(T)^i$.

Note that the composition of two extended endomorphisms is another, and that the tensor product $\phi \otimes_A \phi$ maps $A[[T \otimes 1, 1 \otimes T]]$ to itself by $(\phi \otimes_A \phi)(\sum_{i,j \geq 0} a_{ij} T^i \otimes T^j) = \sum_{i,j \geq 0} \phi(a_{ij}) \phi(T)^i \otimes_A \phi(T)^j$.

If $F(X,Y) \in A[[X,Y]]$ is a [one-parameter, commutative] formal group law over A , then the extended endomorphism ϕ of $A[[T]]$ will be called an extended endomorphism of F provided that the diagram

$$\begin{array}{ccc}
 A[[T]] & \xrightarrow{\Delta_F} & A[[T \otimes 1, 1 \otimes T]] \\
 \left\{ \begin{array}{c} \phi \\ \Delta_F \end{array} \right. & & \left. \begin{array}{c} \phi \otimes_A \phi \\ \Delta_F \end{array} \right. \\
 A[[T]] & \xrightarrow{\Delta_F} & A[[T \otimes 1, 1 \otimes T]]
 \end{array}$$

[defined by $\Delta_F(T) = F(T \otimes 1, 1 \otimes T)$] is commutative.

If $\text{Aut}^*(F)$ denotes the group of extended automorphisms of F [under composition] then it follows from i) and ii) that there is an exact sequence

$$1 \longrightarrow \text{Aut}_A(F) \longrightarrow \text{Aut}_A^*(F) \longrightarrow \text{Aut}_{(\text{rings})}(A)$$

with the terminal group consisting of the continuous ring-automorphisms of A ; the usual automorphisms of F over A [4, I§2] define the group $\text{Aut}_A(F)$.

1.2. We write $\underline{\mathcal{O}}_L$ for the valuation ring of L , and $\hat{\underline{\mathcal{O}}}_L$ for the valuation ring of the completion \hat{L} of a maximal unramified extension L_{nr} of L ; if \mathcal{K} denotes the residue field of $\underline{\mathcal{O}}_L$, and $\bar{\mathcal{K}}$ is the union of the finite fields, then $\hat{\underline{\mathcal{O}}}_L \cong \underline{\mathcal{O}}_L \otimes_{W(\mathcal{K})} W(\bar{\mathcal{K}})$.

If $\pi \in \underline{O}_L$ is a uniformising element, and q is the cardinality of \bar{Y} , then the series

$$\log_{\pi}(\mathbb{T}) = \sum_{i \geq 0} \pi^{-i} q^i$$

defines a formal group law $F_{\pi}(X, Y) = \log_{\pi}^{-1}(\log_{\pi}(X) + \log_{\pi}(Y))$ for which the map $\underline{O}_L^{\times} \ni a \mapsto [a]_{\pi}(\mathbb{T}) = \log_{\pi}^{-1}(a \cdot \log_{\pi}(\mathbb{T})) \in \text{Aut}_{\underline{O}_L}^{\wedge}(F_{\pi})$ is a bijection [1]. By "the" Lubin-Tate group of L , we mean the class of formal group laws over $\hat{\underline{O}}_L$ isomorphic to F_{π} for some (and hence any) choice of π . [If π_0, π_1 are two choices of uniformising element, then [5, lemma 2] there exists an invertible series

$$\phi_0^1(\mathbb{T}) \in \hat{\underline{O}}_L[[\mathbb{T}]] \text{ such that}$$

- i) ϕ_0^1 is an isomorphism of F_{π_0} with F_{π_1} , and
- ii) if σ is the automorphism of \hat{L} defined by the Frobenius operation $x \mapsto x^q$ on the residue field, then

$$\sigma(\phi_0^1(\mathbb{T})) = \phi_0^1([\pi_0^{-1} \pi_1^{-1} \pi_0]_{\pi_0}(\mathbb{T})).$$

We denote the formal group law over \bar{Y} defined by reducing F_{π} modulo the maximal ideal \hat{m}_L of $\hat{\underline{O}}_L$ by \bar{F}_{π} ; its height equals the degree of L over \mathbb{Q}_p [1, lemma 9].

