TOM MEURMAN

A simple proof of Voronoi's identity

Astérisque, tome 209 (1992), p. 265-274
<http://www.numdam.org/item?id=AST_1992__209__265_0>

A SIMPLE PROOF OF VORONOI'S IDENTITY

Tom MEURMAN

1 Introduction

Denote by $d(n)$ the number of divisors of n. Consider the function $\Delta(x)$ satisfying

$$\sum_{n<x} d(n) + \frac{1}{2}d(x) = x\log x + (2\gamma - 1)x + \frac{1}{4} + \Delta(x), \tag{1.1}$$

where $d(x) = 0$ unless x is an integer, and γ is Euler's constant. We have the following well-known and remarkable theorem:

THEOREM. *The function $\Delta(x)$ defined by (1.1) satisfies*

$$\Delta(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{n}\int_0^{\infty}\cos(2\pi u)\sin(2\pi\frac{nx}{u})du. \tag{1.2}$$

Moreover, the series here is boundedly convergent in any interval $[x_1, x_2] \subset (0, \infty)$, and uniformly convergent in any such interval free from integers.

This was proved originally by Voronoi [16], and later by many others, e.g. [1], [3] (or [4, Ch. VIII]), [5] (or [13, Ch. 1]), [6] and [7] combined, [10, Ch. I], [11], [14], [15]. The proofs are usually long and difficult. Most of them depend on the functional equation for the Riemann zeta-function, and the simplest of these is due to Jutila [10]. The simplest proof not depending on the functional equation is due to Landau [11]. Jutila uses the method of Dixon and Ferrar [5] except for the case when x is an integer, where he introduces a substantial simplification. Landau uses Poisson's summation formula and only real analysis. The detailed proof takes about 20 pages.

My purpose is to give a comparatively simple proof not depending on the functional equation. It has some features in common with Landau's proof, but it is shorter and the principle is different. Indeed, the principle can be used (as I shall show in a subsequent paper) in the analogous problem where

$d(n)$ is replaced by $a(n)$, the nth Fourier coefficient of a holomorphic modular cusp form. Landau's method fails here.

The method of this paper works also when $d(n)$ is replaced by $d(n)\exp(2\pi iqn)$ or by $r(n)\exp(2\pi iqn)$, where q is rational and $r(n)$ signifies the number of ways of representing n as the sum of two squares. It probably works also for $a(n)\exp(2\pi iqn)$. Moreover, there are other more obvious generalizations. This may be of some interest, since the modern theory of generalizations is based invariably on the functional equations for the relevant Dirichlet series (see e.g. Berndt [2]).

As to the significance, generalizations or analogues of Voronoi's identity the reader may consult [9, Chapters 3, 13, 15], [10, Ch. I], [12], [2] and [8]. The last two papers contain many references to the literature of the subject.

The integral in (1.2) is usually expressed in terms of the Bessel functions Y_1 and K_1 as follows (see Watson [17]):

$$-\sqrt{nx}(K_1(4\pi\sqrt{nx}) + \frac{\pi}{2}Y_1(4\pi\sqrt{nx})).$$

It is of practical value that this admits sharp approximations involving elementary functions. However, I prefer (1.2) as it stands, since, as will be shown, it is quite easy to obtain such approximations for the integral in (1.2). It would be an unnecessary detour to invoke Watson. This appears not to have been observed before. Landau evaluates the integral in (1.2) within an error $O(1)$ only, and this is not enough for practical purposes.

The principle of the proof of the theorem will be explained in the following section. The most important new idea lies in the use of the weight function $\exp(-v(u + nx/u))$.

