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K-UNIRATIONALITY OF CONIC BUNDLES OVER LARGE ARITHMETIC FIELDS

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In the study of rational surfaces a major role is played by Del Pezzo surfaces and by conic bundles over curves. There are some interesting open questions about their properties. Most important among them is the problem of K-unirationality for conic bundles having a K-rational point. For a more precise exposition of this problem we need some definitions and conventions.

Let K be a field of characteristic $\neq 2$, X an absolutely irreducible variety defined over K, and $\overline{K}$ the algebraic closure of K.

Recall that X is said to be K-rational (respectively, K-unirational) if its function field $K(X)$ is (respectively, is contained in) a purely transcendental extension of K. One says that X is rational if $\overline{X} = X \times_K \overline{K}$ is $\overline{K}$-rational.

DEFINITION 1. A rational K-surface X is called a *conic bundle* over a rational curve C if there exists a K-morphism $f\colon X \to C$ whose generic fibre is a rational curve.

The problem of K-unirationality for conic bundles without a K-rational point clearly has a negative solution. So the following question is natural: are rational conic bundle surfaces with a K-rational point K-unirational? This question is not only of arithmetic interest. Its algebraic significance lies in the connection with the problem of existence of splitting fields of special type for some quaternion algebras. More precisely, V. A. Iskovskikh [8] has established that a conic bundle is K-unirational if and only if the corresponding quaternion algebra over a K-rational field has a K-rational splitting field (here a 'K-rational field' means some rational function field $K(z)$). Thus our K-unirationality question can be formulated as a problem in the theory of central simple algebras.

Further, this problem generalizes to arbitrary finite-dimensional central simple algebras. For a strict formulation we shall need the following definitions.

Let v be a valuation (or a place) of a field F. We shall denote by F_v the completion of F with respect to v (or at v).

DEFINITION 2. Let A be a finite-dimensional central simple algebra over a K-rational field L. Then one says that *A has a K-rational point* if there exist two elements, $k \in K$ and $x \in L$, such that $L = K(x)$ and the algebra $A \otimes_{K(x)} K(x)_{(x-k)}$ (where $(x-k)$ denotes the valuation of $K(x)$ corresponding to $x - k$ with trivial restriction to K) is trivial (i.e. a total matrix algebra over $K(x)_{(x-k)}$).

With the above notation (and definitions), the problem of existence of rational splitting fields for quaternion algebras generalizes as follows.

PROBLEM. *Let A be a finite-dimensional central simple algebra over a K-rational field, and suppose it has a K-rational point. Does A have a K-rational splitting field?*

For some classes of fields this problem has a positive answer. This is trivial in the case of an algebraically closed field K. Actually, in this case, $K(x)$ is a C_1-field (for definitions see e.g. [15]), and the Brauer group of $K(x)$ is trivial (see [15]). Hence any extension of $K(x)$ (in particular, any K-rational field) is a splitting field for any central simple algebra over $K(x)$. The first nontrivial case is that of local fields (i.e. real closed and p-adically closed fields). The case where $K = \mathbb{R}$ was first considered by Iskovskikh [8]. Real closed fields were considered later by the author. As to p-adically closed fields, the author [18] proved that the above problem has a positive solution for Henselian fields K and hence for p-adically closed fields (see [14]). The next natural case for consideration is when K is 'pseudo-closed'. The aim of this paper is an exposition of results on the above problem in this case and of the analogous result for the so-called large arithmetic fields. These results were obtained recently by Yu. Drakokhrust and the author.

The author would like to thank the referee for some useful suggestions (see the Appendix).

§1. The case of pseudo-closed fields

In this section we shall prove that if a central simple algebra over a K-rational field has a K-rational point, then it has a K-rational splitting field, provided K is 'pseudo-closed'. We recall some definitions.

DEFINITION 3. A field K is said to be *formally real* if it has at least one ordering (for details see e.g. [11] and [12]).

DEFINITION 4. A field K is said to be *real closed* if it is formally real and does not admit any proper formally real algebraic extension.