1.3. Now the ring of endomorphisms of a group law of height d over an algebraically closed field of characteristic p is the valuation ring \underline{O}_D of a division algebra D with center \mathbb{Q}_p and invariant $d^{-1} \in \mathbb{Q}/\mathbb{Z} = \text{Br}(\mathbb{Q}_p)$ [4, VI §7.42], and the normalised ordinal valuation of an element of \underline{O}_D is its height as a power series. It follows that the sequence of 1.1 can be continued to the right as

$$1 \longrightarrow \underline{O}_D^{\times} \longrightarrow \text{Aut}_{\bar{Y}}^{\times}(\bar{F}_{\pi}) \longrightarrow \Gamma(\bar{Y}/\mathbb{F}_p) \cong \hat{\mathbb{Z}} \longrightarrow 1 :$$

to construct a lifting of the Frobenius endomorphism $\sigma_0^D(x) = x^p$ of \bar{Y} ,

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let $\theta \in \underline{\mathcal{O}}_D$ be an endomorphism of height 1 [so $\theta(T) = \theta_0(T^p)$ with θ_0 an invertible series]; then $\theta = \sigma_0' \circ \theta_0$ has the desired property. This shows moreover that $\text{Aut}_{\underline{\mathcal{O}}_D}^*(\overline{F}_\pi)$ is isomorphic to the profinite completion of the multiplicative group D^\times of D under the correspondence which sends the endomorphism ϕ [which can be written as $\phi(T) = \phi_0(T^{p^r})$ with ϕ_0 invertible and $r \geq 0$] to the extended automorphism $\sigma_0^r \circ \phi_0$. It suffices to see that the conjugation of a series $a \in \underline{\mathcal{O}}_D^\times$ by θ in $\text{Aut}_{\underline{\mathcal{O}}_D}^*(\overline{F}_\pi)$ agrees with its conjugation by θ in \mathbb{L} , or that $\text{Pot}_0^{\sigma_0'} a \sigma_0^{-1} = a \cdot P$, where $P(T) = T^p$; this is an elementary exercise in the composition of power series.

1.4. It follows similarly that if L is a normal extension of \mathbb{Q}_p , then $\text{Aut}_{\underline{\mathcal{O}}_L}^*(F_\pi)$ is a central topological extension of the Galois group $G(L_{nr}/\mathbb{Q}_p)$ by the multiplicative group $\underline{\mathcal{O}}_L^\times$. To see that the final homomorphism of the sequence in 1.1 is onto, note that if $\pi_0 = \pi$ is a uniformising element and $g \in G(L_{nr}/\mathbb{Q}_p)$ then $\pi_1 = g(\pi)$ is another and $\sum_{i \geq 0} a_i T^i \mapsto \sum_{i \geq 0} g(a_i) (\phi_0^1(T))^i$ defines a (noncanonical!) lift of g to an extended automorphism. Since any automorphism of a formal group law over an integral domain of characteristic 0 is determined by its leading coefficient, the group $G(L_{nr}/\mathbb{Q}_p)$ acts on the subgroup $\underline{\mathcal{O}}_L^\times$ via the canonical homomorphism to $G(L/\mathbb{Q}_p)$.

1.5. Now an extended automorphism of $\hat{\underline{\mathcal{O}}_L}[[T]]$ maps the ideal $\hat{\underline{\mathcal{O}}_L}[[T]]$ to itself, so an extended automorphism ϕ of F_π defines an extended automorphism of \overline{F}_π , which we will denote by

$$\rho : \text{Aut}_{\hat{\underline{\mathcal{O}}_L}}^*(F_\pi) \rightarrow \text{Aut}_{\hat{\underline{\mathcal{O}}_L}}^*(\overline{F}_\pi).$$

Since the reduction of a usual automorphism of F_π is a usual automorphism of \overline{F}_π , we have a commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \underline{\mathcal{O}}_L^{\times} & \longrightarrow & \text{Aut}_{\underline{\mathcal{O}}_L}^*(F_{\pi}) & \longrightarrow & G(L_{nr}/Q_p) \longrightarrow 1 \\
 & & \downarrow \rho_0 & & \downarrow \rho & & \downarrow \\
 1 & \longrightarrow & \underline{\mathcal{O}}_D^{\times} & \longrightarrow & (D^{\times})^{\wedge} & \longrightarrow & G(\bar{L}/F_p) \cong \hat{Z} \longrightarrow 1.
 \end{array}$$