2 The principle

For $0 \le v \le \frac{1}{2}$ define

$$\Delta(x, v) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{d(n)}{n}F(nx, v), \tag{2.1}$$

where

$$F(t, v) = \int_0^\infty e^{-v(u+t/u)}\cos(2\pi u)\sin(2\pi t/u)du. \tag{2.2}$$

Fix t for a moment. Since $\frac{d}{du}(e^{-v(u+t/u)}\sin(2\pi t/u)) \ll u^{-2}$, it follows easily by partial integration that $F(t, v)$ is uniformly convergent in $[0, \frac{1}{2}]$. Moreover, for any $\varepsilon > 0$ we have $\lim_{v\to 0+} e^{-v(u+t/u)} = 1$ uniformly in $[\varepsilon, 1/\varepsilon]$. Hence

$$\lim_{v\to 0+} F(t, v) = F(t, 0). \tag{2.3}$$

Suppose that the following lemma holds:

LEMMA. *For $t \geq t_0 > 0$ and $0 \leq v \leq \frac{1}{2}$ we have*

$$F(t,v) = \frac{1}{2\sqrt{2}} t^{\frac{1}{4}} e^{-2v\sqrt{t}} \cos(4\pi\sqrt{t} - \frac{\pi}{4}) + O(t^{-\frac{1}{4}}), \tag{2.4}$$

where the implied constant in the O-term may depend on t_0 only.

Fix x for a moment. If $\Delta(x,0)$ is convergent then, since $e^{-2v\sqrt{nx}}$ is monotonic, it follows by partial summation that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n} F(nx,0) e^{-2v\sqrt{nx}}$$

is uniformly convergent in $[0, \frac{1}{2}]$. By the lemma,

$$F(nx,v) = F(nx,0)e^{-2v\sqrt{nx}} + O(n^{-\frac{1}{4}}).$$

Hence also the series $\Delta(x,v)$ is uniformly convergent in $[0, \frac{1}{2}]$. By (2.3) this implies that

$$\lim_{v \to 0+} \Delta(x,v) = \Delta(x,0).$$

Then $\Delta(x) = \Delta(x,0)$, i.e. (1.2) holds, if

$$\lim_{v \to 0+} \Delta(x,v) = \Delta(x). \tag{2.5}$$

Therefore, to prove the theorem, we must prove the lemma, the statement (2.5) and the assertions concerning the convergence of $\Delta(x,0)$.

3 Proof of the lemma

The argument to be given here could easily be refined to yield an asymptotic series.

Suppose first that $v > 0$ in (2.2). Clearly

$$F(t,v) = \sqrt{t} \int_0^{\infty} e^{-v\sqrt{t}(u+1/u)} \cos(2\pi\sqrt{t}u) \sin(2\pi\sqrt{t}/u) du.$$

The main term will come from the values of u near 1. Since

$$\cos(\alpha)\sin(\beta) = \frac{1}{2}\mathrm{Im}(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)}),$$

it follows that

$$F(t,v) = \frac{1}{2}\sqrt{t}\mathrm{Im}(F^+ - F^-),$$

where

$$F^{\pm}=\int_0^{\infty} e^{(-v(u+1/u)+2\pi i(u\pm 1/u))\sqrt{t}}du.$$

Consider F^- and denote $g(u)=-v(u+1/u)+2\pi i(u-1/u)$. Integrating twice by parts one gets

$$F^-=\frac{1}{t}\int_0^{\infty}\left(\frac{1}{g'(u)}\left(\frac{1}{g'(u)}\right)'\right)' e^{g(u)\sqrt{t}}du,$$

since, as $v>0$, the integrated terms vanish at each step. We have

$$\left(\frac{1}{g'}\left(\frac{1}{g'}\right)'\right)'=-\left(\frac{g''}{(g')^3}\right)'\ll\left|\frac{g'''}{(g')^3}\right|+\left|\frac{(g'')^2}{(g')^4}\right|\ll\frac{u^2}{(u^2+1)^3}\ll\frac{1}{u^4+1}.$$

Hence $F^-=O(t^{-1})$ uniformly in v.

