W. ABRAMCZUK

# A remark on ellipticity of systems of linear partial differential equations with constant coefficients

*Astérisque*, tome 89-90 (1981), p. 117-128

<http://www.numdam.org/item?id=AST_1981__89-90__117_0>

# A REMARK ON ELLIPTICITY OF SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

by W. ABRAMCZUK (University of Stockholm)

INTRODUCTION.

It is well known (and easy to prove) that a linear partial differential operator with constant coefficients, $P(D)$, is elliptic and has order $N$ if and only if it is a bounded operator with closed range when it acts between the spaces $H_o^m(\Omega)$ and $H_o^{m-N}(\Omega)$, where $H_o^k(\Omega)$ denotes the closure of $C_o^\infty(\Omega)$ in the norm $|\phi|_k = \sum_{|\alpha|\leqq k} |D^\alpha\phi|_{L^2}$, $m$ is an integer such that $m-N \geqq 0$ and $\Omega$ is a bounded open subset of $R^n$.

Here this result is (partially) extended to systems of linear partial differential operators with constant coefficients and to more general spaces of distributions.

Theorem 1 below is a rather straightforward generalisation of the considerations in 10.6 of Hörmander [1]. The first part of Theorem 2 is an easy consequence of the coercivity results in Smith [1] and the second part was inspired by a counter example in Eskin and Shamir [1].

## NOTATION AND DEFINITIONS

To measure the regularity of distributions we use the spaces $L_s^p = L_s^p(R^n)$,

$1 < p < \infty$ , $s \in R$ , of Bessel potentials of $L^p$ functions (Calderon [1]): $u \in L^p_s$ if $u$ is a temperate distribution and $(1 + |\xi|^2)^{s/2} . \hat{u}$ is the Fourier transform of a $L^p$ function denoted here by $J^{-s}u$ . We let $J^{-s}$ transport the $L^p$ norm to $L^p_s$: $|u|_{L^p_s} = |J^{-s}u|_{L^p}$ . When $u$ is a test function this can be made more explicit:

$$|u|_{L^p_s} = |(2\pi)^{-n} \int (1 + |\xi|^2)^{s/2} \hat{u}(\xi) e^{ix\xi} d\xi |_{L^p} .$$

When $\Omega$ is an open subset of $R^n$ we let $L^p_{s,\overline{\Omega}}$ denote the distributions in $L^p_s$ supported in $\overline{\Omega}$ and we put $L^p_s(\Omega) = L^p_s / L^p_{s,R^n\setminus\Omega}$ which we think of as the restriction of $L^p_s$ to $\Omega$ . For technical reasons we assume in what follows that $\Omega$ is also bounded and convex.

When $r = (r_1,\dots,r_K) \in R^K$ we denote the product space $L^p_{r_1} \times \dots \times L^p_{r_K}$ by $L^p_r$ , the space $L^p_{r_1,\overline{\Omega}} \times \dots \times L^p_{r_K,\overline{\Omega}}$ by $L^p_{r,\overline{\Omega}}$ , etc.

By $P(D)$ we denote a matrix of linear differential operators with constant coefficients: $P(D) = (P_{jk}(D))$, $j = 1,\dots,J$, $k = 1,\dots,K$, and by ${}^tP(D)$ the transpose of $P(-D)$ .

<u>Definition 1</u>: The operator $P(D)$ is <u>determined</u> if $P(D)u = 0$ has no non-trivial solutions with compact support (i.e. $P(D):\mathscr{E}'^K \longrightarrow \mathscr{E}'^J$ is injective).

<u>Definition 2</u>: Let $r_k$ and $s_j$, $k = 1,\dots,K$, $j = 1,\dots,J$ , be real numbers such that $r_k - s_j$ are non-negative integers. We call the operator $P(D)$ <u>$(r_k-s_j)$-elliptic</u> if

i) $\deg P_{jk} \leqq r_k-s_j$

ii) $\operatorname{rank} (\overset{\circ}{P}_{jk}(\xi)) = K$ if $0 \neq \xi \in R^n$ ;

here $\overset{\circ}{P}_{jk}$ denotes the part in $P_{jk}$ of degree $r_k-s_j$ .