Any real closed field has a unique ordering, and any formally real field is contained in some real closed one. Moreover, if L_1 and L_2 are two real closed algebraic field extensions of K whose orderings induce the same ordering v on K, then L_1 and L_2 are K-isomorphic and one says that L_1 is a real closure of K (with respect to v). Let us denote by S_K the set of all orderings on K (we do not rule out the possibility that S_K may be empty) and by K_v the real closure of K for each $v \in S_K$.

DEFINITION 5. A field K is called *pseudo-real closed (prc)* if any absolutely irreducible affine K-variety X has a K-rational point if and only if it has a simple K_v-point for every $v \in S_K$.

These definitions imply that the class of pseudo-algebraically closed (pac-) fields K is contained in that of prc-fields. This is a case where S_K is empty. (Pac-fields were introduced by J. Ax and have been systematically investigated in [4], [9], and [17]. As to prc-fields, see e.g. [13].)

The class of p-adically closed fields can be defined in a similar fashion.

DEFINITION 6. Let K be a field of characteristic zero with valuation v, valuation ring Ω, and maximal ideal $\Sigma \subset \Omega$. Suppose the field Ω/Σ is of characteristic p and $\dim_{\mathbb{Z}/p\mathbb{Z}} \Omega/\Sigma = d$. Then K is called a *p-valued field* of p-rank d.

DEFINITION 7. Let K be a p-valued field of p-rank d. Then K is said to be *p-adically closed* if K does not admit any proper p-valued algebraic extension with the same p-rank.

If K is a p-valued field with valuation v, then there exists a maximal p-valued algebraic extension of K having the same p-rank. Any such field K_v is called a *p-adic closure* of K.

DEFINITION 8. Let K be a p-valued field and let M_K be the set of all non-K-isomorphic p-adic closures of K. Then K is said to be *pseudo-p-adically closed (ppc)* if every absolutely irreducible affine K-variety has a K-rational point provided it has a simple L-rational point for every element L of M_K.

REMARK 1. The Brauer groups of prc-fields can be finite but, contrary to the case of pac- and real closed fields, their orders are not uniformly bounded and can even be infinite.

From now on, pac-, prc- and ppc-fields will be called *pseudo-closed fields* for short. The main result of this section is as follows.

THEOREM 1. *Let K be a pseudo-closed field. If a finite-dimensional central simple algebra A over a K-rational field $K(x)$ has a K-rational point, then it has a K-rational splitting field.*

REMARK 2. In the case of pac-fields K, this theorem was proved earlier by I. I. Voronovich [16].

Before proving the theorem, it is convenient to formulate the main result of [18].

THEOREM (∗). *Let A be a central simple (finite-dimensional) algebra over a K-rational field $K(x)$. We assume that A has a K-rational point. If K is Henselian and its characteristic does not divide the index of A, then A has a K-rational splitting field $K(z)$. Furthermore,*

$$x = h(z)/\pi,$$

where $h(z) \in K[z]$ is an irreducible monic polynomial and π is a suitable element in the valuation ideal of K. In addition, a root α of $h(z)$ generates a Galois extension of K, and $K(\alpha)(x)$ is a splitting field of A.

Proof of Theorem 1. Let A be a central division algebra over $K(x)$ and let $\dim_{K(x)} A = n^2$. We denote by P the algebraic closure of $K(t)$ in $K\langle t\rangle$, where $K\langle t\rangle$ is the field of formal power series in t over K. Then P is a Henselian field [2]. By Theorem (∗) the algebra $B = A \otimes_{K(x)} P(x)$ has a P-rational splitting field $P(z)$ such that

$$x = h(z)/t^m.$$

Here n divides m, and $h(z) \in K[z]$ is an irreducible monic polynomial with the property: if $h(\alpha) = 0$ then $K(\alpha)$ is a Galois extension of K and $K(\alpha)(x)$ is a splitting field of A.

