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On the transcendency of the values of the modular function at algebraic points

Hironori SHIGA

Dedicated to Professor Toshio Nishino on his 60th birthday

In this note we study an abelian variety A defined over $\overline{\mathbf{Q}}$. We shall characterize the abelian variety of CM type by its property of periods. The obtained result brings the criterion for algebraicity of the values of the Siegel modular function at algebraic points and of other various modular functions.

Our result can be regarded as the extension of the well known algebraicity criterion (so called Schneider's theorem) for the value of the elliptic modular function (cf. Theorem A-0). The author wishes to express many thanks for valuable advices given by D. Bertrand, F. Beukers and J. Wolfart.

§1 Characterization of the abelian variety of CM type.

We use the following notations.

A: g-dimensional abelian variety defined over $\overline{\mathbf{Q}}$ with a polarization χ.

$End_0\, A = (End\, A) \otimes \mathbf{Q}$,

$\omega_1, \cdots, \omega_g$: a basis system of holomorphic 1-forms on A defined over $\overline{\mathbf{Q}}$,

$\gamma_1, \ldots, \gamma_{2g}$: a basis system of $H_1(A, \mathbf{Z})$,

$\mathfrak{S}_g$: the Siegel upper half space of degree g.

We may change the period matrix ${}^t(\Omega_2, \Omega_1)$ by $M^t(\Omega_2, \Omega_1)$ with a transformation M of $\mathrm{GL}(2g, \mathbf{Z})$. By this procedure we may suppose the polarization χ is given by a $2g \times 2g$ matrix

$$\begin{pmatrix} O & \Delta \\ -\Delta & O \end{pmatrix}$$

with a certain integral diagonal $g \times g$ matrix Δ. Concerning this polarization the period relation is expressed as

$$ {}^t\begin{pmatrix} \Omega_2 \\ \Omega_1 \end{pmatrix} \chi \begin{pmatrix} \Omega_2 \\ \Omega_1 \end{pmatrix} = O $$

$$ \sqrt{-1}\,{}^t\begin{pmatrix} \Omega_2 \\ \Omega_1 \end{pmatrix} \chi \begin{pmatrix} \bar{\Omega}_2 \\ \bar{\Omega}_1 \end{pmatrix} > 0. $$

Hence we obtain a point $\Omega = \Omega_2(\Delta\Omega_1)^{-1}$ of $\mathfrak{S}_g$. The modular group

$$ \Gamma = \mathrm{Sp}(2g, \mathbf{Z}, \Delta) = \{M \in M(2g, \mathbf{Z}) | {}^t g\chi g = \chi\} \tag{1.1} $$

acts on $\mathfrak{S}_g$ by

$$ M \circ \Omega = (A\Omega + B\Delta^{-1})(C\Omega + D\Delta^{-1})^{-1}\Delta^{-1}, \tag{1.2} $$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbf{Z}, \Delta)$. In the following we regard the Γ-equivalence class represented by Ω as the reduced period matrix of A. Our main theorem is stated as the following.

Theorem M.
Let A be a g-dimensional polarized abelian variety defined over $\overline{\mathbf{Q}}$, and let Ω be the reduced period matrix of A. If Ω is an algebraic point of $\mathfrak{S}_g$, then A is an abelian variety of CM type.

We also have the following analogous result.

Theorem M'.
Let A be a g-dimensional polarized abelian variety defined over $\overline{\mathbf{Q}}$. Let ω_i and γ_j $(1 \le i \le 1, 1 \le j \le 2g)$ be as above. Suppose the ratio

$$ \frac{\int_{\gamma_i} \omega_k}{\int_{\gamma_j} \omega_k} $$

is an algebraic number for any indices i, j and k whenever the denominator does not vanish. Then A is an abelian variety of CM type.

Remark 1.1. A simple abelian variety A is said to be of CM type if $End_0\, A$ is isomorphic to a certain number field of degree $2 \times \dim A$. When A is not simple, A is said to be of CM type if every simple component is of CM type.

Remark 1.2. When A is simple and defined over $\overline{\mathbf{Q}}$, the period $\int_{\gamma_i} \omega_k$ does not vanish for every i and k (cf. [W-W]).