If i) and

ii)' rank $(\overset{\circ}{P}_{jk}(\zeta)) = K$ if $0 \neq \zeta \in C^n$

are satisfied we call $P(D)$ <u>$(r_k-s_j)$-very strongly elliptic</u>.

This definition of $(r_k-s_j)$-ellipticity was given in Douglis and Nirenberg [1]. See also Hörmander [1], Ch.X. Systems $(r_k-s_j)$-v.s. elliptic in a similar sense were studied in Smith [1].

<u>Remark</u>: It is easy to see that $(r_k-s_j)$-ellipticity implies the usual one defined, for example, in terms of the characteristic variety of $P(D)$ and that $(r_k-s_j)$-v.s. ellipticity implies that the characteristic variety is discrete. The converse is obviously not true and it is an open problem whether an elliptic $P(D)$ (a $P(D)$ with discrete characteristic variety) becomes $(r_k-s_j)$-elliptic ($(r_k-s_j)$-v.s. elliptic) when multiplied by a non-singular $K \times K$ - matrix with differential operator entries.

<u>Definition 3</u>: Let $1 \leqq m < n$ . Consider $R^n = R^m + R^{n-m}$ and write $x = (x',x")$ , $D = (D',D")$ with the obvious meaning. We say that $P(D)$ is of <u>tensor product type</u> if $P(D) = (P^1(D'),P^2(D")) = (P_1(D'),\ldots,P_I(D'),P_{I+1}(D"),\ldots,P_J(D"))$ is a row matrix with all polynomials $P_j$ , $1 \leqq j \leqq J$ , homogeneous of degree $N > 0$ with no non-trivial relations (i.e. every relation $\Sigma P_j Q_j = 0$ , $Q_j$ polynomials, is of the form $P_j P_k - P_k P_j = 0$ ).

## THEOREMS

Let $r = (r_k)_{k=1,\ldots,K}$, $s = (s_j)_{j=1,\ldots,J}$ . Consider the condition

(*) $\qquad P(D): L^p_{r,\overline{\Omega}} \to L^p_{s,\overline{\Omega}}$ is bounded with closed range.

<u>Theorem 1</u>: For determined $P(D)$ (*) and $(r_k-s_j)$-ellipticity are equivalent.

<u>Theorem 2</u>: (*) is implied by $(r_k-s_j)$-v.s. ellipticity of ${}^tP(D)$ . The converse is true (at least) when $P(D)$ is of tensor product type and $\Omega = \Omega' \times \Omega"$ , where $\Omega'$ and $\Omega"$ are open, bounded and convex sets of $R^m$ and $R^{n-m}$ respectively.

<u>Proof of Theorem 1</u>: We first prove that <u>(*) implies $(r_k-s_j)$-ellipticity</u>: (*) and the injectivity of $P(D)$ give the estimate

$$(1) \qquad C^{-1}\cdot \sum_j \Big| \sum_k P_{jk}(D)u_k \Big|_{L^p_{s_j}} \leqq \sum_k |u_k|_{L^p_{r_k}} \leqq C \cdot \sum_j \Big| \sum_k P_{jk}(D)u_k \Big|_{L^p_{s_j}}$$

for some constant $C > 0$ and all $u_k \in C_o^\infty(\Omega)$ . If we in the first inequality put all $u_k$ but one equal to zero we get

$$(2) \qquad |P_{jk}(D)u|_{L^p_{s_j}} \leqq C \cdot |u|_{L^p_{r_k}}$$

for all $u \in C_o^\infty(\Omega)$, $1 \leqq j \leqq J$, $1 \leqq k \leqq K$ . Putting $u(x) = e^{i\lambda x\eta}\cdot \phi(x)$ with $\lambda > 0$, $0 \neq \eta \in R^n$, $0 \neq \phi \in C_o^\infty(\Omega)$ and using Lemma A7 from the Appendix one easily checks that (2) implies $\deg P_{jk} \leqq r_k - s_j$ .