Let us fix ourselves a $K(x)$-basis $\{u_1, u_2, \dots, u_{n^2}\}$ for A. Then the elements of the standard matrix basis $\{e_{11}, e_{12}, \dots, e_{nn}\}$ of the total matrix algebra $B \otimes_{P(x)} P(z)$ are $P(z)$-linear combinations of $u_1, u_2, \dots, u_{n^2}$. We write $\varphi_i = a_i(z)/b_i(z)$ $(i = 1, 2, \dots, n^4)$ for the coefficients of all these combinations $(a_i(z), b_i(z) \in P[z])$, and $\Phi(z) = S(z)/R(z)$ for the determinant of the matrix N (with $S(z), R(z) \in P[z]$), where $[e_{11}, e_{12}, \dots, e_{nn}] = [u_1, u_2, \dots, u_{n^2}]\, N$. Finally, we write

$$a_i(z) = a_{0i} z^{\deg a_i(z)} + a_{1i} z^{\deg a_i(z)-1} + \dots + a_{\deg a_i(z)\, i},$$

with $a_{0i}, \dots, a_{\deg a_i(z)\, i} \in P$. In quite the same way, we define elements $b_{0i}, \dots, b_{\deg b_i(z)\, i}$; $s_0, \dots, s_{\deg S(z)}$; and $r_0, \dots, r_{\deg R(z)}$. Let L be the extension of K generated by the elements $a_{0i}, \dots, a_{\deg a_i(z)\, i}$, $b_{0i}, \dots, b_{\deg b_i(z)\, i}$ $(i = 1, 2, \dots, n)$ and by t. Then L is a regular extension of K. Since L is algebraic over $K(t)$ and $\operatorname{char} K = 0$, then $L = K(t, s)$, where $f(t, s) = 0$ for some absolutely irreducible polynomial $f(X, Y) \in K[X, Y]$. Now let $\Gamma = \{(\alpha, \beta) \in \mathbb{A}^2 \mid f(\alpha, \beta) = 0\}$. Without loss of generality we may assume that all of $a_{ki}, b_{ki}, r_k, s_k \in K[\Gamma]$, where $K[\Gamma]$ is the ring of regular functions on Γ. Let $g(t, s)$ be the product of all those $a_{ki}, b_{ki}, r_k, s_k, t$ which are not identically zero on Γ, and let E be the affine curve defined by the equations $f(X, Y) = 0$ and $Z\, g(X, Y) = 1$. Since E is isomorphic to an open subset of the absolutely irreducible curve Γ, it is also absolutely irreducible. Further, $L = K(\Gamma) \cong K(E)$, where $K(\Gamma)$ (resp. $K(E)$) is the field of K-rational functions on Γ (resp. on E).

Now, an ordering v of a prc-field K $(v \in S_K)$ can be extended (see [12]) to an ordering w of $K_v\langle t \rangle$. And it is clear that the restriction of w to L induces an extension of v to an ordering of L. Similarly, a valuation v of a ppc-field K can, according to [14], be extended to a p-adic valuation w of $K_v\langle t \rangle$. Hence its restriction to L is a p-adic valuation extending v. Thus we find, according to [13] (resp. [14]), that Γ has a simple K_v-point for every real (resp. p-adic) closure K_v. Hence E has a K-rational point $(a, b) \in \mathbb{A}^2$ such that $a \neq 0$. Further, for every $b_i(z)$ not all $b_{ki}(a, b)$ can vanish, for $S(z)$ not all $s_k(a, b)$, and the same is true for $R(z)$ and the $r_k(a, b)$.