Remark 1.3. Let Ω be a reduced period matrix of a polarized abelian variety (A,χ). The abelian variety A is of CM type if and only if Ω is an isolated fixed point of a certain element of $\mathrm{Sp}(2g,\mathbf{Q},\Delta)$,(cf.[H-I]).

Proof of Theorem M. The following theorem, due to Wüstholz (cf. [Wü]), plays the essential role in our argument.

Theorem W. *Let A be an abelian variety defined over $\overline{\mathbf{Q}}$, and let $T_A \cong \mathbf{C}^g$ be its tangent space. Let* exp *be the exponential map from T_A to $A = \mathbf{C}^g/\Omega_2\mathbf{Z}+\Omega_1\mathbf{Z}$. Let W be a linear subspace of T_A defined over $\overline{\mathbf{Q}}$. If W contains a point $\vec{v}(\neq 0)$ with $\exp(\vec{v}) \in A(\overline{\mathbf{Q}})$, then there exists an algebraic subgroup G of A with the following properties:*
(i) G is defined over $\overline{\mathbf{Q}}$
(ii) the tangent space T_G is a subspace of W and contains $\vec{v}$
(iii) dim *G is positive*

We need the following lemma also.

Lemma 1.
Let A be a simple abelian variety over $\overline{\mathbf{Q}}$. Suppose $A \times A$ contains an algebraic subgroup H defined over $\overline{\mathbf{Q}}$. If we have $\dim H = \dim A$ and $p_i(H) = A$ (p_i is the projection to the i-th component), then H determines an element of $End_0\, A$. Moreover H corresponds to a trivial endomorphism (namely the one identified with a rational number) if and only if the tangent space T_H of H is given by the form

$$\{(\vec{z},\vec{w}) \in \mathbf{C}^g \times \mathbf{C}^g | m\vec{z} = n\vec{w}\}$$

for certain integers m and n.

We proceed our argument in three steps:
(i) We reduce the problem to the simple case
(ii) $D = End_0\, A$ is a $2g$-dimensional vector space over $\mathbf{Q}$,
(iii) D is a commutative division algebra.

Step (i).
Let A and A' be polarized abelian varieties defined over $\overline{\mathbf{Q}}$. Let ${}^t(\Omega_2,\Omega_1)$ and ${}^t(\Omega_2',\Omega_1')$ be the period matrix of A and A', respectively. Let $\Omega(A)$ and $\Omega(A')$ be the reduced period matrix of A andA', respectively. Suppose these two abelian varieties are isogenous, and let φ be the isogeny from A and A'. Then φ is induced from a certain invertible linear transformation U of $\mathbf{C}^g$. Moreover this linear transformation induces a homomorphism of the lattices represented by the form

$$\begin{pmatrix} \Omega_2' \\ \Omega_1' \end{pmatrix} = M \begin{pmatrix} \Omega_2 \\ \Omega_1 \end{pmatrix} \tag{1.3}$$

for a certain element $M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ of $M(2g, \mathbf{Z})$. By (1.1) we have

$$\Omega(A) = \Omega_2 \Delta (\Omega_1)^{-1}, \Omega(A') = \Omega_2' \Delta' {\Omega_1'}^{-1}, \tag{1.4}$$

where Δ and Δ' indicate the polarizations of A and A', respectively. Using (1.4) we obtain

$$\Omega(A') = (P\Omega(A) + Q)(R\Omega(A) + S)^{-1}\Delta'$$

Hence $\Omega(A')$ is an algebraic point of $\mathfrak{S}$ provided $\Omega(A)$ is an algebraic point of $\mathfrak{S}$. This argument shows that it is sufficient to study only the simple case.