We now show that $\operatorname{rank} (\mathring{P}_{jk}(\eta)) < K$ for some $0 \neq \eta \in R^n$ violates the second estimate in (1):

if $\operatorname{rank} (\mathring{P}_{jk}(\eta)) < K$ then, for some $0 \neq (a_1,\dots,a_K) \in C^K$ , $\Sigma_k \mathring{P}_{jk}(\eta)a_k = 0$, $j = 1,\dots,J$, or, since $\mathring{P}_{jk}$ are homogeneous of degree $r_k - s_j$ ,

$$(3) \qquad \sum_k \mathring{P}_{jk}(\lambda\eta)\cdot\lambda^{-r_k}\cdot a_k = 0 \qquad j = 1,\dots,J .$$

Now put in the second estimate in (1) $u_k^\lambda(x) = \lambda^{-r_k}\cdot a_k\cdot e^{i\lambda x\eta}\cdot\phi(x)$ , $0 \neq \phi \in C_o(\Omega)$ . By lemma A7, as $\lambda \to \infty$ ,

$$(4) \qquad \Sigma_k|u_k^\lambda|_{L^p_{r_k}} = \Sigma_k \lambda^{-r_k}\cdot|a_k|\cdot|e^{i\lambda x\eta}\cdot\phi|_{L^p_{r_k}} \longrightarrow \Sigma_k|a_k|\cdot|\phi|_{L^p} > 0 .$$

At the same time it is easy to see that

$$(5) \qquad \Sigma_k|P_{jk}(D)u_k^\lambda|_{L^p_{s_j}} = O(\lambda^{-1}) \quad \text{as} \quad \lambda \to \infty,\ j = 1,\dots,J .$$

Namely, in

$$|\Sigma_k P_{jk}(D)u_k^\lambda|_{L^p_{s_j}} \leqq |\Sigma_k \mathring{P}_{jk}(D)u_k^\lambda|_{L^p_{s_j}} + |\Sigma_k (P_{jk}-\mathring{P}_{jk})(D)u_k^\lambda|_{L^p_{s_j}}$$

the second term of the right hand side is $O(\lambda^{-1})$ by Lemma A7, and as for the first term, observe that

$$\Sigma_k \overset{\circ}{P}_{jk}(D) u_k^\lambda = \Sigma_k \overset{\circ}{P}_{jk}(\lambda\eta) \cdot \lambda^{-r_k} \cdot a_k \cdot e^{i\lambda x\eta} + \Sigma_k \Sigma_l \lambda^{-r_k} \cdot Q_{jk}^l(\lambda\eta)\, e^{i\lambda x\eta} \cdot \psi_{lk}$$

for some homogeneous polynomials $Q_{jk}^l$ of degree $l$, $1 \leqq l < r_k - s_j$, and some $\psi_{lk} \in C_o^\infty(\Omega)$, and then use (3) and Lemma A7.

This ends the proof of the first part of Theorem 1 since (4) and (5) clearly contradict (1).

We now prove that <u>$(r_k-s_j)$-ellipticity implies (*)</u>: that $P(D)$ in (*) is bounded is trivially clear because each $P_{jk}(D): L^p_{r_k,\overline{\Omega}} \to L^p_{s_j,\overline{\Omega}}$ is bounded if $\deg P_{jk} \leqq r_k - s_j$.

To show that $P(D)$ has closed range we first observe that the set $(P(D)\mathscr{E}'^K_{\overline{\Omega}}) \cap L^p_{s,\overline{\Omega}}$ is closed in $L^p_{s,\overline{\Omega}}$ (it is well known that $P(D)\mathscr{E}'^K_{\overline{\Omega}}$ is closed in $\mathscr{E}'^J_{\overline{\Omega}}$ and the topology of $L^p_{s,\overline{\Omega}}$ is stronger than that of $\mathscr{E}'^J_{\overline{\Omega}}$) and then we prove that $(P(D)\mathscr{E}'^K_{\overline{\Omega}}) \cap L^p_s = P(D) L^p_{r,\overline{\Omega}}$.