We now consider the extension $K(y)$ of $K(x)$, where $x = h(y)/a^m$, and the algebra $C = A \otimes_{K(x)} K(y)$. There exists a place μ of $K(x)(t, s)$ to $K(x)$, which is trivial on $K(x)$ and such that $\mu(t) = a$, $\mu(s) = b$ (see e.g. [3]). Let ν be a place of $K(z)(t, s)$ to $K(y)$ extending μ and such that $\nu(z) = y$. The construction of E and the choice of (a, b) imply:

$$\nu(S(z)) \neq 0, \quad \nu(R(z)) \neq 0, \quad \nu(b_i(z)) \neq 0 \quad (i = 1, 2, \dots, n^4).$$

Let $[v_{11}, v_{12}, \dots, v_{nn}] = [u_1, u_2, \dots, u_{n^2}]\, \nu(N)$. Then the matrix $\nu(N)$ is nonsingular and $\{v_{11}, v_{12}, \dots, v_{nn}\}$ is a basis for C. Finally, on comparing the structure constants of the algebras B and C with respect to $\{e_{11}, e_{12}, \ldots, e_{nn}\}$ and $\{v_{11}, v_{12}, \ldots, v_{nn}\}$, we are led to the conclusion that the triviality of B implies that of C. □

COROLLARY 1. *Let X be a conic bundle defined over a pseudo-closed field K. Then X is K-unirational if and only if X has a K-point.*

§2. The case of large arithmetic fields

In this section a 'closed field' means an algebraically closed, real closed or p-adically closed field.

Let $\mathbb{Q}$ be the field of rational numbers, $\overline{\mathbb{Q}}$ its algebraic closure, and let $\mathbb{Q}_v^{\mathrm{alg}} = \mathbb{Q}_v \cap \overline{\mathbb{Q}}$ be the p-adic closure of $\mathbb{Q}$ if $v = p$, and the real closure of $\mathbb{Q}$ if v is real. We denote by $G(\mathbb{Q})$ the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}$.

Let $v_1, v_2, \ldots, v_r$ be a finite set of (not necessarily distinct) absolute values of $\mathbb{Q}$. We write: $\bar{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_r) \in G(\mathbb{Q})^r$ and

$$\mathbb{Q}_{\bar{\sigma}} = \sigma_1(\mathbb{Q}_{v_1}^{\mathrm{alg}}) \cap \sigma_2(\mathbb{Q}_{v_2}^{\mathrm{alg}}) \cap \ldots \cap \sigma_r(\mathbb{Q}_{v_r}^{\mathrm{alg}}). \qquad (*)$$

For two elements $\bar{\sigma}, \bar{\lambda} \in G(\mathbb{Q})^r$ we now define $\overline{\sigma\lambda}$ to be the element $(\sigma_1\lambda_1, \ldots, \sigma_r\lambda_r)$. In what follows, we use the term 'almost all' in the sense of the Haar measure on $G(\mathbb{Q})^r$. With the above notation, one has the following two lemmas, which are analogous to Lemmas 12.4 and 12.5 of [7].

LEMMA 1. *Let $\bar{\tau} \in G(\mathbb{Q})^r$ and let $L \subset \mathbb{Q}_{\bar{\tau}}$ be a finite extension of $\mathbb{Q}$. Then, for almost all $\bar{\lambda} \in G(\mathbb{Q})^r$, the following holds: if $f(X,Y) \in L[X,Y]$ is an absolutely irreducible polynomial such that, for every $i = 1, 2, \ldots, r$, there exists a point $(a_{0i}, b_{0i}) \in (\tau_i(\mathbb{Q}_{v_i}^{\mathrm{alg}}))^2$ with $f(a_{0i}, b_{0i}) = 0$ and $\frac{\partial f}{\partial Y}(a_{0i}, b_{0i}) \neq 0$, then the curve $f(X,Y) = 0$ has a $\mathbb{Q}_{\overline{\lambda\tau}}$-rational point.*

Proof. Let $\tau_1(\mathbb{Q}_{v_1}^{\mathrm{alg}}), \ldots, \tau_m(\mathbb{Q}_{v_m}^{\mathrm{alg}})$ $(m \leq r)$ be all L-nonisomorphic closed fields from $(*)$. Using [5], since L is a Hilbertian field, one has sequences $\{a_j\}, \{b_j\}$ $(a_j \in L, b_j \in \overline{\mathbb{Q}})$ such that

i) a_j lies near a_{0i} in $\tau_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$, $i = 1, 2, \ldots, m$;

ii) $f(a_j, Y)$ is irreducible over L and we have: $\deg f(a_j, Y) = \deg_Y f(X,Y)$ and $f(a_j, b_j) = 0$;

iii) for $L_j = L(b_j)$ the sequence $L_1, L_2, \ldots$ is linearly disjoint over L.

