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THE NON-VANISHING OF GROSS' p-ADIC REGULATOR GALOIS COHOMOLOGICALLY

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In this paper we give an expository discussion of Galois cohomological interpretations of the nonvanishing of Gross' p-adic regulator that may be helpful in proving this nondegeneracy or in understanding possible connections or parallels with Leopoldt's conjecture. When the underlying CM number field K is an abelian extension of $\mathbb{Q}$, the nonvanishing has been demonstrated by Gross (3) using transcendental methods and our equivalences yield new interpretations of this deep result[1].

Fix a prime p and a CM number field K. Let K^+ denote the maximal totally real subfield of K, $Gal(K/K^+)=\{1,J\}$. If M is a $Gal(K/K^+)$-module, let $M^-=\{m\in M \mid m^{1+J}=1\}$. Let X denote the group of divisors of K above p (i.e. $X=\{\sum_{\mathfrak{P}/p} a_{\mathfrak{P}}\cdot\mathfrak{P} \mid a_{\mathfrak{P}}\in\mathbb{Z}\}$). If L is any number field, let U(L) denote the group of p-units of L ($U(L)=\{\alpha\in L^\times \mid \mathrm{ord}_{\mathfrak{P}}\alpha=0\ \forall\ \mathfrak{P}\nmid p\}$), $\mu(L)$ the group of roots of unity of L, and $\bar{L}$ the algebraic closure of L. A p-adic analog of the map of the Dirichlet S-unit theorem is given by

$$\lambda : U(K)^- \to \mathbb{Q}_p \otimes X^-$$
$$\varepsilon \to \sum_{\mathfrak{P}/p} \log_p (\mathbb{N}_{K_{\mathfrak{P}}/\mathbb{Q}_p} \varepsilon).\mathfrak{P}$$

where the p-adic logarithm is normalized by Iwasawa's convention $\log_p(p)=0$. Taking tensor products over $\mathbb{Z}$, we have induced maps

$$\lambda_{\mathbb{Q}} : \mathbb{Q}\otimes U(K)^- \to \mathbb{Q}_p\otimes X^-$$
$$\lambda_{\bar{\mathbb{Q}}} : \bar{\mathbb{Q}}\otimes U(K)^- \to \mathbb{Q}_p\otimes X^-$$
$$\lambda_{\mathbb{Q}_p} : \mathbb{Q}_p\otimes U(K)^- \to \mathbb{Q}_p\otimes X^- .$$

It is known (3) that $\lambda_{\mathbb{Q}}$ and $\lambda_{\bar{\mathbb{Q}}}$ are always injections. Gross (3) has conjectured that $\lambda_{\mathbb{Q}_p}$ is also an injection.

To look at $\lambda_{\mathbb{Q}_p}$ in a nicer arithmetic way, namely in terms of a determinant (or "regulator"), we define the map

1. The conjectures of Gross and of Leopoldt are also discussed in the article of J.-F. Jaulent "Sur les conjectures de Leopoldt et de Gross", in this volume.

$$\phi : U(K)^- \to X^-$$
$$\varepsilon \to \sum_{\mathfrak{P}|\mathfrak{p}} \text{ord}_p \left(N_{K_{\mathfrak{P}}/\mathbb{Q}_p} \varepsilon \right) . \mathfrak{P} \quad .$$

Taking tensor products with $\mathbb{Q}_p$ (over $\mathbb{Z}$) induces a map

$$\phi_{\mathbb{Q}_p} : \mathbb{Q}_p \otimes U(K)^- \to \mathbb{Q}_p \otimes X^- .$$

The map $\phi_{\mathbb{Q}_p}$ is an isomorphism (as can easily be shown by writing down an explicit inverse $\phi_{\mathbb{Q}_p}{}^{-1}$ using the finiteness of the class number of K (2)). Gross' p-adic regulator is defined to be

$$R_K = \det (\lambda_{\mathbb{Q}_p} \circ \phi_{\mathbb{Q}_p}{}^{-1} \mid \mathbb{Q}_p \otimes X^-).$$

It is the determinant of an r x r matrix where

$r=r(K,p)$ = the number of primes of K^+ which divide p and split in K. Gross' conjecture that $\lambda_{\mathbb{Q}_p}$ is an injection is equivalent with the inequality $R_K \neq 0$. This nonvanishing has been proved by transcendental methods if $K/\mathbb{Q}$ is abelian (3) and it is also known to hold if $r \leq 1$ (2). We remark that if L is any number field, not necessarily CM, Gross (3) has associated with it a p-adic regulator R_L. - The definition is more complicated but $R_L \neq 0$ for all number fields L if and only if $R_K \neq 0$ for all CM number fields K. Let $\chi : Gal(K/K^+) \to \{\pm 1\}$ denote the odd character corresponding to K/K^+ and ω the Teichmüller character. The determinant R_K is conjecturally related to the coefficient of s^r in the Taylor expansion of the p-adic L-function $L_p(\chi\omega, s)$ around s=0 (2). In particular, this coefficient is conjectured to be non-zero if and only if $R_K \neq 0$.