<u>Proposition</u>: If $u_k \in \mathscr{E}'$, $k = 1,\ldots,K$, $\Sigma_k P_{jk}(D) u_k = f_j \in L^p_{s_j}$, $j = 1,\ldots,J$, and $P(D)$ is $(r_k-s_j)$-elliptic, then $u_k \in L^p_{r_k}$.

<u>Proof</u>: First we show how we can reduce the proof to the case of a square system, then we prove that case.

Denote by $\Delta$ the polynomial $\sum_{k=1}^n x_k^2$ and the corresponding differential operator: $\Delta = \Delta(D) = \sum_{k=1}^n D_k^2$. Let $N$ be some integer $\geqq \max_{j,k} (r_k - s_j)$. From $\Sigma_k P_{jk}(D) u_k = f_j$, $j = 1,\ldots,J$, we get

$$\Sigma_j \overline{P}_{jl}(D) \Delta^{N-(r_l-s_j)} \Sigma_k P_{jk}(D) u_k = \Sigma_j \overline{P}_{jl}(D) \Delta^{N-(r_l-s_j)} f_j, \quad l = 1,\ldots,K \quad ,$$

where $\overline{P}_{jk}$ denotes the polynomial obtained by complex conjugation of the coefficients of $P_{jk}$. After changing the order of summation and putting

$Q_{lk} = \Sigma_j \overline{P}_{jl} P_{jk} \Delta^{N-(r_l-s_j)}$ and $\phi_l = \Sigma_j \overline{P}_{jl}(D)\Delta^{N-(r_l-s_j)} f_j$ we see that $u_k$, $k = 1,\ldots,K$, satisfy a square system of differential equations:

$$(6) \qquad \Sigma_k Q_{lk}(D) u_k = \phi_l, \quad l = 1,\ldots,K .$$

This system is $(2N+r_k-r_l)$-elliptic: $\deg Q_{lk} \leqq \deg (\overline{P}_{jl} P_{jk} \Delta^{N-(r_l-s_j)}) \leqq (r_l-s_j)+(r_k-s_j)+2N-2(r_l-s_j) = 2N+r_k-r_l$. Denote by $\mathring{Q}_{lk}$ the part in $Q_{lk}$ of degree $2N+r_k-r_l$ and observe that $\mathring{Q}_{lk} = \Sigma \overline{\mathring{P}}_{jl} \mathring{P}_{jk} \Delta^{N-(r_l-s_j)}$. Now, if rank $(\mathring{Q}_{lk}(\xi)) < K$ for some $0 \neq \xi \in R^n$, then for some $0 \neq (a_1,\ldots,a_K) \in C^n$

$$\Sigma_k \Sigma_j \overline{\mathring{P}}_{jl}(\xi) \mathring{P}_{jk}(\xi) |\xi|^{2(N-(r_l-s_j))} \cdot a_k = 0, \quad l = 1,\ldots,K .$$

Putting $\xi = \lambda \cdot \eta$, $\lambda > 0$, $|\eta| = 1$, this gives

$$(7) \qquad \Sigma_k \Sigma_j \overline{\mathring{P}}_{jl}(\eta) P_{jk}(\eta) \cdot (\lambda^{r_k} \cdot a_k) = 0, \qquad l = 1,\ldots,K .$$

On the other hand it is easy to show that if rank $(\mathring{P}_{jk}(\eta)) = K$ then also rank $(\Sigma_j \overline{\mathring{P}}_{jl}(\eta) \mathring{P}_{jk}(\eta)) = K$ what clearly contradicts (7) thus proving that rank $(Q_{lk}(\xi)) = K$ if $0 \neq \xi \in R^n$.

Observe now that $\phi_l$ in (6) are in $L^p_{r_l-2N}$ and so, if the Proposition is true in the special case when $K = J$, it follows that

$$u_k \in L^p_{r_l-2N+(2N+r_k-r_l)} = L^p_{r_k}$$

and the Proposition is true in the general case.