Step (ii)

Let A be a simple abelian variety defined over $\overline{\mathbf{Q}}$. Suppose the reduced period matrix

$$\Omega(A) = (\lambda_{ij})_{1 \le i,j \le g}$$

is an algebraic point of $\mathfrak{S}$. Set

$$\begin{pmatrix} \vec{\beta}_1 \\ \cdot \\ \cdot \\ \cdot \\ \vec{\beta}_g \end{pmatrix} = \begin{pmatrix} \int_{\gamma_{g+1}} \vec{\omega} \\ \cdot \\ \cdot \\ \cdot \\ \int_{\gamma_{2g}} \vec{\omega} \end{pmatrix}, \qquad \begin{pmatrix} \vec{\alpha}_1 \\ \cdot \\ \cdot \\ \cdot \\ \vec{\alpha}_g \end{pmatrix} = \begin{pmatrix} \int_{\gamma_1} \vec{\omega} \\ \cdot \\ \cdot \\ \cdot \\ \int_{\gamma_g} \vec{\omega} \end{pmatrix}.$$

Then we have $\Omega_1 = \begin{pmatrix} \vec{\alpha}_1 \\ \cdot \\ \cdot \\ \cdot \\ \vec{\alpha}_g \end{pmatrix}$ and $\Omega_2 = \begin{pmatrix} \vec{\beta}_1 \\ \cdot \\ \cdot \\ \cdot \\ \vec{\beta}_g \end{pmatrix}$, hence it holds

$$\begin{pmatrix} \vec{\beta}_1 \\ \cdot \\ \cdot \\ \cdot \\ \vec{\beta}_g \end{pmatrix} = \Omega \begin{pmatrix} \vec{\alpha}_1 \\ \cdot \\ \cdot \\ \cdot \\ \vec{\alpha}_g \end{pmatrix}. \tag{1.5}$$

Set

$$G = \underbrace{A \times \cdots \times A}_{2g \text{ times}},$$

and let $\vec{u}_i$ and $\vec{z}_i$ $(i = 1, \cdots, g)$ be the coordinate for the tangent space T_A of i-th and $g+i$ th component of G, respectively. We regard them as row vectors. We suppose the exponential map is given by the projection

$$\mathbf{C}^g \longrightarrow \mathbf{C}^g / \Omega_2 \mathbf{Z}^g + \Omega_1 \mathbf{Z}^g.$$

Set the linear subspace

$$W = \{(\vec{z}_1, \cdots \vec{z}_g, \vec{u}_1, \cdots \vec{u}_g) | \begin{pmatrix} \vec{u}_1 \\ \cdot \\ \cdot \\ \cdot \\ \vec{u}_g \end{pmatrix} = \Omega \begin{pmatrix} \vec{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \vec{z}_g \end{pmatrix} \} \tag{1.6}$$

in $\mathbf{C}^{2g} = T_G$. It is a g^2-dimensional linear subspace. By (1.5) W contains a point

$$P_0 = (\vec{\alpha}_1, \cdots, \vec{\alpha}_g, \vec{\beta}_1, \cdots, \vec{\beta}_g).$$

Let us note that $\exp(P_0) = (0, \ldots, 0)$ in G, especially this is an algebraic point on G. By virtue of Theorem W we can find an algebraic subgroup H of G defined over $\overline{\mathbf{Q}}$. Also the tangent space T_H of H has the property:

$$P_0 \in T_H \subset W. \tag{*}$$

Set $h = \dim H$. Because A is simple the projection map p_i from G to the i-th component A is always surjective. By considering the following exact sequence

$$O \longrightarrow \ker p_i \longrightarrow H \xrightarrow{p_i} A \longrightarrow O$$

we know $\ker p_i$ is $(h-g)$-dimensional. By the iteration of this argument we know that $h = kg$ for a certain positive integer k.

Let π_k be the projection of T_H to the $(\vec{z}_1, \ldots, \vec{z}_k)$-space (the first k-components of T_G). After an appropriate change of order in $\vec{z}_1, \ldots, \vec{z}_g$, if necessary, it becomes surjective. Therefore T_H is represented by $g-k$ relations of the form

$$\begin{cases} \vec{z}_{k+1} = c_{1,k+1}\vec{z}_1 + \cdots + c_{k,k+1}\vec{z}_k \\ \quad\quad \cdots\cdots \\ \quad\quad \cdots\cdots \\ \quad\quad \cdots\cdots \\ \vec{z}_g = c_{1,g}\vec{z}_1 + \cdots + c_{k,g}\vec{z}_k \end{cases} \tag{1.7}$$

together with (1.6).