Consider F^+ and substitute $y=\operatorname{sgn}(u-1)\sqrt{u+1/u-2}$. Then

$$F^+=e^{-2a}\int_{-\infty}^{\infty}f(y)e^{-ay^2}dy,$$

where $f(y)=y+\sqrt{y^2+4}-2/\sqrt{y^2+4}$ and $a=(v-2\pi i)\sqrt{t}$. Replace $f(y)$ by its Maclaurin polynomial $f_3(y)$, say, of degree 3, and estimate the error in the way F^- was estimated. This gives $O(t^{-1})$ uniformly in v, since

$$\left(\frac{1}{y}\left(\frac{f(y)-f_3(y)}{y}\right)'\right)'\ll\frac{1}{y^2+1}.$$

The terms of $f_3(y)$ of odd degree may clearly be omitted. The integral corresponding to the term of degree 2 is integrated (once) by parts. This gives

$$F^+=e^{-2a}(1+O(t^{-\frac{1}{2}}))\int_{-\infty}^{\infty}e^{-ay^2}dy+O(t^{-1}).$$

It is well-known that the integral here equals $\sqrt{\pi/a}$. Hence

$$F^+=\left(\frac{\pi}{a}\right)^{\frac{1}{2}}e^{-2a}+O(t^{-\frac{3}{4}}).$$

We have

$$\left(\frac{\pi}{a}\right)^{\frac{1}{2}}-\frac{1}{\sqrt{2}}t^{-\frac{1}{4}}e^{\pi i/4}=O(vt^{-\frac{1}{4}})=O(t^{-\frac{3}{4}}e^{v\sqrt{t}}).$$

Hence

$$F^+=\frac{1}{\sqrt{2}}t^{-\frac{1}{4}}e^{-2v\sqrt{t}+(4\pi\sqrt{t}+\pi/4)i}+O(t^{-\frac{3}{4}}).$$

Hence, uniformly in v,

$$F(t,v)=\frac{1}{2\sqrt{2}}t^{\frac{1}{4}}e^{-2v\sqrt{t}}\sin(4\pi\sqrt{t}+\frac{\pi}{4})+O(t^{-\frac{1}{4}}).$$

However, by (2.3) this holds even for $v=0$.

4 $\Delta(x,v)$ as an integral

Suppose that $v > 0$. From the lemma it follows that the series in (2.1) is absolutely convergent. Hence, and by (2.2), it may be represented as the double series

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{mk}\int_0^{\infty} e^{-v(u+mkx/u)}\cos(2\pi u)\sin(2\pi mkx/u)du$$

$$= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty} e^{-v(mu+kx/u)}\cos(2\pi mu)\sin(2\pi kx/u)du = \sum\sum\int f_{m,k}(u),$$

say. This is absolutely convergent, since

$$|f_{m,k}(u)| \le \exp(-\frac{1}{2}v(u + x/u + \sqrt{mkx})).$$

Moreover, the double series $\sum\sum f_{m,k}(u)$ is absolutely and uniformly convergent in any interval $[u_1, u_2] \subset (0, \infty)$. Therefore the summations may be taken behind the integral sign. This gives (as Im(log) = arg)

$$\Delta(x,v) = \frac{2}{\pi}\int_0^{\infty}\mathrm{Re}\left(\frac{1}{e^{pu}-1}\right)\arg(1-e^{-px/u})du,$$

where

$$p = v + 2\pi i.$$

5 Truncation and decomposition

Let N be an integer, $N > 2x$ and $R = N + \frac{1}{2}$. Denote

$$\Delta_1 = -\frac{1}{\pi}\int_0^R \arg(1-e^{-px/u})du,$$

$$\Delta_2 = \frac{1}{\pi}\int_0^R \mathrm{Re}\left(\frac{2}{e^{pu}-1}+1\right)\arg(1-e^{-px/u})du$$

and $\Delta_3 = \Delta(x,v) - \Delta_1 - \Delta_2$. We estimate Δ_3 by partial integration. We have

$$\int \mathrm{Re}\left(\frac{1}{e^{pu}-1}\right)du = \mathrm{Re}\left(\frac{1}{p}\log(1-e^{-pu})\right). \tag{5.1}$$

In any case this is $O(1)$. But for $u = R$ it is $O(v)$. Hence

$$\Delta_3 = \frac{2}{\pi}\int_R^{\infty}\mathrm{Re}\left(\frac{1}{e^{pu}-1}\right)\arg(1-e^{-px/u})du \ll v + \int_R^{\infty}|\frac{d}{du}\arg(1-e^{-px/u})|du.$$