So assume from now on that $K = J$. Let $\phi \in C_o^\infty(R^n)$ be $= 1$ on the (real) zeroes of $\det (P_{jk})$. This is possible since the set is bounded. Using the matrix notation we then have

$$\hat{u} = (\phi+(1-\phi)P^{-1}P) \cdot \hat{u} = \phi \cdot \hat{u} + (1-\phi)P^{-1} f\hat{~} ,$$

or, denoting by $({}^{co}P_{kj})$ the matrix formed by the cofactors in $(P_{jk})$,

$$(8)\qquad \hat{u}_k = \phi\cdot\hat{u}_k + \Sigma_j (1-\phi)\cdot \frac{{}^{co}P_{kj}}{\det P}\cdot \hat{f_j}, \quad k = 1,\dots,K .$$

Now put $m_s(\xi) = (1+|\xi|^2)^{s/2}$. It follows from (8) that

$$(9)\qquad m_{r_k}\cdot\hat{u}_k = \phi\cdot m_{r_k}\cdot\hat{u}_k + \Sigma_j (1-\phi)\frac{{}^{co}P_{kj}}{\det P}\cdot m_{r_k-s_j}\cdot m_{s_j}\cdot\hat{f_j}, \quad k = 1,\dots,K .$$

The first term in each of the sums of (9) is a $C_o^\infty$ function and therefore its inverse Fourier transform is in every $L^p$, $1 \leqq p \leqq \infty$. The inverse Fourier transform of every other term is also in $L^p$, $1 < p < \infty$, because the functions

$$(1-\phi)\,\frac{{}^{co}P_{kj}}{\det P}\, m_{r_k-s_j}$$

are easily seen to be multipliers on $L^p$, $1 < p < \infty$.

This ends the proof of the Proposition and thus of Theorem 1.

Proof of Theorem 2. $(r_k-s_j)$-v.s. ellipticity of ${}^tP(D)$ implies (*): $\deg p_{jk} = \deg ({}^tP)_{kj} \leqq r_k-s_j$ $(= -s_j-(-r_k))$ and so $P(D)$ is bounded. To see that $P(D)$ has closed range it is enough to see that the adjoint of $P(D)$, ${}^tP(D)\colon L^q_{-s}(\Omega) \longrightarrow L^q_{-r}(\Omega)$, $1/p + 1/q = 1$, has closed range. Now, it is a well known fact that, for any ${}^tP(D)$, ${}^tP(D)\mathscr{D}'(\Omega)^J$ is closed in $\mathscr{D}'(\Omega)^K$ and since $L^q_{-r}(\Omega) \subset \mathscr{D}'(\Omega)^K$ topologically, $({}^tP(D)\mathscr{D}'(\Omega)^J) \cap L^q_{-r}(\Omega)$ is closed in $L^q_{-r}(\Omega)$. But $({}^tP(D)\mathscr{D}'(\Omega)^J) \cap L^q_{-r}(\Omega) = {}^tP(D)L^q_{-s}(\Omega)$ when ${}^tP(D)$ is $(r_k-s_j)$-v.s. elliptic by Theorem 8.15 in Smith [1].

(*) implies $(r_k-s_j)$-v.s. ellipticity of ${}^tP(D)$ when $P(D)$ is of tensor product type and $\Omega = \Omega' \times \Omega''$ : by duality this amounts to proving the following assertion:

Let $P(D)\colon u \to (P^1(D')u, P^2(D'')u) = (P_1(D')u,\dots,P_I(D')u,P_{I+1}(D'')u,\dots,P_J(D'')u)$ as an operator from $L^p_s(\Omega)$ to $L^p_{s-N}(\Omega)^J$ have closed range for some $s \in R$,

$1 < p < \infty$ . Then the polynomials $P_j$, $j = 1,\dots,J$ , have no common complex non-trivial zero.