Condition i), Krasner's lemma, Sturm's theorem [11] and the condition $f(a_{0i}, b_{0i}) = 0$ imply that $f(a_j, Y)$ has a root in $\tau_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$, $i = 1, 2, \ldots, r$. Then there exist $\lambda_{j1}, \lambda_{j2}, \ldots, \lambda_{jr} \in G(\mathbb{Q})$ such that $\lambda_{ji}(L_j) \subset \tau_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$, $i = 1, 2, \ldots, r$.

Let $\bar{\lambda}_j = (\lambda_{j1}, \ldots, \lambda_{jr})$. Condition iii) implies (see [10]) that, for almost all $\bar{\lambda} \in G(\mathbb{Q})^r$, there exists j such that $\overline{\lambda \lambda_j} \in G(L_j)^r$. Then $\lambda_i^{-1}(L_j) = \lambda_{ji}(L_j) \subset \tau_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$ $(i = 1, 2, \ldots, r)$, i.e. $L_j \subset \lambda_i \tau_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$ $(i = 1, 2, \ldots, r)$. Hence $L_j \subset \mathbb{Q}_{\overline{\lambda\tau}}$, and (a_j, b_j) is a $\mathbb{Q}_{\overline{\lambda\tau}}$-rational point of the curve $f = 0$.

LEMMA 2. *For almost all $\bar{\sigma} \in G(\mathbb{Q})^r$ one has: if $f(X,Y)$ is an irreducible polynomial in $\mathbb{Q}_{\bar{\sigma}}[X,Y]$ and there exists, for every $i = 1, 2, \ldots, r$, a $\sigma_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$-rational point (a_{0i}, b_{0i}) such that $f(a_{0i}, b_{0i}) = 0$ and $\frac{\partial f}{\partial Y}(a_{0i}, b_{0i}) \neq 0$, then the curve $f = 0$ has a $\mathbb{Q}_{\bar{\sigma}}$-rational point.*

Proof. Let T be a countable dense subset of $G(\mathbb{Q})^r$ and let $L \subset \mathbb{Q}_{\bar{\sigma}}$ be the finite extension of $\mathbb{Q}$ generated by all coefficients of $f(X,Y)$. We consider the element $\bar{\tau} \in T \cap G(\mathbb{Q})^r\, \bar{\sigma}$. Since the fields $\sigma_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$, $\tau_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$ are isomorphic over L, we see that $f(X,Y)$ has $\tau_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$-rational points (c_{0i}, d_{0i}) such that $\frac{\partial f}{\partial Y}(c_{0i}, d_{0i}) \neq 0$ $(i = 1, 2, \ldots, r)$. Then by Lemma 1 we see that $f(X,Y)$ has a $\mathbb{Q}_{\bar{\sigma}}$-point, unless $\bar{\sigma}$ belongs to a zero subset of $G(L)^r \bar{\tau}$. Since the set of all polynomials $f(X,Y) \in \overline{\mathbb{Q}}[X,Y]$ is countable and a countable union of zero sets is again a zero set, the lemma is proved.