The conjectured connection of R_K with p-adic L-functions and the Main Conjecture of Iwasawa theory combine to suggest that the nonvanishing of R_K can be studied algebraically using Iwasawa theory. This was first done by Federer and Gross in (2) and following the introduction of some notation, we begin our algebraic study with one of their results. Let K_n denote the n-th layer of the basic $\mathbb{Z}_p$-extension K_∞ of K, $G_n = Gal(K_n/K) \simeq \mathbb{Z}/p^n\mathbb{Z}$, $\Gamma = Gal(K_\infty/K)$, and $\Gamma_n = Gal(K_\infty/K_n)$. Write N_n for the norm map from K_n to K.

Proposition 1: (2, (4.7)):
$R_K \neq 0$ if and only if the index $(U(K)^- \cap \mathbb{N}_n(K_n): \mu(K)(U(K)^-)^{p^n})$ is bounded as $n \to \infty$. //

For n sufficiently large, say for $n \geq N$, $U(K_n)^-/\mu(K_n)$ does not depend on n and hence the boundedness condition of Proposition 1 is equivalent to $(U(K)^- \cap \mathbb{N}_n(K_n^x): \mathbb{N}_n(U(K_n)^-))$ being bounded as $n \to \infty$. We therefore have

Proposition 2:
$R_K \neq 0$ if and only if $\#(\mathrm{Ker}(U(K)^-/\mathbb{N}_n(U(K_n)^-) \xrightarrow{i_n} K^x/\mathbb{N}_n(K_n^x))$ is bounded as $n \to \infty$ (where the map i_n is the natural map induced by inclusion).//

If A is a Γ-module and $m \geq n \geq 0$, define

$$\pi(n,m,A) : A^\Gamma/\mathbb{N}_n(A^{\Gamma_n}) \to A^\Gamma/\mathbb{N}_m(A^{\Gamma_m})$$
$$a \bmod \mathbb{N}_n(A^{\Gamma_n}) \to a^{p^{m-n}} \bmod \mathbb{N}_m(A^{\Gamma_m}).$$

In particular we have $\pi(N,m,U(K_\infty)^-)$: $U(K)^-/\mathbb{N}_n(U(K_n)^-) \to U(K)^-/\mathbb{N}_m(U(K_m)^-)$ and $\pi(n,m,K_\infty^x): K^x/\mathbb{N}_n(K_n^x) \to K^x/\mathbb{N}_m(K_m^x)$. Observe that $\pi(n,m,K_\infty^x) \circ i_n = i_m \circ \pi(n,m,U(K_\infty)^-)$ and hence we may take a direct limit of the ker (i_n) with respect to the $\pi(n,m,U(K_\infty)^-)$. Moreover, since $U(K)^- = \mu(K_n)U(K_N)^-$ for $n \geq N$, Proposition 2 yields

Proposition 3:
$R_K \neq 0$ if and only if $\ker(\varinjlim i_n) = \varinjlim (\ker i_n)$ is finite.//

Next note that the Dirichlet S-unit theoren and the equalities $U(K_n)^- = \mu(K_n)U(K_N)^-$ $(n \geq N)$ and $\{\pm 1\}\mathbb{N}_n(\mu(K_n)) = \mu(K)$ allow us to compute the structure of $\varinjlim_{\pi(n,m,U(K_\infty)^-)} U(K)^-/\mathbb{N}_n(U(K_n)^-)$. More precisely we have

Proposition 4:
$\varinjlim_{\pi(n,m,U(K_\infty)^-)} U(K)^-/\mathbb{N}_n(U(K_n)^-) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r$. Hence (by Proposition 3),
$R_K \neq 0$ if and only if $\mathrm{Im}(\varinjlim i_n) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r$.//

Taking direct limits with respect to maps $\pi(n,m,A)$ and the condition of Proposition 4 are made more natural by introducing the language of Galois cohomology. To do this we use the following lemma.:

Lemma:
If A is a Γ-module and $m \geq n \geq 0$, then there is a commutative diagram

$$\begin{array}{ccc} A^\Gamma/\mathbb{N}_n(A^{\Gamma_n}) & \xrightarrow{\pi(n,m,A)} & A^\Gamma/\mathbb{N}_m(A^{\Gamma_m}) \\ \downarrow & & \downarrow \\ H^2(G_n,A^{\Gamma_n}) & \xrightarrow{\text{inflation}} & H^2(G_m,A^{\Gamma_m}) \end{array}$$

where the lefthand map is cup product by a generator σ_n *of* $H^2(G_n,\mathbb{Z})$, *the righthand map is cup product by a generator* σ_m *of* $H^2(G_m,\mathbb{Z})$, *and the inflation of* σ_n *in* $H^2(G_m,\mathbb{Z})$ *is equal to* $\sigma_m{}^{p^{m-n}}$. *Hence* $\varinjlim_{\pi(n,m,A)} A^\Gamma/\mathbb{N}_n(A^{\Gamma_n}) \simeq \varinjlim_{\text{inflation}} H^2(G_n,A^{\Gamma_n}) \simeq H^2(\Gamma,A)$. *Moreover, if* $\phi: A \to B$ *is a homomorphism of* Γ-*modules, then the diagram*

$$\begin{array}{ccc} \varinjlim_{\pi(n,m,A)} A^\Gamma/\mathbb{N}_n(A^{\Gamma_n}) & \xrightarrow{\sim} & H^2(\Gamma,A) \\ \downarrow & & \downarrow \\ \varinjlim_{\pi(n,m,B)} B^\Gamma/\mathbb{N}_n(B^{\Gamma_n}) & \xrightarrow{\sim} & H^2(\Gamma,B) \end{array}$$

is commutative where the horizontal isomorphisms are each defined with respect to the same sequence $\{\sigma_n\}$ *and the vertical maps are the natural maps induced by* ϕ. //

Applying this lemma to the situation of Proposition 4, we find

Proposition 5:
$R_K \neq 0$ *if and only if* $\mathrm{Im}(H^2(\Gamma,U(K_\infty)^-) \xrightarrow{f} H^2(\Gamma,K_\infty{}^\times)) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r$ *where the cohomology map f is induced by the natural inclusion* $U(K_\infty)^- \hookrightarrow K_\infty{}^\times$.//

We note that the equivalence of Proposition 5 was first observed by Iwasawa using a beautiful but more complicated argument (1, §28), (also see (4), (5)).

The homomorphism f factors through $H^2(\Gamma,K_\infty{}^{\times -})$ and the natural map $H^2(\Gamma,K_\infty{}^{\times -}) \to H^2(\Gamma,K_\infty{}^\times)$ has kernel and cokernel killed by 2 (it is an isomorphism for $p\neq 2$.). Hence $\mathrm{Im}(f)\simeq(\mathbb{Q}_p/\mathbb{Z}_p)^r$ precisely when $\mathrm{Im}(g)\simeq(\mathbb{Q}_p/\mathbb{Z}_p)^r$ where g is the Γ-cohomology map induced by the inclusion $U(K_\infty)^- \hookrightarrow K_\infty{}^{\times -}$. The advantage of working with the map g rather than f is that, as we next explain, g appears naturally in an exact sequence of Γ-cohomology groups.

Define the ring $\theta(n,p) = \{\alpha \in K_n \mid \mathrm{ord}_{\mathfrak{p}}\, \alpha \geq 0 \ \forall\ \mathfrak{p} \nmid p\}$ and let $\theta(n) = \{\alpha \in K_n \mid \mathrm{ord}_{\mathfrak{p}}\, \alpha \geq 0\ \forall \mathfrak{p}\}$, the ring of integers of K_n. Let $P(n,p)$ denote the group of principal $\theta(n,p)$-ideals and $P(n)$ the group of principal $\theta(n)$-ideals. Then we have exact sequences of G_n-modules

$$\text{(i)} \qquad 1 \to U(K_n)^- \to K_n^{\times -} \to P(n,p)^{1-J} \to 1$$
$$\alpha \to \alpha\theta(n,p)$$

and

$$\text{(ii)} \qquad 1 \to \mu(K_n) \to K_n^{\times -} \to P(n)^{1-J} \to 1$$
$$\alpha \to \alpha\theta(n)$$

For $n=\infty$, the map g appears in the Γ-cohomology sequence associated to (i) and exactness yields $\mathrm{Im}(g) \simeq \ker\,(H^2(\Gamma,K_\infty^\times) \xrightarrow{h} H^2(\Gamma,P(\infty,p)^{1-J})$. The natural map h factors through $H^2(\Gamma,\ P(\infty)^{1-J})$ and since $\#(H^i(\Gamma,\mu(K_\infty))) = 1$ or 2 ($\forall\, i\geq 1$), the exactness of the Γ-cohomology sequence associated to (ii) tells us that $\ker(h) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r$ precisely if the natural cohomology map $H^2(\Gamma,P(\infty)^{1-J}) \to H^2\,(\Gamma,\ P(\infty,\ p)^{1-J})$ has kernel isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$. Recalling the lemma preceeding Proposition 5 and the assertion of Proposition 5, we therefore find

Proposition 6:

Let Ψ: $\varinjlim_{\pi(n,m,P(\infty)^{1-J})} (P(\infty)^{1-J})^\Gamma \,/\, \mathbb{N}_n((P(\infty)^{1-J})^{\Gamma_n}) \longrightarrow$

$\varinjlim_{\pi(n,m,P(\infty)^{1-J})} (P(\infty,p)^{1-J})^\Gamma / \mathbb{N}_n\,((P(\infty,p)^{1-J})^{\Gamma_n})$ *signify the homomorphism induced by the natural surjection* $P(\infty)^{1-J} \twoheadrightarrow P(\infty,p)^{1-J}$.

