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## BACKWARD PARABOLIC EQUATIONS

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### I. INTRODUCTION

This paper is devoted to the study of the singularities of the solutions of backward parabolic pseudo-differential equations.

Let $\mathbb{R}^n$ denote the n-dimensional euclidean space and write $x = (x', x_n) \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$. Let $\Omega'$ be an open subset of $\mathbb{R}^{n-1}$ and $S$ a positive constant. Suppose that the extendible distribution $\vec{\tau} = (\tau_1, \ldots, \tau_N)$ of $D^*(\Omega' \times ]0,S[)$ satisfies

(I.1) $\quad \vec{\tau} \cdot [(D_{x_n} + Q(x, D_{x'}))] \in C_\infty(\Omega' \times [0,S[)$

where $Q(x, D_{x'})$ is a first order properly supported $(N \times N)$ pseudo-differential operator in $\Omega'$ depending smoothly on $x_n \in [0,S[$ and with principal symbol $Q_1(x, \xi')$ homogeneous of degree 1 in $\xi'$.

It follows that

$$\vec{\tau} \cdot \vec{\phi} = \int \vec{\tau}_{x_n} \cdot \vec{\phi} \, dx_n$$

with $\vec{\tau}_{x_n} \in C_\infty([0,S[\; ; D^*(\Omega'))$.

We assume that the operator $D_{x_n} + Q$ is backward parabolic at $(x'_o, \xi'_o) \in T^*(\Omega') \backslash 0$, that is

(I.2) $\quad$ all the eigenvalues of the matrix $Q_1(x'_o, 0, -\xi'_o)$ have positive real parts.

By extension, we say that the equation (I.1) is backward parabolic at $(x'_o,\xi'_o)$. The condition (I.2) still holds if $(x,\xi')$ belongs to a conic neighborhood $\omega' \times [0,s[ \times \gamma$ of $(x'_o,0,-\xi'_o)$.

We examine the behaviour of the singularities of $\vec{\tau}$ near $(x'_o,\xi'_o)$. As is well known, [4], $\vec{\tau}$ is microlocally $C_\infty$ if $x_n > 0$ ; more precisely,

$$WF\ \vec{\tau} \cap [(\omega' \times ]0,s[) \times (-\gamma \times \mathbb{R})] = \emptyset \ .$$

Moreover, all the traces of $\vec{\tau}$ are regular at $(x'_o,\xi'_o)$. This is the main result of the present paper which we prove in section III. We obtain it by constructing in section II a microlocal parametrix at $(x'_o,\xi'_o)$ for the Cauchy problem

$$\text{(I.3)} \qquad \begin{cases} D_{x_n}\vec{u} + Q(x,D_{x'})\vec{u} = 0 \ , \\ \vec{u}\big|_{x_n=0} = \vec{g}(x') \ . \end{cases}$$

J. Polking has obtained in [2] other regularity theorems for parabolic operators, using $L^2$ methods, (see also [3]).

## II. CONSTRUCTION OF A MICROLOCAL PARAMETRIX

We first introduce an auxiliary space.

Let us set

$$q(x,\xi',W) = \mathrm{dtm}\ (Q_1(x,\xi')+iWI_N) \ , \ W \in \mathbb{C} \ .$$

It follows from (I.2) that all the roots $W$ of $q$ have positive imaginary parts when $(x,\xi') \in \omega' \times [0,s[ \times \gamma$. We denote by $\phi_{x,\xi'}$ a closed curve containing these roots in its interior.

<u>Definition II.1.</u>: The space $\Sigma_m$ is the linear hull of the functions

$$\frac{W^j A_k(x,\xi')}{[q(x,\xi',W)]^l}\ ,\ j+k-lN \leq m,\ j,l \in \mathbb{N}\ ,$$

where $A_k$ is a classical $(N \times N)$ symbol of order $k$ in $\omega' \times [0,s[$ with support in $\xi'$ contained in a closed subcone of $\gamma$ .

The essential property of this space is presented in the following theorem.

Theorem II.1.: If $F$ is an element of $\Sigma_m$ , then the function

$$A(x,\xi') = \int_{\phi_{x,\xi'}} e^{ix_n W} F(x,\xi',W)\,dW$$

belongs to the space

$$\mathcal{S}_{m+1} = S^{\rho,\sigma}_{m+1}(\omega' \times [0,s[ \times \mathbb{R}^n) \cap S_{-\infty}(\omega' \times ]0,s[ \times \mathbb{R}^n)$$

with $\rho = (1,\ldots,1)$ , $\sigma = (0,\ldots,0,1)$ , [1].