Using the above two lemmas we can prove the following

THEOREM 2. *For almost all $\bar{\sigma} \in G(\mathbb{Q})^r$, if a finite-dimensional central simple algebra over a $\mathbb{Q}_{\bar{\sigma}}$-rational field has a $\mathbb{Q}_{\bar{\sigma}}$-rational point, then it has a $\mathbb{Q}_{\bar{\sigma}}$-rational splitting field.*

Proof. Let M be a curve similar to the one occurring in the proof of Theorem 1. In fact it is enough to establish that, for almost all fields $\mathbb{Q}_{\bar{\sigma}}$, the curve M has a $\mathbb{Q}_{\bar{\sigma}}$-rational point. Then $L = \mathbb{Q}_{\bar{\sigma}}(\Gamma) \cong \mathbb{Q}_{\bar{\sigma}}(E)$ is contained in $\mathbb{Q}_{\bar{\sigma}}\langle t\rangle$. Since $\mathbb{Q}_{\bar{\sigma}} \subset \sigma_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$, it has an absolute value v_i. By [12] and [14], v_i extends to an absolute value w_i on $\sigma_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})\langle t\rangle$ and the restriction of w_i to L yields an extension of v_i to L. Hence we see that Γ has a $\sigma_i(\mathbb{Q}_{v_i}^{\mathrm{alg}})$-rational point (a_{0i}, b_{0i}) for each v_i, $i = 1, 2, \ldots, r$. By Lemma 2 it follows that, for almost all $\bar{\sigma} \in G(\mathbb{Q})^r$, the curve Γ given by the equation $f(X,Y) = 0$ has a $\mathbb{Q}_{\bar{\sigma}}$-rational point. Now, the existence of a $\mathbb{Q}_{\bar{\sigma}}$-rational point on M follows from the proof of Lemma 1. Indeed, its statement remains true if we exclude from the infinite sequence of fields L_j a finite number of fields, which correspond to points (a_j, b_j) lying on Γ but not on M. The end of the proof of our theorem is similar to that of Theorem 1.

COROLLARY 2. *For almost all $\bar{\sigma} \in G(\mathbb{Q})^r$, the $\mathbb{Q}_{\bar{\sigma}}$-unirationality of a conic bundle X is equivalent to the existence of a $\mathbb{Q}_{\bar{\sigma}}$-rational point on X.*

REMARK 3. A field of the form $\mathbb{Q}_{\bar\sigma}$ is not pseudo-closed in general. In fact, let m, n be two distinct odd prime numbers. By the Chinese Remainder Theorem we can find two integers u_m, u_n such that $(u_m/m) = (u_n/n) = -1$ and $(u_m/n) = (u_n/m) = 1$ (where $(../..)$ denotes the Legendre symbol). Besides, for any odd prime number $p \neq m, n$ there exists $v_p \in \mathbb{Z}$ such that $(v_p/p) = 1$ and $(v_p/m) = -1$. Finally, let $a \in \mathbb{Z}$ be such that $a \equiv 1 \pmod 8$ and $(a/n) = -1$. Let us consider the absolutely irreducible quadrics defined by the polynomials: $f_m(X,Y) = X^2 - u_mY^2 - m$, $f_n(X,Y) = X^2 - u_nY^2 - n$, $f_p(X,Y) = X^2 - v_pY^2 - p$, and $f_a(X,Y) = X^2 - aY^2 - n$. Then the quadric $f_m = 0$ has simple points over $\mathbb{Q}_n$ and over the real closure of $\mathbb{Q}$, but it has no points over $\mathbb{Q}_m$. Hence, $\mathbb{Q}_{(m,n)}$ is neither a prc- nor a pnc-field. The same argument with $f_n = 0$ shows that $\mathbb{Q}_{(m,n)}$ is not a pmc-field either. Now the quadric $f_p = 0$ has no simple points over $\mathbb{Q}_m$, so $\mathbb{Q}_{(m,n)}$ is not a ppc-field. Finally, the quadric $f_a = 0$ has no simple points over $\mathbb{Q}_n$, so $\mathbb{Q}_{(m,n)}$ is not a p2c-field.