Now the proof goes as follows: for any $g \in C_o(\Omega')$, $P^1(D')g \neq 0$ , and $h \in L^p_{s-N}(\Omega")$, $P^2(D")h = 0$ , put $f = (P^1(D')g \otimes h, 0)$ , i.e. $f_j = P_j(D')g \otimes h$ if $1 \leqq j \leqq I$ and $f_j = 0$ if $j > I$ . Using Lemma A4 of the Appendix, one easily checks that $f$ is in the closure of the range of $P(D)$ . Assume now that $f = P(D)u$ for some $u \in L^p_s(\Omega)$ . Since also $f = P(D)(g \otimes h)$ , we must have

$$(10) \qquad u = g \otimes h + v$$

for some $v \in \mathscr{D}'(\Omega)$, $P(D)v = 0$ . Let $\phi_1 \in C^\infty_o(\Omega')$ separate $g$ and the kernel of $P^1(D')$ in $\mathscr{D}'(\Omega')$: $(g,\phi_1) = 1$ and $(v_1,\phi_1) = 0$ if $P^1(D')v_1 = 0$ . Apply to (10) the operator $T_{\phi_1}$ of Lemma A5: it follows that $h = T_{\phi_1}(u)$ and is thus in $L^p_s(\Omega")$ . In this way the assumption that the range of $P(D)$ is closed leads to the implication: if $h \in L^p_{s-N}(\Omega")$ and $P^2(D")h = 0$ , then $h \in L^p_s(\Omega")$ . By Lemma A6 it now follows that the polynomials $P_j$, $I+1 \leqq j \leqq J$ , have no common non-trivial complex zero. If we let $P^1$ and $P^2$ change roles we see that the same is also true about $P_j$ , $1 \leqq j \leqq I$ . Theorem 2 is proved.

Example: Let $P(D)$ in the proof of the second part of Theorem 2 be the $\bar{\partial}$ operator and let $\Omega$ be a polydisc in $C^n \cong R^{2n}$ . The result is that $\bar{\partial}u = f$ cannot, in general, be solved with gain of one derivative in the $L^p_s$-space meaning in a polydisc in $C^n$ , $n > 1$ .

## APPENDIX

We first introduce some additional notation. The set of exponential polynomials in $R^n$ is denoted by $EXP(R^n) = EXP$ . Given two matrices of polynomials $P$ and $Q$ , we say that $Q$ is a compatibility matrix for $P$ if the rows of $Q$

generate the module of relations between the rows of $P$. We call a matrix of polynomials homogeneous if all its elements are homogeneous of the same degree. By $\Phi_P$ we denote the set of solutions to $P(D)u = 0$ in a space $\Phi$ of (tuples of) distributions; when $\Phi$ is a cartesian product of $K$ copies of some space $\Psi$, we write $\Psi_P$ instead of $(\Psi^K)_P$. For $\phi \in C_0^\infty(R^n)$ and $\varepsilon > 0$ define $\phi_\varepsilon(x) = \phi(\varepsilon x)$ and $\phi^\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x)$. For a distribution $u$ define $u_\varepsilon$ by $(u_\varepsilon, \phi) = (u, \phi^\varepsilon)$, $\phi \in C_0(R^n)$. Note that when $u$ is a tempered distribution we have $u_\varepsilon\hat{} = \hat{u}^\varepsilon$.

Lemma A1: For $1 < p < \infty$ and $s \in R$ $f \to f_\varepsilon$ is a bounded operator in $L_s^p$ and $|f_\varepsilon - f|_{L_s^p} \to 0$ as $\varepsilon \to 1$.

Lemma A2: Let $\phi \in C_0^\infty(R^n)$, $\phi \geqq 0$, $\int\phi = 1$, $1 < p < \infty$, $s \in R$. Then $f \to \phi^\delta * f$ is a bounded operator in $L_s^p$ and $|f - \phi^\delta * f|_{L_s^p} \to 0$ as $\delta \to 0$.

These lemmas are common knowledge when $s$ is a non-negative integer; for the general case see Abramczuk [1].

Lemma A3: For a homogeneous matrix $Q$ (the restriction to $\Omega$ of) $EXP_Q$ is dense in $L_s^p(\Omega)_Q$ (in the topology of $L_s^p(\Omega)$).

