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**A Geometric Interpretation
of Lannes' Functor T .**

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1. Introduction. In this note we are concerned with a question raised by [Lannes 2.3]. In what follows R will denote a finite field of the form $\mathbf{Z}/p\mathbf{Z}$, homology and cohomology are always taken with coefficient in R and denoted by H_*X etc. For a space X let $\{R_s X\}_s$ denote the Bousfield-Kan localization tower. We denote by $B\tau$ the classifying space of the underlying abelian group of R . Let $P_s X$ denote the s -Postnikov section of X . By a "space" we mean a Kan complex or a C.W. complex.

1.1 Theorem: *If $H^i X < \infty$ for all $i \geq 0$, then $TH^*X \cong \varinjlim_s H^*(P_s R_s X)^{B\tau}$, where T is Lannes' functor (see below). If, in addition, X is nilpotent then $TH^*X \cong \varinjlim$*

$$H^*(P_s X)^{B\tau} \cong \varinjlim H^*(P_s R_\infty X)^{B\tau}$$

The proof of this theorem yields a new proof for Lannes theorem 1.5 below that essentially asserts 1.1 for dimension zero and was a the motivation for his question [Lannes 2.3]. The proof of theorem is based on the following technical proposition:

1.2 Proposition: *Let $G \rightarrow E \rightarrow B$ be a principal fibration where G is a (topological or simplicial) group. Assume that in each dimension the R -cohomology of the mapping spaces E^{B^t} and B^{B^t} is finite. Then if the relation $TH^*W \cong H^*W^{B^t}$ is satisfied by $W = E$ and $W = B$ then it is also satisfied by $W = G$.*

Remark: The finiteness assumption, noted by the referee, is necessary in order to use cohomological Eilenberg-Moore spectral sequence.

Corollary: *If W is a nilpotent space of finite type with $\pi_i W = 0$ for $i \gg 0$ or a R -localization thereof then*

$$TH^*W \cong H^*W^{B\tau}.$$

Further, as a direct corollary of 1.1 and 9.3 of [Bousfield] one gets the following interesting special case due to Lannes [4].

1.3 Corollary. *Let $H^i X < \infty$ for all $i \geq 0$ for nilpotent space X of finite type. Assume that a given algebraic component $T_c H^* X$ of $TH^* X$ is finite in all dimensions and vanishes in dimension one. Then $T_c H^* X \cong H^* X_c^{B\tau} \cong H^*(R_\infty X)_c^{B\tau}$*

where $X_c^{B\tau}$ is the corresponding component.

Another example where the main result (1.1) implies a result on $H^* \text{map}(B\tau, X)$ is when the latter has a finite homotopy group in each dimension.

1.4 Corollary: *Let X be nilpotent space of finite type with $\pi_1 X$ finite. Assume that for a given $f : B\tau \rightarrow X$ the groups $\pi_i \text{map}(B\tau, P_n X)_{f_n}$ are finite for all $i, n \geq 0$. Then $H^* \text{map}(B\tau, X) \cong T_c H^* X$ where T_c is algebraic component of T that corresponds to f .*

The referee also notes that theorem 1.1 gives a new proof of the following result [Lannes, 0.4].

1.5 Corollary: *If Y is a nilpotent space with $H^n(Y, R)$ finite for all n , then the natural map*

$$[Bt, Y] \cong [Bt, R_\infty Y] \rightarrow \text{Hom}_K(H^* Y, H^* B_i)$$

is an isomorphism of profinite sets.

Proof: This follows directly from 1.1 above in light of the algebraic fact [Lannes 3.5] and the old result of [Dror] about the tower $R_* Y$.

The authors would like to thank W. Dwyer for his suggestion to consider the tower $R_* X$ as a starting point for a geometric interpretation of T , and to H. Miller for several useful discussions. The authors would also like to thank the referee for his careful reading and for correcting a non-fatal error in an earlier version of this paper. The referee notes that if one considers the fibration $\Omega X \rightarrow * \rightarrow X$ for X being the infinite wedge of $\mathbb{R}P^\infty / \mathbb{R}P^{2n+2}$ over the integers, the formula in 1.2 holds for $W = \Omega X$ but not for X itself. Therefore one cannot turn around 1.2 to conclude that either E or B satisfy the property $TH^* W = H^* W^{B_i}$, assuming the other two spaces do.