If we recall Lemma 1 we can show that every coefficient c_{ij} in (1.7) determines a $\mathbf{Q}$-endomorphism of A. Set $D = End_0 A$, it is an algebra over $\mathbf{Q}$. We claim that D is a $2g$-dimensional vector space over $\mathbf{Q}$. So we assume the contrary, namely

$$(**) \qquad \dim_{\mathbf{Q}} D \leq g$$

If it holds $k = g$, there is no relation of the form (1.7). By using Lemma 1 we can find every λ_{ij} belongs to D.

Then we obtain a number field of degree at most g by attaching λ_{ij}'s to $\mathbf{Q}$. According to the following Lemma 2 it contradicts the simpleness of A.

Lemma 2.

Let $\Omega = (\lambda_{ij})$ be an element of $\mathfrak{S}_g$. If every λ_{ij} belongs to a certain number field of degree at most g, then the corresponding abelian variety A is not simple.

Proof. By the assumption we can find the linear relation

$$(+) \qquad b_{i0} + b_{i1}\lambda_{i1} + \cdots + b_{ig}\lambda_{ig} = 0 \quad (i = 1, \ldots, g)$$

over $\mathbf{Q}$. By considering the transformation

$$I_{2g} + (E_{ij} - E_{g+j,g+i})$$

of $\mathrm{Sp}(g, \mathbf{Z})$ we can assume $b_{ii} \neq 0 \quad (i = 1, \ldots, g)$, where E_{ij} indicates the matrix element with the (i, j)-element is equal to 1.

Set

$$A_1 = \begin{pmatrix} b_{11} & b_{12} & \cdots & \cdots & b_{1g} \\ 0 & 1 & & & \\ & & \cdots & \cdots & 0 \\ 0 & & & & 1 \end{pmatrix},$$

$$B_1 = C_1 = O, D_1 = {}^t A_1^{-1}$$

and set

$$A_2 = D_2 = I_g, B_2 = b_{10}E_{11}, C_2 = O.$$

By putting

$$M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix},$$

we obtain two elements M_1 and M_2 of $\mathrm{Sp}(g, \mathbf{Q})$. Put

$$\Omega' = M_2 M_1(\Omega).$$

So we find

$$\begin{pmatrix} b_{10} + b_{11}\lambda_{11} + \ldots + b_{1g}\lambda_{1g} \\ \lambda_{2g} \\ \cdot \\ \cdot \\ \lambda_{gg} \end{pmatrix}$$

as the g-th column vector of Ω'. By (+) it means the $(1,g)$-element (and the $(g,1)$-element) of Ω' is equal to 0. By the iteration of this procedure Ω is deformed to a matrix of $\mathfrak{S}_g$ with a direct sum decomposition. This contradicts the simpleness of A. q.e.d.

Next we suppose $k = g - 1$. In this case we get one relation of the form (1.7):

$$\vec{z}_g = c_1\vec{z}_1 + \cdots + c_{g-1}\vec{z}_{g-1} \tag{1.7'}$$

$c_i \in D$. By substituting it in (1.6) we obtain

$$\left\{\begin{array}{l} \vec{u}_1 = (\lambda_{11} + c_1\lambda_{1g})\vec{z}_1 + \cdots + \quad (\lambda_{1,g-1} + c_{g-1}\lambda_{1,g})\vec{z}_{g-1} \\ \cdots \\ \vec{u}_g = (\lambda_{g1} + c_1\lambda_{gg})\vec{z}_1 + \cdots + \quad (\lambda_{g,g-1} + c_{g-1}\lambda_{g,g})\vec{z}_{g-1} \end{array}\right. \tag{1.8}$$

The coefficients in (1.8) are also $\mathbf{Q}$-endomorphisms. Namely every $\lambda_{ij} + c_j\lambda_{ig}$ belongs to D. By using the fact $\lambda_{ij} = \lambda_{ji}$ we know that every λ_{ij} is contained in the D-module $D \oplus \lambda_{gg}D$. On the other hand we have a certain $\mathbf{Q}$-linear relation

$$b_0 + b_1(\lambda_{11} + c_1\lambda_{1g}) + \cdots + b_g(\lambda_{g1} + c_1\lambda_{gg}) = 0$$

between $g+1$ endomorphisms

$$1, \lambda_{11} + c_1\lambda_{1g}, \ldots, \lambda_{g1} + c_1\lambda_{gg},$$

for we assumed (**). Therefore λ_{gg} itself belongs to D. So we find that every λ_{ij} is also an element of D. Hence it occurs a linear relation of the form (+) over $\mathbf{Q}$ between

$$1, \lambda_{11}, \ldots, \lambda_{1g}.$$

It is a contradiction again.