The last integrand is $O(xu^{-2})$, since for $t > 0$ we have

$$\frac{d}{dt}\arg(1-e^{-pt}) = \operatorname{Im}\left(\frac{p}{e^{pt}-1}\right), \tag{5.2}$$

and for $\operatorname{Re}(z) \geq 0$ and $|\operatorname{Im}(z)| \leq \pi$ we have

$$\operatorname{Im}\left(\frac{z}{e^z-1}\right) = \operatorname{Im}\left(\frac{z+1-e^z}{e^z-1}\right) \ll |z|. \tag{5.3}$$

It follows that $\Delta_3 \ll v + x/R$, or equivalently,

$$\Delta(x,v) = \Delta_1 + \Delta_2 + O(v + xR^{-1}).$$

6 The integral Δ_1

Denote $\psi(t) = t - [t] - \frac{1}{2}$. Writing $\arg(1-e^{-2\pi i t})$ as a power series we recognize the Fourier series of $\psi(t)$. Thus $\arg(1-e^{-2\pi i t}) = -\pi\psi(t)$ for $t \notin \mathbf{Z}$. For $t > 0$ we have

$$\frac{d}{dv}\arg(1-e^{-pt}) = \operatorname{Im}\left(\frac{t}{e^{pt}-1}\right) \ll \frac{t}{\| t \|},$$

where $\| t \|$ denotes the distance of t from the nearest integer. Hence,

$$\arg(1-e^{-pt}) + \pi\psi(t) \ll \min\left(\frac{vt}{\| t \|}, 1\right) \ll \left(\frac{vt}{\| t \|}\right)^{\frac{3}{4}} \tag{6.1}$$

for $t \notin \mathbf{Z}$ and $t > 0$. Then an easy calculation gives (as $R > x$)

$$\begin{aligned} \Delta_1 &= x\int_{x/R}^{\infty} \psi(t)t^{-2}dt + O\left(x\int_{x/R}^{\infty}\left(\frac{vt}{\| t \|}\right)^{\frac{3}{4}} t^{-2}dt\right) \\ &= x\left(\log\left(\frac{R}{x}\right) + 1 - \gamma\right) - \frac{R}{2} + O(Rv^{\frac{3}{4}}). \end{aligned}$$

7 The integral Δ_2

Since $Re((e^{pu}+1)(e^{\bar{p}u}-1)) = e^{2vu} - 1 \ll vu$ for $vu \ll 1$, and since $|e^{pu}-1|^2 \gg (vu)^2 +$
$\| u \|^2 \gg (vu)^{\frac{1}{2}} \| u \|^{\frac{3}{2}}$, it follows that

$$Re\left(\frac{2}{e^{pu}-1}+1\right) = Re\left(\frac{e^{pu}+1}{e^{pu}-1}\right) = \frac{e^{2vu}-1}{|e^{pu}-1|^2} \ll (vu)^{\frac{1}{2}} \| u \|^{-\frac{3}{2}}. \tag{7.1}$$

Trivially $\arg(1-e^{-px/u}) \ll e^{-vx/u} \ll (vx/u)^{-\frac{1}{4}}$. Hence the subintegral $\int_0^{1/2}$ of Δ_2 is $O((v/x)^{\frac{1}{4}})$. Hence (recall that $R = N + \frac{1}{2}$)

$$\Delta_2 = \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\frac{1}{2}}^{\frac{1}{2}} Re\left(\frac{2}{e^{p(n+u)}-1}+1\right)A_p(u)du + O\left(\left(\frac{v}{x}\right)^{\frac{1}{4}}\right),$$

where

$$A_q(t) = \arg(1 - e^{-qx/(n+t)}).$$

CASE 1: $x/n \notin \mathbf{Z}$. For $|u| \le \frac{1}{2}$ we have

$$A_p(u) = A_p(0) + O\left((\| \frac{x}{n} \|^{-1} N|u|)^{\frac{3}{4}}\right). \tag{7.2}$$