Proof: If $K = K' \times [\varepsilon_o,\varepsilon_1]$ is a compact subset of $\omega' \times [0,s[$ , we have, uniformly for $x \in K$ ,

$$|A(x,\xi')| \leq \begin{cases} C|\xi'|^{m+1} & \text{if } \varepsilon_o = 0\ , \\ \dfrac{C'_N}{|\xi'|^N}\ ,\ \forall N, & \text{if } \varepsilon_o > 0\ . \end{cases}$$

Let $\gamma'$ denote a closed subcone of $\gamma$ containing $[F(x,.,W)]$ .

It clearly suffices to prove that

$$\text{(II.1)} \qquad \sup_{x \in K} \left| \int_{\phi_{x,\xi'}} \frac{e^{ix_n W} W^j}{[q(x,\xi',W)]^l}\,dW \right| \leq \begin{cases} C|\xi'|^{j-lN+1} & \text{if } \varepsilon_o = 0\ , \\ \dfrac{C'_N}{|\xi'|^N}\ ,\ \forall N, & \text{if } \varepsilon_o > 0\ , \end{cases}$$

in $\gamma'$ .

Note that there exists a closed curve $\phi$ enclosing the compact set

$$\{W : \exists (x,\xi') \in K \times \gamma'\ , |\xi'| = 1 : q(x,\xi',W) = 0\}$$

and contained in

$\{W : \text{Im}\, W > c > 0\}$ .

Hence, for $(x,\xi') \in K \times \gamma'$ , we obtain

$$\int_{\phi_{x,\xi'}} \frac{e^{ix_n W} W^j}{[q(x,\xi',W)]^l}\, dW = \int_{|\xi'|\phi} \frac{e^{ix_n W} W^j}{[q(x,\xi',W)]}\, dW =$$

$$= |\xi'|^{j-lN+1} \int_{\phi} \frac{e^{ix_n|\xi'|W} W^j}{[q(x,\frac{\xi'}{|\xi'|},W)]^l}\, dW$$

The absolute value of this expression is bounded by

$$C\, e^{-c\, \varepsilon_o |\xi'|}\, |\xi'|^{j-lN+1}$$

We then easily obtain (II.1).

It follows that the expression

$$D_{x'}^{\alpha'} D_{x_n}^{\alpha_n} D_{\xi'}^{\beta'} A(x,\xi') = \sum_{p=0}^{\alpha_n} C_{\alpha_n}^{p} \left(\int e^{ix_n W} W^p D_{x'}^{\alpha'} D_{x_n}^{\alpha_n - p} D_{\xi'}^{\beta'} F\, dW\right)$$

gives the required estimate since

$$W^p D_{x'}^{\alpha'} D_{x_n}^{\alpha_n - p} D_{\xi'}^{\beta'} F \in \Sigma_{m+p-|\beta'|} \subset \Sigma_{m+\alpha_n-|\beta'|} \quad .$$

Now, we shall construct a microlocal parametrix at $(x'_o,\xi'_o)$ for the Cauchy problem (I.3).

<u>Theorem II.2.</u>: There exists of smooth family of $(N \times N)$ pseudo-differential operators in $\omega'$ of order $o$

$$P(x,D_{x'})\, \vec{\psi} = ⨍\!\!\int e^{i(x'-y')\cdot\xi'} A(x,\xi')\, \vec{\psi}\,(y')\, dy'\, d\xi'$$

with

$$x_n \in [0,s[ \;,\; A \in \mathcal{S}_0 \quad ,$$

and such that

(i) $(D_{x_n} I_N + Q)P$ is an integral operator with kernel in $C_\infty(\omega' \times [0,s[ \times \omega')$ ,

(ii) $P(x',0,D_{x'})$ is elliptic at $(x'_o,-\xi'_o)$ .

<u>Proof</u>: Let us define the amplitude by

$$A(x,\xi') \sim \sum_{p,q=0}^{\infty} A_{pq}(x,\xi')$$

where $A_{pq} \in \mathcal{S}_{-(p+q)}$ .

More precisely, we set

$$A_{pq}(x,\xi') = \int_{\phi_{x,\xi'}} e^{ix_n W} F_{pq}(x,\xi',W)dW$$

with $F_{pq} \in \Sigma_{-1-(p+q)}$ .

In particular, we take

$$F_{oq} = (Q_1(x,\xi') + i\, W\, I_N)^{-1} F_q(x',\xi') \quad , \tag{II.2}$$

with $F_q \in S_{-q}(\omega' \times \mathbb{R}^n)$ .