2 First examples.

Let \mathcal{U} denote the category of unstable modules over the algebra \mathcal{A} of Steenrod reduced powers relative to a prime $p = \text{char } R$. Let \mathcal{K} denote the category of unstable \mathcal{A} -algebra. In [Lannes] a left adjoint T is defined to the functor $- \otimes H^* B\tau$, where the latter is taken

either as a functor from \mathcal{U} to itself or from \mathcal{K} to itself. If one regards an element $A \in \mathcal{K}$ as an element of \mathcal{U} , the value of T does not change. Thus the defining property of T is $hom_C(TM, N) = hom_C(M, N \otimes H^*B\tau)$ where C is either \mathcal{U} or \mathcal{K} .

2.1 Three basic properties [Lannes]: (i) T is exact. (ii) T commutes with tensor products. (iii) T commutes with direct limits.

2.2 Motivation: It can be seen from 1.1, 1.2, 1.3 that the construction of T is motivated by attempts to describe the cohomology of $X^{B\tau} \equiv map(B\tau, X)$ in terms of H^*X , when the latter is given as an object in \mathcal{K} . Lannes proves the relation between the homotopy class $[B\tau, X]$ and $(TH^*X)^0$ and X as in 1.3, see [Lannes 7.1.1]. [Miller] proves it for $dim X < \infty$.

2.3 Example. It is easy to calculate directly from the adjointness relation that if V is a finite dimensional vector space over R then

$$TH^*K(V, n) \cong \bigotimes_{n \geq i \geq 0} H^*K(V, i) \cong H^*map(B\tau, K(V, n)).$$

Here $map(X, Y)$ denotes the space of maps $X \rightarrow Y$ otherwise denoted by Y^X . Similar calculation holds for a finite products of $K(V_i, n_i)$ with $dim_R V_i < \infty$. However it turns out that for homotopically large space one cannot, in general, interpret TH^*X as the cohomology of $map(B\tau, X)$, (see 2.5 below).

2.4. Example. An important class of spaces on which T behaves nicely are finite Postnikov pieces of nilpotent spaces. The prime examples of such spaces are $K(\mathbf{Z}, n)$ spaces for $n \geq 0$.

Proposition: For any $n \geq 0$ there is an isomorphism $TH^*K(\mathbf{Z}, n) \cong H^*map(B\tau, K(\mathbf{Z}, n))$.

Proof: For $p = 2$ we show by a direct computation that $TH^*K(\mathbf{Z}, n) \cong H^* \prod_{i=1}^{\lfloor n/2 \rfloor} K(\mathbf{Z}/2\mathbf{Z}, 2i)$. For $p > 2$ the argument is similar. Now since $H^*K(\mathbf{Z}, n) \cong P(S_q^I \mid I \text{ admissible with } e_1(I) \geq 2 \text{ and } e(I) \leq n-1)$ a map of the algebra $H^*K(\mathbf{Z}, n)$ over A is given by the image of the generator in dimension n . Thus

$$hom_{\mathcal{K}}(K(\mathbf{Z}, n), K) = ker \beta : K_n \rightarrow K_{n+1}$$

where K is any object in \mathcal{K} and β is The Bockstein operation. Now compute:

$$\begin{aligned}
 \text{hom}_{\mathcal{K}}(TH^*K(\mathbf{Z}, n), K) &\cong \text{hom}_{\mathcal{K}}(H^*K(\mathbf{Z}, n), K \otimes H^*B\tau) \\
 &\cong \ker\beta : K \otimes H^*B\tau)_n \rightarrow (K \otimes H^*B\tau)_{n+1} \\
 &\cong \ker\beta : \bigoplus_{i+j=n} K_i \otimes H^j B\tau \rightarrow \bigoplus_{i+j=n+1} K_i \otimes H^j B\tau \\
 &\cong \bigoplus_{\substack{i+j=n \\ j \text{ even}}} K_i = \\
 &\cong \text{hom}_{\mathcal{K}}(H^* \prod_{i=1}^{\lfloor n/2 \rfloor} K(\mathbf{Z}(2\mathbf{Z}, i), K).
 \end{aligned}$$

This together with the adjointness property of T completes the proof. Similarly let \mathbf{Z}_p denotes the p -adic numbers $\mathbf{Z}_p = \text{invlim } \mathbf{Z}/p^k\mathbf{Z}$. Then [B - K VI 6.4] one has an R homology equivalence $K(\mathbf{Z}, n) \rightarrow K(\mathbf{Z}_p, n)$ for all $n \geq 0$. There is a pro-isomorphism on R -homology of $K(\mathbf{Z}, n) \rightarrow (K(\mathbf{Z}/p^k\mathbf{Z}, n))_n$. Therefore

$$H^*K(\mathbf{Z}, n) \cong H^*K(\mathbf{Z}_p, n) = \varinjlim_k H^*K(\mathbf{Z}/p^k\mathbf{Z}, n)$$