Now the final vertical arrow fits in an exact sequence

$$1 \longrightarrow I(L/Q_p) \longrightarrow G(L_{nr}/Q_p) \longrightarrow G(\bar{L}/F_p) \cong \hat{Z} \longrightarrow 1$$

which defines the inertia group of L over Q_p , and the homomorphism ρ_0 is injective since \bar{F}_{π} is of finite height. It follows that ρ_0 is injective, for $I(L/Q_p)$ acts effectively on $\underline{\mathcal{O}}_L^{\times}$.

It will simplify matters to pull our commutative diagram back along the dense embedding $Z \longrightarrow \hat{Z}$: the effect is to replace $(D^{\times})^{\wedge}$ with D^{\times} , $G(L_{nr}/Q_p)$ with the open dense subgroup $W(L_{nr}/Q_p)$, and $\text{Aut}_{\underline{\mathcal{O}}_L}^*(F_{\pi})$ with an open dense subgroup which we will denote Aut^0 ; the original diagram can be recovered by profinite completion.

1.6. It remains to identify the image of ρ . We observe first that because \bar{F}_{π} has coefficients in \mathcal{K} , the extended automorphism $\rho([\pi]_{\pi}) = \sigma_0^d = \sigma$ commutes with elements of $\rho_0(\underline{\mathcal{O}}_L^{\times})$ in D^{\times} . It follows that $\rho_0(\underline{\mathcal{O}}_L^{\times})$ and σ generate a (normal) subgroup of Aut^0 isomorphic to L^{\times} , and that the image of ρ is therefore contained in the normaliser $N(L)$ of L^{\times} in D^{\times} . But now the Weyl group of L^{\times} in D^{\times} is $G(L/Q_p)$ if L is normal [8, appendix III§7] so we have a commutative diagram

$$\begin{array}{ccccccc}
 & & 1 & & & & \\
 & & \downarrow & & & & \\
 & & \text{Aut}^0 & \longrightarrow & W(L_{nr}/Q_p) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & L^{\times} & \longrightarrow & N(L) & \longrightarrow & G(L/Q_p) \longrightarrow 1 \\
 & & & & & & \downarrow \\
 & & & & & & 1
 \end{array}$$

with exact rows and columns. If $x \in N(L)$ then there is some y in Aut^0 such that $z = y^{-1}x$ lies in L^X , so $x = yz$ lies in Aut^0 .

This completes the proof of

1.7. proposition: The morphism ρ maps an open dense subgroup of $\text{Aut}_{\hat{\mathcal{O}}_L}^*(F_\pi)$ onto the normaliser $N(L)$ of L^X in D^X .

§2. some corollaries

2.1. If $\delta \in N(L)$, then we write $\bar{\rho}^{-1}(\delta)(T) = \omega(\delta)T + \text{higher order terms}$ for the action of the extended automorphism $\bar{\rho}^{-1}(\delta)$ on the formal parameter T ; here $\omega(\delta)$ is a unit of $\hat{\mathcal{O}}_L^X$. The composition $N(L) \longrightarrow W(L_{ab}/Q_p) \longrightarrow W(L_{nr}/Q_p)$ defines an action of $N(L)$ on $\hat{\mathcal{O}}_L$ which we will denote by juxtaposition. With this notation, we have

$$\omega(\delta_0 \delta_1) = \delta_1(\omega(\delta_0)) \cdot \omega(\delta_1) ;$$

in other words, ω is a crossed antihomomorphism from $N(L)$ to $\hat{\mathcal{O}}_L^X$.

Note that if $\delta \in \mathcal{O}_L^X$, then $\omega(\delta) = \delta^{-1}$ [7, III§A4].