Even for the case $k \leq g-2$ we can proceed this argument.

Step (iii). By the same argument in Step (ii) we can find every λ_{ij} belongs to D, and they constitute a generator system of D. So the commutativity is obvious. q.e.d.

§2. Applications.

0) Transcendency of $j(\tau)$.

We have a new proof of the following well-known fact.

Theorem A-0. *Let $j(\tau)$ be the elliptic modular function. Suppose $\tau \in \overline{\mathbf{Q}}$, then $j(\tau)$ is an algebraic number if and only if τ is an imaginary quadratic number.*

Proof.

If part is the conclusion from the classical complex multiplication theory. Only if part is immediate from Theorem M for $g = 1$. q.e.d.

1) The transcendency of the Igusa-Rosenhain modular mapping on $\mathfrak{S}_2$.

Let $\lambda : \mathfrak{S}_2 \longrightarrow \mathbf{P}^3$ be the Igusa-Rosenhain modular mapping defined by

$$\lambda(\Omega) = [\xi_0, \cdots, \xi_3]$$

$$= [\vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vartheta^2 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vartheta^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}],$$

where the thetas indicate the Jacobi theta constants (cf. [I]). It is a modular mapping with respect to $\Gamma(2)$, the principal congruence subgroup of $\mathrm{Sp}(2, \mathbf{Z})$ with level 2. Moreover λ gives a birational correspondence between $\mathfrak{S}_2/\Gamma(2)$ and $\mathbf{P}^2$.

Theorem A-1. *Suppose Ω is an algebraic point of $\mathfrak{S}_2$, then $\lambda(\Omega)$ is an algebraic point of $\mathbf{P}^3$ if and only if Ω is a CM-point.*

Remark 2.1. The mapping λ is the inverse of the period mapping for the family of the curves of genus 2 given by the Legendre normal form

$$C(\xi) : y^2 = x \prod_{i=0}^{3} (x - \xi_i),$$

where $[\xi_0, \cdots, \xi_3]$ is a parameter on $\mathbf{P}^3$.

Proof of Theorem A-1.

If part is the consequence of the complex multiplication theory (cf. [S-T]), so we consider only if part. Let $A = Jac(C)$ be the jacobian variety of $C(\xi)$. By the assumption $C(\xi)$ is defined over $\overline{\mathbf{Q}}$. So A is also defined over $\overline{\mathbf{Q}}$. According to Theorem M A must be of CM type. Recalling Remark 1.3 we know the period $\Omega(A)$ is a CM point of $\mathfrak{S}_2$. q.e.d.

2) The Picard modular mapping.

We use the following notations in this argument.

$\zeta = \exp(2\pi i/3)$, $\quad K = \mathbf{Q}(\zeta)$, $\quad \mathcal{O}_K = \mathbf{Z} \bigoplus \mathbf{Z}\zeta$,

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let D be the domain defined by

$$D = \{\eta \in \mathbf{P}^2(\mathbf{C}) : {}^t\bar{\eta}H\eta < 0\} = \{(u,v) \in \mathbf{C}^2 : 2\Re v + |u|^2 < 0\}$$

(by putting $v = \eta_1/\eta_0, u = \eta_2/\eta_0$), this is biholomorphically equivalent to the 2 dimensional hyperball. Let Γ be the modular group defined by

$$\Gamma = U(H, \mathcal{O}_K) = \{g \in M(3, \mathcal{O}_K) : {}^t\bar{g}Hg = H\}.$$