Moreover it follows by a spectral sequence argument that the tower $\{map(B\tau, K(\mathbf{Z}/p^k\mathbf{Z}, k))\}_*$ is an R completion tower for the function complex $map(B\tau, K(\mathbf{Z}, n))$, since all function complexes involved here are R -nilpotent. Again using comparison of spectral sequences converging to homology we see that there is an R -homology (hence R -cohomology) equivalence $map(B\tau, K(\mathbf{Z}, n)) \rightarrow map(B\tau, K(\mathbf{Z}_p, n))$. Therefore the R -cohomology of the last range is isomorphic to the limit of the R -cohomologies $\varinjlim_k H^*map(B\tau, K(\mathbf{Z}/p^k\mathbf{Z}, n))$. But since the functor T commutes with direct limits we get the desired example:

$$TH^*K(\mathbf{Z}_p, n) \cong H^*map(B\tau, K(\mathbf{Z}_p, n)).$$

The second example of $K(\mathbf{Z}_p, n)$ is in reality equivalent to the first using the isomorphism of cohomologies $H^*(B\tau, \mathbf{Z}) \cong H^*(B\tau, \mathbf{Z}_p)$. Since the function complexes $hom(B\tau, K(\mathbf{Z}, n))$ and $hom(B\tau, K(\mathbf{Z}_p, n))$ are built out of these cohomology groups, the map $\mathbf{Z} \rightarrow \mathbf{Z}_p$ induces a homotopy equivalence between them. Now since $TH^*K(\mathbf{Z}, n) \cong TH^*K(\mathbf{Z}_p, n)$ one satisfies 2.4 if and only if the other does.

2.5 Example. It is not hard to check that if V is an infinite dimensional vector space over R then 2.3 fails to hold.

Similarly, let $RB\tau$ be the free (simplicial) R -module generated by $B\tau$, then R has non-trivial homotopy groups in all dimensions and $H^0\text{map}(B\tau, RB\tau)$ is larger than $T^0H^*B\tau$ which is countable.

3. Proof of 1.2. The main observation of this note is (1.2) from which (1.1) and (1.3) follow. We use the Eilenberg Moore spectral sequence (EMSS) to gain information on H^*W as an object in \mathcal{U} , i.e. as an unstable module over the Steenrod algebra \mathcal{A} . D. Rector, L. Smith, A. Heller and others showed that there is a natural action on the Eilenberg-Moore spectral sequence $E^r(W \rightarrow E \xrightarrow{u} B)$ by \mathcal{A} making the differentials \mathcal{A} -linear and such that E_∞ is a graded \mathcal{A} -module associated to a filtration:

3.1. $H^*W \supseteq \dots \supseteq F^{-2} \supseteq F^{-1} \supseteq F^0 \supseteq 0 \supseteq 0 \supseteq \dots$ of H^*W by \mathcal{A} -submodules. We shall need the following result of [Dwyer] that gives a necessary and sufficient condition for a strong convergences of the spectral sequence: For every n the above filtration of H^*W is *finite* iff $\pi_1 B$ operates nilpotently on H^i (fibre) for all i .

3.2 Observe that if $p : E \rightarrow B$ is a fibre map with B not necessarily connected and with $\pi_1(B, *)$ operates nilpotently on $H^i(p^{-1}(*))$, then EMSS of $(* \rightarrow B \leftarrow E)$ will be identical to the one associated to the connected component of $* \in B$ and therefore will likewise converge strongly to $H^*(p^{-1}(*))$. This is because the functor Tor_{H^*B} appearing in E_2 ‘eliminates’ all the components of H^*E not hitting the component of $* \in B$ in H^*B , due to the trivial action of off base point elements in H^*B on $H^*(b) = R$.

Claim: If L is any space of then the Eilenberg-Moore spectral sequence for the fibre square

$$(3.3) \quad \begin{array}{ccc} \text{map}(L, W) & \rightarrow & \text{map}(L, E) \\ \downarrow & & \downarrow \bar{u} \\ \text{map}(L, *) & \rightarrow & \text{map}(L, B) \end{array}$$

induced by the fibration in (3.1) converges strongly.