Proof: Consider the inclusions: $EXP_Q|_\Omega \subset \bigcup_{\Omega \subset\subset \Omega'} C^\infty(\Omega')_Q|_\Omega \subset L_s^p(\Omega)_Q$. The range of the first one is dense in the $C^\infty(\overline{\Omega})$-topology by the known (local) density results. We show that the range of the second inclusion is dense if $Q$ is homogeneous: given $u \in L_s^p(\Omega)_Q$ it is clear that $Q(D)u_\varepsilon = \varepsilon^{\deg Q}\cdot(Q(D)u)_\varepsilon$ so $u_\varepsilon \in L_s^p(\varepsilon^{-1}\cdot\Omega)_Q$ and $\Omega \subset\subset \varepsilon^{-1}\cdot\Omega$ if $\varepsilon < 1$ and $0 \in \Omega$ what can be assumed without loss of generality. With $\phi$ like in Lemma A2 $u_\varepsilon * \phi^\delta \in C^\infty(\Omega_{\varepsilon,\delta})_Q$ for some $\Omega_{\varepsilon,\delta} \supset\supset \Omega$ if $\delta$ is small enough. The proof ends by using the two preceding lemmas on

$$|u_\varepsilon * \phi^\delta - u|_{L_s^p(\Omega)} \leqq |u_\varepsilon * \phi^\delta - u_\varepsilon|_{L_s^p} + |u_\varepsilon - u|_{L_s^p} .$$

Lemma A4: Let $P(D): L_r^p(\Omega) \to L_s^p(\Omega)$, $r \in R^K$, $s \in R^J$. If $P$ has a homogeneous compatibility matrix $Q$ then the range of $P(D)$ is dense in $L_s^p(\Omega)_Q$.

Proof: $EXP_Q|_\Omega = P(D)(EXP^K|_\Omega) \subset P(D)L^p_r(\Omega) \subset L^p_s(\Omega)_Q$ and use Lemma A3.

Lemma A5: Let $\Omega = \Omega' \times \Omega''$ be like in Theorem 2. For $\phi_1 \in C^\infty_o(\Omega')$ and $u \in \mathscr{D}'(\Omega)$ let $T_{\phi_1}(u)$ be the linear functional on $C^\infty_o(\Omega'')$ defined by $T_{\phi_1}(u): \phi_2 \to (u, \phi_1 \otimes \phi_2)$ . Then

i) the operator $u \to T_{\phi_1}(u)$ maps $L^p_s(\Omega)$ into $L^p_s(\Omega'')$

ii) if $P(D): u \to (P^1(D')u, P^2(D'')u)$ and $\phi_1$ vanishes on $\mathscr{D}'(\Omega')_{P^1}$ then $T_{\phi_1}$ vanishes on $\mathscr{D}'(\Omega)_P$ .

Proof: We only prove ii): it is easily seen that $\phi_1 = \Sigma_j P_j(-D')\psi_{1_j}$ for some $\psi_{1_j} \in C^\infty_o(\Omega')$ . But then, if $v \in \mathscr{D}'(\Omega)_P$ , $(v, \phi_1 \otimes \phi_2) = (v, (\Sigma_j P_j(-D')\psi_{1_j}) \otimes \phi_2) = \Sigma_j (v, P_j(-D')(\psi_{1_j} \otimes \phi_2)) = \Sigma_j (P_j(D')v, \psi_{1_j} \otimes \phi_2) = 0$ .

Lemma A6: Let $1 < p < \infty$ and $s \in R^K$ . If for some K-tuple of positive integers $N = (N_1, \dots, N_K)$ every solution to $P(D)u = 0$ in $L^p_s(\Omega)$ is actually in $L^p_{s+N}(\Omega)$ then the linear space of distribution solutions to $P(D)u = 0$ in $\Omega$ is finite dimensional.