Applying $D_{x_n} + Q$ to $P$ yields, [3],

$$(D_{x_n} + Q)P\, \vec{\psi} = \iint e^{i(x'-y')\cdot\xi'}[D_{x_n} A(x,\xi') + B(x,\xi')]\, \vec{\psi}\, (y')dy'd\xi'$$

where $B(x,\xi')$ is a symbol of $\mathcal{S}_1$ defined by the following asymptotic expansion

$$B(x,\xi') \sim \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} D^{\alpha}_{\xi'} Q(x,\xi')\, D^{\alpha}_{x'} A(x,\xi')$$

Writing for large $\xi'$,

$$Q = Q_1 + Q_0 \quad ,$$

with $Q_0 \in S_0$ , we obtain

$$D_{x_n} A + B \sim \sum_{k=0}^{\infty} T_{1-k} A$$

where

$$T_1(x,\xi',D_{x_n}) = D_{x_n} + Q_1 \quad ,$$

$$T_0(x,\xi',D_{x'}) = Q_0 + \sum_{|\alpha|=1} \frac{i^{-|\alpha|}}{\alpha!} D^{\alpha}_{\xi'} Q \; D^{\alpha}_{x'} \quad ,$$

$$T_{1-k}(x',\xi',D_{x'}) = \sum_{|\alpha|=k} \frac{i^{-|\alpha|}}{\alpha!} D^{\alpha}_{\xi'} Q \; D^{\alpha}_{x'} \quad , \quad \text{if} \quad k \geq 2 \quad ,$$

are differential operators which map $\mathscr{S}_m$ into $\mathscr{S}_{m+1-k}$ .

Noting that

$$A \sim \sum_{r=0}^{\infty} \left( \sum_{p+q=r} A_{pq} \right)$$

we get

$$D_{x_n} A+B \sim \sum_{r=0}^{\infty} \left( \sum_{k=0}^{r} \sum_{p+q=r-k} T_{1-k} A_{pq} \right) \quad .$$

In order to realize condition (i), we annihilate each term of the asymptotic expansion of $D_{x_n} A+B$ . We obtain

$$\text{(II.3)} \qquad \sum_{q=0}^{r-1} \sum_{k=0}^{r-q} T_{1-k} A_{r-q-k,q} = 0 \quad , \quad \text{for} \quad r \geq 1 \quad ,$$

if we remark that

$$T_1 A_{\circ q} = \left( \int_{\phi_{x,\xi'}} e^{ix_n W} (iWI_N + Q_1)(iWI_N + Q_1)^{-1} dW \right) F_q = 0 \quad .$$

The conditions (II.3) are satisfied if the functions $F_{pq}$ are given by

$$\text{(II.4)} \qquad F_{pq} = -(iWI_N + Q_1)^{-1} \sum_{k=1}^{p} T_{1-k} F_{p-k,q} \; , \; p \geq 1 \; , \; q \in \mathbb{N} \; .$$

These relations determine $F_{pq}$ from $F_{oq}$ .

Furthermore, we have

$$P(x',0,D_{x'}) \vec{\psi} = \iint e^{i(x'-y')\cdot\xi'} A(x',0,\xi') \vec{\psi}(y') dy' d\xi' \quad .$$

Here $A(x',0,\xi')$ is a classical symbol of order $0$ having the following asymptotic expansion

$$A(x',0,\xi') \sim A_{oo}(x',0,\xi') + \sum_{q=1}^{\infty} \left(\sum_{p=1}^{\infty} A_{p,q-1}(x',0,\xi') + A_{oq}(x',0,\xi')\right)$$

The condition (ii) is satisfied if we take

$$\begin{cases} A_{oo}(x',0,\xi') = \alpha(x')\ \chi\ (\xi')\ I_N \ , \\ A_{oq}(x',0,\xi') = -\sum_{p=1}^{\infty} A_{p,q-1}(x',0,\xi') \quad , \text{ for } \ q \geq 1 \ , \end{cases}$$

where $\alpha \in D(\omega')$ is equal to $1$ in a neighborhood of $x'_o$ and $\chi \in C_\infty(\mathbb{R}^n)$ is homogeneous of degree $o$ for large $\xi'$ , equal to $1$ in a conic neighborhood of $-\xi'_o$ for $|\xi'| \geq \frac{1}{2}|\xi'_o|$ and with support contained in a closed subcone $\gamma'$ of $\gamma$ . Noting that

$$A_{oq}(x',0,\xi') = \left(\int_{\phi_{x',0,\xi'}} (Q_1(x',0,\xi') + iWI_N)^{-1} dW\right) F_q(x',\xi') = 2\pi F_q(x',\xi')$$

we obtain

$$\text{(II.5)} \qquad \begin{cases} F_0 = \frac{1}{2\pi}\ \alpha(x')\chi(\xi')\ I_N \ , \\ F_q = -\ \frac{1}{2\pi} \sum_{p=1}^{\infty} A_{p,q-1}(x',0,\xi') \quad , \text{ for } \ q \geq 1 \ . \end{cases}$$