Since $\text{map}(L, *) = *$ the above pull back square is, in fact, a fibre map \bar{u} with a non-connected space $\text{map}(L, b)$ as the base and with $\text{map}(L, G)$ as the fibre are the component

of the null homotopic maps in the base.

3.4 Lemma: *If $G \rightarrow E \rightarrow B$ is a principal fibration of spaces where G is a group, then for any space L the map $\text{map}(L, E) \rightarrow \text{map}(L, B; E)$, where the range is the space of maps $L \rightarrow B$ liftable to E , is a principal fibration with the group being $\text{map}(L, G)$ and the action is pointwise.*

Proof: One checks directly that the pointwise action is a transitive action of $\text{map}(L, G)$ on the fibres of the above maps.

It follows from the above that the EMSS of (3.3) converges strongly, and as argued above the E^s -terms are the same for the fibrations $\text{map}(L, E; \text{null homotopic on } B) \rightarrow \text{map}(L, B; \text{null homotopic})$.

Now we wish to compare the Eilenberg-Moore spectral sequence of (3.4) to the spectral sequence gotten by applying T to the Eilenberg-Moore sequence of the given fibration $W \rightarrow E \xrightarrow{u} B$. Let $E_r(u)$ be the spectral sequence of the fibration u .

For each $r \geq 1$ and $s \leq 0$ the \mathbf{Z} -graded objects $E_1^{s,*}, E_2^{s,*}$ are unstable modules over the Steenrod algebra since the first one is, being the cohomology of the space

$B \times B \times \cdots \times B \times E$ (product taken s times). (Notice, however, that if we grade $\{E_r^{p,q}\}$ by the total degree $p+q$, we do not get an unstable module, but rather a stable one - e.g. Sq^i can operate non-trivially on elements of bi-degree $(-s, s)$, for any $s \geq i > 0$). Therefore, we can form a spectral sequence $\{TE_r; Td_r\}$ by applying T to each $E_r^{-s,*}(u)$ as an object in U , to get another object in U namely $TE_r^{-s,*}$.

3.5. Claim. $TE_r(u)$ converges to TH^*W in the sense that $TE_\infty^{s,*}(u)$ is associated graded \mathcal{A} -module to the \mathcal{A} -filtration TF^i with $\varinjlim TF^i \cong TH^*W$.

To see why notice (2.1) that T is exact so it converts an exact couple to an exact couple, and since all the terms in spectral sequence $E_r^{s,*}(u)$ are \mathcal{A} -modules in \mathcal{U} and all derivations \mathcal{A} -maps one can apply the functor T to get another spectral sequence. Since T is a left adjoint it commutes with direct limits so that $H^*W = \varinjlim F^i$ implies what we need.

Let $E_r(\bar{u})$ be the spectral sequence of the fibre-square (3.4) for $L = B\tau$. One can construct a comparison map $TE_r(u) \rightarrow E_r(\bar{u})$ using the adjointness properties of T : the

evaluation map $B\tau \times \text{map}(B\tau, X) \rightarrow X$ induces [Lannes] a map

$$TH^*X \rightarrow H^*\text{map}(B\tau, X).$$

Therefore there is a natural map of \mathcal{A} -modules

$$TE_r(u) \rightarrow E_r(\bar{u}).$$

Claim: Under the assumption of lemma 1.2, this map is an isomorphism.

Proof: First notice that if $K \in \mathcal{K}$ and $M, N \in \mathcal{U}$ are K -modules then $T(A \otimes_{\mathcal{K}} B) = TA \otimes_{TK} TB$. This is because $A \otimes_{\mathcal{K}} B$ is the cokernel of a difference map $A \otimes K \otimes B \rightarrow A \otimes B$ of the two operation of K . Now T commutes with \otimes in \mathcal{U} so TN, TM are TK -modules and again by commutation and (right) exactness of T (2.3) we get the tensor product. Next notice that since the unstable \mathcal{A} -model $Tor_s(M, N)$ is the s -homology group of a chain complex consisting in degree s of $M \otimes K \otimes K \otimes \dots \otimes K \otimes N$ and since T preserves tensor products one has for all $s \geq 0$.

$$T(Tor_K^s(M, N)) = Tor_{TK}^s(TM, TN).$$

By assumption on the space E and by (2.3) we get the desired result. Thus we have an isomorphism for $r = 2$ thus for all r .