To prove this, suppose that $4N|u| \le \|x/n\|$ (the other case being trivial). Suppose that $|t| \le |u|$. Then

$$|\frac{x}{n+t} - \frac{x}{n}| \le 2x|t| \le 2N|u| \le \frac{1}{2} \| \frac{x}{n} \| .$$

Hence $\|x/(n+t)\| \gg \|x/n\|$. Using (5.2) we get

$$A_p'(t) \ll \| \frac{x}{n+t} \|^{-1} n^{-2}x \ll \| \frac{x}{n} \|^{-1} N.$$

Hence $A_p(u) - A_p(0) \ll \| x/n \|^{-1} N|u|$, and (7.2) follows.

CASE 2: $x/n \in \mathbf{Z}$. Put $q = v - 2\pi iu/n$. For $|u| \le \frac{1}{2}$ we have

$$A_p(u) = A_q(0) + O\left((N|u|)^{\frac{3}{4}}\right). \tag{7.3}$$

To prove this, suppose that $4N|u| \le 1$ (the other case being trivial). Suppose that $|t| \le |u|$. Then $|\mathrm{Im}(qx/(n+t))| \le \pi$. Hence we may use (5.2) and (5.3) to get

$$A_q'(t) \ll |q|n^{-2}x \ll n^{-2}x \ll N,$$

since(5.2) holds also when p is replaced by q. Hence $A_q(u) - A_q(0) \ll N|u|$. But $A_p(u) = A_q(u)$, since $(p-q)x/2\pi i(n+u) \in \mathbf{Z}$. Hence (7.3) follows.

Denote

$$Q_x = \begin{cases} \| x \| & for \;\; x \notin \mathbf{Z}, \\ 1 & for \;\; x \in \mathbf{Z}. \end{cases} \tag{7.4}$$

To estimate the contribution of the O-terms in (7.2) and (7.3) we use (7.1) and the fact that $\| x/n \| \ge Q_x/n$ if $x/n \notin \mathbf{Z}$. This gives

$$\ll v^{\frac{1}{2}}N^{\frac{3}{4}} \left(\sum_{x/n \notin \mathbf{Z}} n^{\frac{1}{2}} \| \frac{x}{n} \|^{-\frac{3}{4}} + \sum_{x/n \in \mathbf{Z}} n^{\frac{1}{2}} \right) \ll v^{\frac{1}{2}}N^3 Q_x^{-\frac{3}{4}}.$$

Consider the contribution of $A_q(0)$. The function $A_q(0)$ is odd in u, while

$$Re((e^{p(n+u)-vu} - 1)^{-1})$$

is even. Moreover,

$$\left(e^{p(n+u)} - 1\right)^{-1} - \left(e^{p(n+u)-vu} - 1\right)^{-1} \ll \frac{vu}{(vn)^2 + u^2} \ll \left(\frac{v}{n|u|}\right)^{\frac{1}{2}}.$$

It follows that

$$\sum_{x/n\in\mathbf{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} Re\left(\frac{2}{e^{p(n+u)}-1}+1\right) A_q(0)du \ll (Nv)^{\frac{1}{2}}.$$

Consider the contribution of $A_p(0)$. We have

$$\sum_{x/n\notin\mathbf{Z}} A_p(0) \int_{-\frac{1}{2}}^{\frac{1}{2}} Re\left(\frac{2}{e^{p(n+u)}-1}\right) du \ll Nv,$$

since by (5.1) the integral in the summand equals

$$2Re(\frac{1}{p})\log\left(\frac{1+e^{-v(n+\frac{1}{2})}}{1+e^{-v(n-\frac{1}{2})}}\right) \ll Re(\frac{1}{p}) \ll v.$$