An extended automorphism of F_π defines an extended automorphism of $F_\pi \otimes_Z Q$, and it follows that

$$\bar{\rho}^{-1}(\delta)(T) = \log_{\delta}^{-1}(\omega(\delta) \cdot \log_{\pi}(T)) ;$$

consequently the crossed antihomomorphism ω specifies the action of $N(L)$ on $\hat{\mathcal{O}}_L[[T]]$.

2.2. The 1-cocycle $\delta \mapsto \omega(\delta^{-1})$ of $N(L)$ with values in the right $N(L)$ module $(\hat{\mathcal{O}}_L^X)^{\text{op}}$ [defined by $x^{\text{op}}\delta = (\delta^{-1}x)^{\text{op}}$] defines a class in a continuous cochain cohomology group isomorphic to $H_c^1(W(L_{ab}/Q_p); \hat{\mathcal{O}}_L^X)$.

The Hochschild-Serre spectral sequence of the topological extension

$$E : 1 \longrightarrow W(L_{ab}/L_{nr}) \longrightarrow W(L_{ab}/Q_p) \longrightarrow W(L_{nr}/Q_p) \longrightarrow 1$$

yields an exact sequence

$$\begin{aligned} \dots \rightarrow H_c^1(W(L_{ab}/Q_p); \hat{\underline{O}}_L^X) \rightarrow H_c^0(W(L_{nr}/Q_p); H_c^1(\underline{O}_L^X; \underline{O}_L^X) \cong G(L/Q_p)) - \\ \text{invariants of } \text{Hom}_c(\underline{O}_L^X, \underline{O}_L^X) \xrightarrow{d_2} H_c^2(W(L_{nr}/Q_p); \hat{\underline{O}}_L^X) \cong H_c^2(G(L/Q_p); \hat{\underline{O}}_L^X) \end{aligned}$$

of terms of low degree. The existence of the cocycle ω implies $d_2 = 0$; since $d_2(x) = -x \cup [E]$ [3, theorem 4] it follows that the inclusion $\underline{O}_L^X \rightarrow \hat{\underline{O}}_L^X$ induces the zero map from $H_c^2(W(L_{nr}/Q_p); \underline{O}_L^X)$ to $H_c^2(W(L_{nr}/Q_p); \hat{\underline{O}}_L^X)$. A direct proof of this might suggest a construction for ω .

2.3. The isomorphism $G(L_{ab}/Q_p)$ with $\text{Aut}_{\hat{\underline{O}}_L}^*(F_\pi)$ defined in §1.7 respects an implicit proalgebraic group structure, which may be made explicit by observing that $G(L_{ab}/Q_p)$ is isomorphic to the semidirect product $I(L_{ab}/Q_p) \cdot G(\bar{Y}/F_p)$, in which $I(L_{ab}/Q_p)$ is the inertia group of L_{ab} over Q_p . In particular, $I(L_{ab}/Q_p)$ admits a continuous action of $G(\bar{Y}/F_p)$, and may therefore be regarded as a proetale groupscheme over F_p [2, II§5]. On the other hand \underline{O}_D^X is represented by a group of power series with coefficients in \bar{Y} , and has an obvious structure as proetale groupscheme defined [a priori] over \bar{Y} , in which the generator of $G(\bar{Y}/\bar{Y})$ acts on \underline{O}_D^X by π -conjugation in D^X . The maximal compact subgroup $N^0(L)$ of $N(L)$ inherits this structure.

However, if the uniformising element π of \underline{O}_L is chosen to satisfy an Eisenstein equation with coefficients in $\hat{\underline{Z}}_p$, then \bar{F}_π has coefficients in F_p , and $\theta(X) = X^p$ defines an endomorphism of \bar{F}_π which maps to an f^{th} root of π in $\text{Aut}_{\hat{\underline{Z}}_p}^*(\bar{F}_\pi)$, where $q = p^f$. It follows that $N^0(L)$ is in fact a proetale groupscheme defined over F_p , and is isomorphic as such to $I(L_{ab}/Q_p)$. Consequently the group of F_p -valued points of $I(L_{ab}/Q_p)$ can be identified with the automorphisms of \bar{F}_π defined over $\hat{\underline{Z}}_p$, which leads to the

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corollary: $I(L_{ab}/Q_p)(F_p) \cong \hat{\mathbb{Z}}_p^x$

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