It follows that one has an isomorphism $TE_\infty(u) \cong E_\infty(\bar{u})$. Now we get for each submodule in the filtration an isomorphism:

$$TF^i(u) \cong F^i(\bar{u})$$

and taking direct limits, noting again that T commutes with direct limits, we get the desired result by comparison of spectral sequence, namely

$$TH^*W \cong H^*\text{map}(B\tau, W).$$

Thus using the bar construction of the EMSS we saw that $E_1^{s,*}$ is an unstable module over A . This proves 1.2.

4. Proof of 1.1 and 1.3. If $H^i X < \infty$ for all i , then $\pi_i P_s R_s X < \infty$ for all $s, i \geq 0$ where P_s is the Postnikov section. This means that the space $P_s R_s X$ satisfies the conditions of (1.2) and we have $TH^* P_s R_s X = H^* \text{map}(B\tau, P_s R_s X)$. But since $H_i X \rightarrow \{H_i R_s X\}$ is a pro-isomorphism of towers [Dror] of finite groups, we have $H^* X \cong \varprojlim H^* P_s R_s X$ therefore (2.3)(iii) $TH^* X \cong \varprojlim TH^* P_s R_s X = \varprojlim H^* \text{map}(B\tau, H^* P_s R_s X)$, This gives 1.1.

Now one uses the following lemma of [Bousfield 9.3]. Consider a tower of fibrations of pointed R -nilpotent spaces $\{X_m\}$

4.1. Lemma. *If $\{\tilde{H}_i(X_m, R)\}_m$ are pro-trivial for $i \leq 1$ and pro-constant for all i , then $\varprojlim X_m = X_\infty$ is simply connected and the map $H_i(X_\infty, R) \rightarrow \{H_i(X_m, R)\}$ is a pro-isomorphism for all i .*

We use (4.1) with $X_m = P_m R_m X$.

Notice that if $\{H_m\}$ is an inverse tower of finite groups with $A_\infty = \varprojlim A_m$ a finite group then the map $A_\infty \rightarrow \{A_m\}_m$ is a pro-isomorphism, because $\varprojlim^1 (-)$ vanishes on tower of finite groups and \varprojlim is left exact. Consider the tower $H_i(X_s^{B\tau})_c = (H^i(X_s^{B\tau}))^*_c$ where $(-)^*$ denotes the R -dual. By 1.2 this is a tower of finite groups since one considers only a component $(X_s^{B\tau})_c$ for which T_c is finite in all dimension. Therefore $H_0(X_s^{B\tau})_c \cong R$ the tower $H_1(X_s^{B\tau})_c$ is pro-trivial, being pro-isomorphic to $(T_c^1 H^*(X))^*$. Therefore by lemma 4.1 the tower $\{H_i(X_s^{B\tau})_c\}_{s \geq 0}$ is pro-isomorphic to $H_i(\varprojlim (X_s^{B\tau})_c)$.

Since for any tower $\cdots \rightarrow Y_{m+1} \rightarrow Y_m \rightarrow \cdots \rightarrow Y_0$ of fibrations taking inverse limit commutes with taking function complex $\text{map}(L, -)$ the desired conclusion follow from $\varprojlim (X_s)^{B\tau} \sim (\varprojlim X_s)^{B\tau} = X^{B\tau}$, since X is assumed to be R -nilpotent.

4.2. Proof of 1.4. By [Lannes 7.1.1] we have again $H^0 \text{map}(B\tau, X) \cong T^0 H^* X$ so that as in 1.3. $T_c H^* X$ is a well defined component corresponding to $[f] \in [B\tau, X]$. We have $X_f^{B\tau} = \varprojlim \text{map}(B\tau, P_n X)_{f_n}$ where f_n is the obvious composition $B\tau \xrightarrow{f} X \rightarrow P_n X$. Since all the relevant homotopy groups are finite, one gets vanishing \varprojlim^1 - term and thus $\pi_i X_f^{B\tau} \cong \varprojlim \pi_i(\text{map}(B\tau, P_n X), f_n)$. But, again this means that there is a pro-isomorphism $\tau_i X_f^{B\tau} \cong \{\pi_i(\text{map}(B\tau, P_n X), f_n)\}_n$ for each i , so that the constant tower $X_f^{B\tau}$ is pro-equivalent to the tower $\{P_n X\}_{f_n}^{B\tau}$. Thus, they have the same R - cohomology.

But the R - cohomology of the latter is pro-isomorphic to $TH^* X$ as needed.

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