Finally, (6.1) with $t = x/n$ implies that

$$\frac{1}{\pi}\sum_{x/n\notin\mathbf{Z}} A_p(0) = -\sum_{x/n\notin\mathbf{Z}} \psi(\frac{x}{n}) + O(v^{\frac{3}{4}}N^{\frac{7}{4}}Q_x^{-\frac{3}{4}}),$$

and we have

$$\begin{aligned}\sum_{x/n\notin\mathbf{Z}} \psi(\frac{x}{n}) &= \sum_{n=1}^{N} \psi(\frac{x}{n}) - \sum_{x/n\in\mathbf{Z}} \psi(\frac{x}{n}) \\ &= x\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\sum_{m\le x/n} 1 - \frac{N}{2} + \frac{1}{2}d(x) \\ &= x(\log N+\gamma) - \sum_{n<x} d(n) - \frac{1}{2}d(x) - \frac{N}{2} + O(\frac{x}{N}).\end{aligned}$$

Collecting the relevant formulas in this and the two previous sections and using (1.1) we get

$$\Delta(x,v) = \Delta(x) + O(xN^{-1} + v^{\frac{1}{4}}N^3Q_x^{-\frac{3}{4}}) \tag{7.5}$$

for $N > x$, $N \in \mathbf{Z}$. This implies (2.5).

8 Convergence of $\Delta(x, 0)$

Denote

$$f(n,x) = x^{\frac{1}{4}}n^{-\frac{3}{4}}\cos(4\pi\sqrt{nx} - \frac{\pi}{4})$$

and

$$S = \sum_{a<n\le b} d(n)f(n,x),$$

where $2 \le a < b$. By (2.1) and (2.4) it is sufficient to consider S. Let $0 < \varepsilon < \frac{1}{4}$. We set out to prove that

$$S \ll x^{\frac{3}{4}} a^{-\frac{1}{4}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} a^{-\frac{1}{2}} Q_x^{-1}, \tag{8.1}$$

$$S \ll x^{\frac{3}{4}}. \tag{8.2}$$

The symbol Q_x was defined by (7.4).

By partial summation and (1.1) we find that S equals

$$-\int_a^b \left(\frac{\partial}{\partial t} f(t,x)\right) \Delta(t)dt + \int_a^b f(t,x)(\log t + 2\gamma)dt + \left[f(t,x)(\Delta(t) + \frac{1}{2}d(t))\right]_a^b. \tag{8.3}$$

The integrated term is $O(x^{\frac{1}{4}} a^{-\frac{1}{4}})$, since $\Delta(t) \ll t^{\frac{1}{2}}$. The second integral is $O((xa)^{-\frac{1}{4}} \log a)$, by partial integration.

Consider the first integral in (8.3). We have

$$\frac{\partial}{\partial t} f(t,x) = -2\pi x^{\frac{3}{4}} t^{-\frac{5}{4}} \sin(4\pi\sqrt{tx} - \frac{\pi}{4}) + O(x^{\frac{1}{4}} t^{-\frac{7}{4}}).$$

The error term here contributes $\ll x^{\frac{1}{4}} a^{-\frac{1}{4}}$. Then replace $\Delta(t)$ by $\Delta(t,v)$, where $v = b^{-28}$. This gives an error $O(x^{\frac{3}{4}} a^{-\frac{5}{4}})$, since (7.5) with $N = [b^2]$ implies that

$$\int_a^b |\Delta(t) - \Delta(t,v)| dt \ll 1.$$

Then replace $\Delta(t,v)$ by

$$\frac{1}{\pi\sqrt{2}} \sum_{n=1}^{\infty} d(n) f(n,t) e^{-2v\sqrt{nt}} = \Delta^*(t,v),$$

say, which in view of (2.4) differs from $\Delta(t,v)$ by $\ll t^{-\frac{1}{4}}$. This difference gives an error $O(x^{\frac{3}{4}} a^{-\frac{1}{2}})$.