We also consider the modular group with the level structure by $\sqrt{-3}$

$$\Gamma(\sqrt{-3}) = \{g \in \Gamma : g \equiv 1_3 \mod (\sqrt{-3})\}.$$

Set

$$\Omega = \Omega(u,v)$$
$$= \begin{pmatrix} (u^2 + 2\omega^2 v)/(1-\omega) & \omega^2 u & (\omega u^2 - \omega^2 v)/(1-\omega) \\ \omega^2 u & -\omega^2 & u \\ (\omega u^2 - \omega^2 v)/(1-\omega) & u & (u^2 + 2v)/(\omega - \omega^2) \end{pmatrix}.$$

This Ω gives an embedding of D into $\mathfrak{S}_3$. Using above notations we define the mapping $\lambda : D \longrightarrow \mathbf{P}^2$ by $\lambda(u,v) = [\xi_0, \xi_1, \xi_2] =$

$$[\vartheta^3 \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 \end{bmatrix}(\Omega), \vartheta^3 \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}(\Omega), \vartheta^3 \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ \frac{2}{3} & \frac{1}{6} & \frac{2}{3} \end{bmatrix}(\Omega)].$$

Remark 2.2. The mapping λ is the inverse of the period mapping for the following family of algebraic curves of genus 3 (cf. [S])

$$C(\xi) : y^3 = x\prod_{i=0}^{2}(x-\xi_i),$$

where $[\xi_i]$ is a parameter in

$$\mathbf{P}^2 - \{\prod_{i=0}^{2}\xi_i \prod_{j,k=0}^{2}(\xi_j-\xi_k) \neq 0\}.$$

Moreover λ induces the biholomorphic correspondence between the compactification of $D/\Gamma(\sqrt{-3})$ and $\mathbf{P}^2$.

Theorem A-2. *Suppose the point $P=(u,v)$ varies on the algebraic points of D. Then $\lambda(u,v)$ is an algebraic point if and only if P is an isolated fixed point of an element in*

$$U(H,K)=\{g\in M(3,K) : {}^t\bar{g}Hg=H\}.$$

Proof. We can obtain the above conclusion by the similar argument as 1). q.e.d.

3) The inverse of the Schwarz function for the Gauss hypergeometric differential equation.

Let $F(\alpha,\beta,\gamma)$ be the Gauss hypergeometric function and $D(\alpha,\beta,\gamma)$ be the corresponding hypergeometric differential equation:

$$x(x-1)y''+\{\gamma+(1+\alpha+\beta)x\}y'-\alpha\beta y=0 \tag{2.1}$$

Always the parameters α, β and γ are supposed to be rational numbers. Set

$$\lambda=1-\gamma,\quad \mu=\beta-\alpha,\quad \nu=\gamma-\alpha-\beta$$

Let N be the least common multiplier of the denominators of α, β and γ. Put

$$1-\alpha=A/N,\quad \alpha+1-\gamma=B/N,\quad \beta=C/N$$

We assume the following condition for λ, μ and ν

$$\text{(2.2)} \qquad \begin{cases} 1/\lambda,\ 1/\mu,\ 1/\nu \in \mathbf{Z} \cup \{0\}, \\ |\lambda| + |\mu| + |\nu| < 1. \end{cases}$$

Set

$$\text{(2.3)} \quad I(\infty, z) = \Gamma(\gamma)\{\Gamma(\alpha)\Gamma(\beta)\}^{-1} \int_0^\infty x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-zx)^{-\beta}dx$$

$$\text{(2.3')} \quad I(1, z) = \Gamma(\gamma)\{\Gamma(\alpha)\Gamma(\beta)\}^{-1} \int_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-zx)^{-\beta}dx.$$

Under the condition (2.2) $I(\infty, z)$ and $I(1, z)$ are independent solutions of (2.1). Moreover the Schwarz function

$$\sigma(z) = I(\infty, z)/I(1, z)$$

has a single valued inverse function, say $\lambda(\tau)$, defined on a bounded domain D (namely it is biholomorphically equivalent to the upper half plane H).