The relations (II.2), (II.4), (II.5) determine the functions $F_{pq}$ . It is easy to prove by induction that $F_{pq} \in \Sigma_{-1-(p+q)}$ .
Let us remark that the support in $(x',\xi')$ of $F_{pq}$ is contained in $[\alpha] \times \gamma'$ ; hence

$$[A(.,x_n,.)] \subset [\alpha] \times \gamma' \ .$$

Furthermore, if $x_n > 0$ , $P(x,D_{x'})$ is an integral operator with kernel $\in C_\infty(\omega' \times ]0,s[ \times \omega')$ .

## III. MAIN THEOREM

Lemma III.1.: If the distribution $\vec{\tau}$ satisfies the equation (I.1), we have

(i) $$D_{x_n}\vec{\tau}_{x_n}\cdot\vec{\phi}' - \vec{\tau}_{x_n}\cdot Q(x,D_{x'})\vec{\phi}' + \int \vec{f}\cdot\vec{\phi}' dx' = 0 \ , \text{ if } x_n \in [0,S[ \ , \ \vec{\phi}' \in D(\Omega')$$

and where $\vec{f} \in C_\infty(\Omega' \times [0,S[\ )$ ,

(ii) $$\int_0^{+\infty} \vec{\tau}_{x_n}\cdot(D_{x_n} + Q(x,D_{x'})\vec{\phi}\ dx_n + \iint_0^{+\infty}\vec{f}\cdot\vec{\phi}dx = -\vec{\tau}_0\cdot\vec{\phi}(x',0) \quad , \text{ for every}$$

$\vec{\phi} \in D(\Omega' \times ]-S,S[\ )$ .

Proof: Integrating by parts, we obtain

(III.1) $$\int_0^{+\infty}\vec{\tau}_{x_n}\cdot(D_{x_n} + Q(x,D_{x'}))\vec{\phi}dx_n = \int_0^{+\infty}[-D_{x_n}\vec{\tau}_{x_n}\cdot\vec{\phi} + \vec{\tau}_{x_n}\cdot Q(x,D_{x'})\vec{\phi}]\ dx_n + \\ - \vec{\tau}_0\cdot\vec{\phi}(x',0) \quad .$$

In particular, if we take

$$\vec{\phi} = \psi\ \vec{\phi}' \quad , \vec{\phi}' \in D(\Omega') \quad , \psi \in D(]0,S[) \quad ,$$

we obtain

$$\int\psi\ dx_n \int\vec{f}\cdot\vec{\phi}'\ dx' = \int_0^{+\infty}\psi[-D_{x_n}\vec{\tau}_{x_n}\cdot\vec{\phi}' + \vec{\tau}_{x_n}\cdot Q(x,D_{x'})\vec{\phi}']\ dx_n \quad ,$$

where $\vec{f} \in C_\infty(\Omega' \times [0,S[)$ .

Hence we deduce (i) and using (III.1), we get (ii).

Theorem III.1.: If the equation (I.1) is backward parabolic at $(x_0',\xi_0')$ , all the traces of the distribution $\vec{\tau}$ are regular at $(x_0',\xi_0')$ .

Proof: Let us introduce in the relation (ii) of Lemma III.1 the function

$$\alpha(x_n)\ P(x,D_{x'})\ \vec{\psi}$$

where $P$ is the microlocal parametrix constructed in Theorem II.2 and $\alpha$ is a function in $D(]-s,s[)$ equal to $1$ in a neighborhood of the origin.

We obtain

$$\vec{\tau}\cdot(D_{x_n}\alpha)\ P(x,D_{x'})\vec{\psi} + \int \vec{g}\cdot\vec{\psi}\ dx' = -\vec{\tau}_0\cdot P(x',0,D_{x'})\vec{\psi} \quad ,$$

where $\vec{g} \in C_\infty(\omega')$ .

Hence

$$\vec{\tau}_0\cdot\ P(x',0,D_{x'}) \in C_\infty \quad .$$

Since $P(x',0,D_{x'})$ is elliptic at $(x'_o,-\xi'_o)$ , it follows that

$$(x'_o,\xi'_o) \notin WF\ \vec{\tau}_0 \quad .$$

To complete the proof, it remains to note that

$$WF\ \vec{\tau}_0 = \bigcup_{k=0}^{\infty} WF\ D^k_{x_n}\ \vec{\tau}_{x_n}\Big|_{x_n=0}$$

by relation (i) of Lemma III.1.

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