We are now left with

$$x^{\frac{3}{4}} \int_a^b t^{-\frac{5}{4}} \sin(4\pi\sqrt{tx} - \frac{\pi}{4}) \Delta^*(t,v) dt.$$

This is, on integrating Δ^* term-by-term,

$$\ll x^{\frac{3}{4}} \sum_{n=1}^{\infty} d(n) n^{-\frac{3}{4}} \left| \int_a^b t^{-1} \sin(4\pi\sqrt{tx} - \frac{\pi}{4}) \cos(4\pi\sqrt{nt} - \frac{\pi}{4}) e^{-2v\sqrt{nt}} dt \right|.$$

Here $\sin(\ldots)\cos(\ldots)$ is a linear combination of

$$\sin(4\pi\sqrt{t}(\sqrt{x} - \sqrt{n})) \quad \text{and} \quad \cos(4\pi\sqrt{t}(\sqrt{x} + \sqrt{n})),$$

so that, integrating by parts, the integral is $\ll a^{-\frac{1}{2}} |\sqrt{x} \pm \sqrt{n}|^{-1}$. Therefore, excluding the term involving $\sqrt{x} - \sqrt{n}$, in which n is the (or a) positive integer minimizing $|n - x|$, the rest of the expression is $\ll x^{\frac{1}{2}+\varepsilon} a^{-\frac{1}{2}}$.

Denote by I_x the integral in the excluded term. If $x \notin \mathbf{Z}$ then $I_x \ll x^{\frac{1}{2}}a^{-\frac{1}{2}}\|x\|^{-1}$. Otherwise $I_x = 0$. This completes the proof of (8.1).

To finish the proof of (8.2) we may assume, by (8.1), that $0 < \|x\| < a^{-\frac{1}{2}}$. Split up I_x at $c = \min(b, \|x\|^{-2})$. Then, estimating as before, $\int_c^b \ll x^{\frac{1}{2}}$, and by the general inequality $|\sin y| \leq |y|$, we have $\int_a^c \ll x^{-\frac{1}{2}}$.

References

1. B.C. Berndt, Identities involving the coefficients of a class of Dirichlet series I, Trans. Amer. Math. Soc. 137 (1969), 345–359.
2. B.C. Berndt, The Voronoi summation formula, in "The theory of arithmetic functions," Lect. Notes Math. 251, pp. 21–36, Springer, 1972.
3. K. Chandrasekharan and R. Narasimhan, Functional equations with multiple gamma factors and the average order of arithmetical functions, Ann. Math. 76 (1962), 93–136.
4. K. Chandrasekharan, "Arithmetical functions," Springer, 1970.
5. A.L. Dixon and W.L. Ferrar, Lattice point summation formulae, Quart. J. Math. 2 (1931), 31–54.
6. G.H. Hardy, On the expression of a number as the sum of two squares, Quart. J. Math. 46 (1915), 263–283.
7. G.H. Hardy, On Dirichlet's divisor problem, Proc. London Math. Soc. 15 (1916), 1–25.
8. D.A. Hejhal, A note on the Voronoi summation formula, Mh. Math. 87 (1979), 1–14.
9. A. Ivić, "The Riemann zeta-function," John Wiley & Sons, 1985.
10. M. Jutila, "Lectures on a method in the theory of exponential sums," Tata Institute of Fundamental Research, Lectures on mathematics and physics 80, Springer, 1987.
11. E. Landau, Über das Konvergenzgebiet einer mit der Riemannschen Zetafunktion zusammenhängenden Reihe, Math. Ann. 97 (1927), 251–290.
12. T. Meurman, On exponential sums involving the Fourier coefficients of Maass wave forms, J. reine angew. Math. 384 (1988), 192–207.
13. Y. Motohashi, "Lectures on the Riemann–Siegel formula," Ulam. Seminar, Dept. Math., Colorado Univ., Boulder, 1987.
14. A. Oppenheim, Some identities in the theory of numbers, Proc. London Math. Soc. 26 (1927), 295–350.
15. W. Rogosinski, "Neue Anwendung der Pfeifferschen Methode bei Dirichlets Teilerproblem," Thesis, Göttingen, 1922.
16. M.G. Voronoi, Sur une fonction transcendante et ses applications à la sommation de quelques séries, Ann. Ecole Norm. Sup. 21 (1904), 207–267, 459–533.
17. G.N. Watson, "A treatise on the theory of Bessel functions (2nd ed.)," Cambridge Univ. Press, 1951.

Department of Mathematics, University of Turku, SF–20500 Turku, Finland