Here we use the notations according to [Wo]. The hypergeometric function $F(\alpha, \beta, \gamma : z) = I(\infty, z)$ can be considered as a period integral on the algebraic curve

$$X(N, z) : y^N = x^A(1-x)^B(1-zx)C.$$

Let us denote its Jacobian variety by $Jac(X(N, z))$. Set

S= the system of linearly independent differential forms of the form $\omega_n = \frac{P(x)dx}{y^n}$ for a certain positive integer n.

Then we have $r = \sharp S = \varphi(N)$. So we put $S = \{\omega^{(1)}, \cdots, \omega^{(r)}\}$ and $\omega^{(i)} = \omega_{n_i}$.

Let ζ be the primitive N-th root of unity, and set $K = \mathbf{Q}(\zeta)$. We obtain a lattice Λ in $\mathbf{C}^r$ by putting

$$\Lambda = \{(\rho_i(a) \int_0^1 \omega^{(i)} + \rho_i(b) \int_0^\infty \omega^{(i)})_{1 \le i \le r} : a, b \in \mathcal{O}_K\},$$

where $\mathcal{O}_K$ is the ring of integers in K and ρ_i is the automorphism of K with $\rho_i(\zeta) = \zeta^{n_i}$. Then we obtain an abelian variety $T = \mathbf{C}^r/\Lambda$. According to Wolfart $Jac(X(N, z))$ is isogenous to

$$T \bigoplus \sum_{D|N, D \neq N} Jac(X(D, z)).$$

Theorem A-3. *Suppose τ is algebraic. Then $\lambda(\tau)$ is an algebraic number if and only if T is an abelian variety of CM type.*

Proof.

i) The case T is simple. Let us consider $G = T \times T$. In the tangent space $T_G \cong \mathbf{C}^r \times \mathbf{C}^r$ of G we define the linear subspace

$$W = \{(\vec{z}, \vec{w}) \in \mathbf{C}^r \times \mathbf{C}^r : z_1 \int_0^\infty \omega_1 = w_1 \int_0^1 \omega_1\}$$

We have a natural action of the cyclotomic field $\mathbf{Q}(\zeta)$, where ζ indicates the N-th primitive root of unity, on the lattice Λ. Therefore $End_0 T$ contains the algebraic number field $\mathbf{Q}(\zeta)$ of degree $r = \phi(N) = \dim T$. So it is sufficient to show that $End_0 T$ contains a $\mathbf{Q}$-endomorphism which is not contained in $\mathbf{Q}(\zeta)$.

The subspace W contains a point

$$P_0 = (\vec{z}, \vec{w}) = \begin{pmatrix} \int_0^1 \omega_1 & \int_0^\infty \omega_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \int_0^1 \omega_r & \int_0^\infty \omega_r \end{pmatrix}$$

which is sent to $0 \in G(\overline{\mathbf{Q}})$ by the exponential map. According to Theorem W there exists an algebraic subgroup H satisfying the three conditions in the statement. Because T is simple H must be r-dimensional. Obviously H does not coincide with the component of T. By Lemma 1 we can find a $\mathbf{Q}$-endomorphism, say σ.

Suppose σ is contained in $\mathbf{Q}(\zeta)$. On multiplying by a certain integer, σ can be supposed to lie in the ring of integers of $\mathbf{Q}(\zeta)$. By the definition of the lattice Λ the endomorphism σ carries the vector $\int_0^1 \vec{\omega}$ to a certain $\mathbf{Z}$-linear combination of

$$\alpha_1 \int_0^1 \vec{\omega}, \ldots, \alpha_r \int_0^1 \vec{\omega}$$

where $\{\alpha_1, \ldots, \alpha_r\}$ is the generator system of the ring of integers. On the other hand if we examine the construction of σ, it carries the vector $\int_0^1 \vec{\omega}$ to $\int_0^\infty \vec{\omega}$. This is a contradiction.

ii) The case T is not simple.

Suppose we have the decomposition $T = A_1 \times \cdots \times A_k$, where every A_i is the same simple abelian variety A. By the same arguments as in i) also in this case we have a certain subvariety H in $G = T \times T$. If we restrict it to a certain product $A_i \times A_j$ in $T \times T$, we can find a $\mathbf{Q}$-endomorphism σ of $End_0 A$. So we can proceed the same discussion as the first case.

q.e.d.

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