REMARK 4. Let K be a finite extension of $\mathbb{Q}$ and let Σ be a finite subset of the set $P(K)$ of all inequivalent places (or valuations) of K. For any $g \in \Sigma$ let $K_g^{\mathrm{alg}} = K_g \cap \overline{\mathbb{Q}}$ and $K^\Sigma = \bigcap_{g\in\Sigma}(\bigcap_{\sigma\in G(K)}(\sigma(K_g^{\mathrm{alg}})))$, where $G(K)$ is the Galois group of $\overline{\mathbb{Q}}$ over K. Let Ω be an algebraic extension of K^Σ (for instance, K^Σ). Recently F. Pop informed me that he proved the following

THEOREM. *An absolute irreducible variety V defined over Ω has an Ω-rational point if and only if V has a simple Ω_v-rational point for all inequivalent extensions v of the elements of Σ to Ω.*

Using this fact and modifying the proof of Theorems 1 and 2 one can prove the following

THEOREM. *Let Ω be as above. If a finite-dimensional central simple algebra over an Ω-rational field has an Ω-point, then it has an Ω-rational splitting field.*

COROLLARY. *Any conic bundle defined over Ω is Ω-unirational if and only if it has an Ω-rational point.*

Appendix

By using the language of finitely presented morphisms, one can give another nice form to the arguments in §1. Here is a brief exposition of this point of view, which I owe to Per Salberger. (Most facts on finitely presented morphisms can be found in [6].)

PROPOSITION. *Let K be a pseudo-closed field and P the algebraic closure of $K(t)$ in $K\langle t\rangle$. Let $f\colon X \to Y$ be a proper dominant K-morphism between smooth projective geometrically connected K-varieties. Suppose there exists a finitely presented P-morphism $s_P\colon Y_P \to X_P$ such that the composition $f_P \circ s_P\colon Y_P \to Y_P$ with the induced P-morphism $f_P\colon X_P \to Y_P$ is finite and surjective. Then there exists a K-morphism $s\colon Y \to X$ such that $f \circ s\colon Y \to Y$ is finite and surjective.*

Sketch of proof. Using standard results on finitely presented morphisms [6], we see that there exists a finite extension field L of $K(t)$ in $K\langle t\rangle$ such that $s_P\colon Y_P \to X_P$ is defined over L. In this way we obtain an L-morphism $s_L\colon Y_L \to X_L$ with $f_L \circ s_L\colon Y_L \to Y_L$ finite and surjective. Since $L \subset K\langle t\rangle$ and $[L : K(t)] < \infty$, it is further clear that L is the function field of a smooth projective geometrically connected K-curve C. Once more we apply the standard results in [6] on finitely presented morphisms and extend s_L to a U-morphism $s_U\colon Y_U \to X_U$ such that $f_U \circ s_U\colon Y_U \to Y_U$ is finite and surjective for some open K-subset U of C. Now $U(K)$ is nonempty since K is pseudo-closed and $L = K(U) \subset K\langle t\rangle$ (extend orderings and valuations as in §1). So we may specialize s_U at some K-point on U to obtain the desired K-morphism.

COROLLARY. *Let K, P be as above and let X be a smooth K-variety. Let $f\colon X \to \mathbb{P}^1_K$ be a proper dominant K-morphism whose generic fibre is a Severi-Brauer variety over the function field of $\mathbb{P}^1_K$ (e.g. a conic bundle surface). Then X is K-unirational if and only if X_P is P-unirational.*

Proof. X is K-unirational if and only if there exists a K-morphism $g\colon \mathbb{P}^1_K \to X$ such that $f \circ g\colon \mathbb{P}^1_K \to \mathbb{P}^1_K$ is finite and surjective, i.e. if and only if there exists a P-morphism $g_P\colon \mathbb{P}^1_P \to X_P$ such that $f_P \circ g_P\colon \mathbb{P}^1_P \to \mathbb{P}^1_P$ is finite and surjective, i.e. if and only if X_P is P-unirational.

It is known that any Severi-Brauer variety over the function field of $\mathbb{P}^1_K$ extends to a proper morphism $f\colon X \to \mathbb{P}^1_K$ from a smooth K-variety X (cf. [1]). So there is no reason to restrict our unirationality results to the case where f is of relative dimension one. The splitting field result for central simple algebras is equivalent to the existence of $g\colon \mathbb{P}^1_K \to X$ with $f \circ g$ finite and surjective.

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