Proof: The assumption implies that $L^p_s(\Omega)_P \subset \bigcap_r L^p_r(\Omega) = C^\infty(\bar{\Omega})^K$ . Now $L^p_s(\Omega)_P$ is closed in $L^p_s(\Omega)$ and in the stronger topology of $C^\infty(\bar{\Omega})^K$ . Hence $L^p_s(\Omega)_P$ is a Fréchet space in two comparable topologies. By the closed graph theorem, the two topologies coincide. One of these is a Banach space topology and the other is a Montel space topology and it is known that these coincide only on finite dimensional spaces. Hence $\dim L^p_s(\Omega)_P < \infty$ and $\mathscr{D}'(\Omega)_P = L^p_s(\Omega)_P$ by a density argument.

Lemma A7: Let $\phi \in C^\infty_o(R^n)$, $\lambda > 0$, $\eta \in R^n$, $s \in R$, $1 < p < \infty$ . Let $P$ be a polynomial of degree $m$ with principal part $P_m$ . Then

$$\lim_{\lambda\to\infty} \lambda^{-(s+m)} |P(D)(e^{i\lambda x\eta}\cdot\phi)|_{L^p_s} = |P_m(\eta)|\cdot|\eta|^s\cdot|\phi|_{L^p} .$$

Proof: Consider first the case $m = 0$ :

$$|e^{i\lambda x\eta}\cdot\phi|_{L^p_s} = |(2\pi)^{-n}\int \hat{\phi}(\xi-\lambda\eta)(1+|\xi|^2)^{s/2}e^{ix\xi}d\xi|_{L^p} = |(2\pi)^{-n}\int \hat{\phi}(\xi)\, e^{ix\xi}\cdot$$

$$\cdot(1+|\xi+\lambda\eta|^2)^{s/2}d\xi|_{L^p} = \lambda^s|(2\pi)^{-n}\int \hat{\phi}(\xi)\, e^{ix\xi}\,(\lambda^{-2}+|\lambda^{-1}\xi+\eta|^2)^{s/2}d\xi|_{L^p} .$$

Now multiply by $\lambda^{-s}$ and let $\lambda \to \infty$ . If $m \neq 0$ $P(D)(e^{i\lambda x\eta}\cdot\phi) =$

$$\sum_\alpha P^{(\alpha)}(\lambda\eta)\cdot e^{i\lambda x\eta}\cdot\frac{D^\alpha\phi}{\alpha!} = \lambda^m P_m(\eta)\, e^{i\lambda x\eta} + \sum_{1\leq j<m} \lambda^j\cdot Q_j(\eta)\cdot e^{i\lambda x\eta}\cdot\psi_j$$ for some homogeneous polynomials $Q_j$ , $\deg Q_j = j$ , and test functions $\psi_j$ . After multiplication of the last equality by $\lambda^{-(s+m)}$ the $L^p_s$-norm of the first term has the limit $|P_m(\eta)|\cdot|\eta|^s\cdot|\phi|_{L^p}$ as $\lambda \to \infty$ by the previous case and the second term $\to 0$ .

## REFERENCES


ABRAMCZUK, W.: [1] On the range of linear partial differential operators in Lebesgue spaces of distributions, Research report, University of Stockholm, 1980.

CALDERON, A.P.: [1] Lebesgue spaces of differentiable functions and distributions, Proc. Symp. on Pure Math., Partial Differential Equations, 33-49 (1961).

DOUGLIS, A. and NIRENBERG, L.: [1] Interior estimates for elliptic systems of partial differential equations, Comm. Pure Appl. Math. 8, 503-538 (1955).

ESKIN, M. and SHAMIR, E.: [1] Elliptic overdetermined differential systems in convex cones, Journal d'Analyse Mathématique, vol. 32 (1977).

HÖRMANDER, L.: [1] Linear partial differential operators, Springer, 1969.

HÖRMANDER, L.: [2] An introduction to complex analysis in several variables, North Holland, 1973.

PALAMODOV, V.P.: [1] Linear differential operators with constant coefficients, Moscow, 1967.

SMITH, K.T.: [1] Formulas to represent functions by their derivatives, Math. Ann. 188, 53-77 (1970).

Wojciech ABRAMCZUK,
Department of Mathematics,
University of Stockholm,
Box 6701,
S-113 85 Stockholm,
Sweden.