Then $R_K \neq 0$ *if and only if* $\mathrm{Ker}\,(\Psi) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r$.//

Our reason for having passed back to direct limits is that we find it more natural to describe the kernel of Ψ than the kernel of the corresponding map of Γ-cohomology groups. On the otherhand, while the preceeding discussion could have been carried out without mentioning Galois cohomology (i.e. always working with limits), the author believes that the cohomology makes things more natural and suggests possible interpretations of Gross' conjecture in terms of group extensions or division algebras.

Let I signify the set of places of K^+ which lie above p and split in K. It is a finite set with r elements. Let h denote the class number of K. If $\mathfrak{p}\in I$ with $\theta(K)=\mathfrak{P}\mathfrak{P}^J$, then $\sigma_{\mathfrak{p}} = (\mathfrak{P}^h\theta(\infty))^{1-J}$ is a nontrivial element of $(P(\infty)^{1-J})^\Gamma$. One may define $b_{\mathfrak{p}}$ to be the smallest non-negative integer satisfying

$$\sigma_{\mathfrak{p}}^{p^n} \notin \mathbb{N}_{b_{\mathfrak{p}}+n}\,((P(\infty)^{1-J})^{\Gamma_{b_{\mathfrak{p}}+n}}) \text{ for all } n\geq 0.$$

If $n \geq b_{\mathfrak{p}}$, define

$$\varepsilon_{\mathfrak{p},n} = (a_i)_{i \geq n} \in \varinjlim_{\pi(i,j,P(\infty)^{1-J})} ((P(\infty)^{1-J})^{\Gamma}) / \mathbb{N}_i((P(\infty)^{1-J})^{\Gamma_i})$$

by

$$a_i = \alpha_{\mathfrak{p}}^{p^{i-n}} \mod \mathbb{N}_i\,((P(\infty)^{1-J})^{\Gamma_i}).$$

Then $\varepsilon_{\mathfrak{p},n}$ is an element of order $p^{n-b_{\mathfrak{p}}+1}$ and if $m \geq n \geq b_{\mathfrak{p}}$, then $\varepsilon_{\mathfrak{p},m}{}^{p^{m-n}} = \varepsilon_{\mathfrak{p},n}$. Hence, letting n run over all $n \geq b_{\mathfrak{p}}$, the $\varepsilon_{\mathfrak{p},n}$ generate a group $\langle\varepsilon_{\mathfrak{p},n}\rangle \simeq (\mathbb{Q}_p/\mathbb{Z}_p)$. Note that while the definition of $\alpha_{\mathfrak{p}}$, and hence of $\varepsilon_{\mathfrak{p},n}$, depends on a choice of which prime dividing p is called $\mathfrak{P}$, the group $\langle\varepsilon_{\mathfrak{p},n}\rangle$ is independent of this choice. Moreover, since $\mathfrak{p}$ divides p, $\langle\varepsilon_{\mathfrak{p},n}\rangle$ is a subgroup of $\ker(\Psi)$. Let $\langle\varepsilon_{\mathfrak{p},n} | n \geq b_{\mathfrak{p}}, \mathfrak{p} \in I\rangle$ denote the group generated by the $\varepsilon_{\mathfrak{p},n}$ as p runs over I. -It is a subgroup of finite index in $\ker(\Psi)$ as may be seen from the exactness of the Γ-cohomology sequences attached to (i) and (ii). We therefore have

Proposition 7:

$R_K \neq 0$ if and only if $\langle\varepsilon_{\mathfrak{p},n} | \, n \geq b_{\mathfrak{p}}, \mathfrak{p} \in I\rangle \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r$.//

The author hopes to give an account in a subsequent paper of what Proposition 7 tells us about what a contradiction to Gross' conjecture would imply. In particular, a contradiction to $\langle\varepsilon_{\mathfrak{p},n} \mid n \geq b_{\mathfrak{p}}, \mathfrak{p} \in I\rangle \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^r$ allows the construction of a non-basic $\mathbb{Z}_p$-extension and makes one think of Leopoldt's conjecture.

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