Astérisque

FLORIN RĂDULESCU

A type III_λ factor with core isomorphic to the von Neumann algebra of a free group, tensor $B(H)$

Astérisque, tome 232 (1995), p. 203-209

<http://www.numdam.org/item?id=AST_1995__232__203_0>

© Société mathématique de France, 1995, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (http://smf4.emath.fr/Publications/Asterisque/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

A TYPE III_λ FACTOR WITH CORE ISOMORPHIC TO THE VON NEUMANN ALGEBRA OF A FREE GROUP, TENSOR $B(H)$.

FLORIN RĂDULESCU

In this paper we obtain a type III_λ factor by using the free product construction from [Vo1,Vo2] and show that its core ([Co]) is $\mathcal{L}(F_\infty) \otimes B(H)$. We will prove that

$$M_2(\mathbb{C}) * L^\infty([0,1],\nu)$$

is a type III_λ factor if $M_2(\mathbb{C})$ is endowed with a nontracial state. Moreover we will show that the core ([Co]) of this type III_λ factor (when tensorized by $B(H)$) is $\mathcal{L}(F_\infty) \otimes B(H)$ and we will give an explicit model for the associated (trace scaling) action of $\mathbb{Z}$ on the core (cf. [Co], [Ta]). Here $B(H)$ is the space of all linear bounded operators on a separable, infinite dimensional Hilbert space H.

Recall from [Vo1], that a family $(A_i)_{i\in I}$ of subalgebras in a von Neumann algebra M with state ϕ, is free with respect to ϕ if $\phi(a_1 a_2 ... a_k) = 0$ whenever

$$\phi(a_i) = 0, a_i \in A_{j_i}, i = 1, 2, ...k, j_1 \neq j_2, ...j_{k-1} \neq j_k.$$

Reciprocally given a family $(A_i, \phi_i), i \in I$ of von Neumann algebras with faithful normal states ϕ_i , one may construct (see[Vo1]) the (reduced) free product von Neumann algebra $*A_i$, which contains $A_i, i \in I$ and has a faithful normal state ϕ so that $\phi|_{A_i} = \phi_i$ and so that the algebras $(A_i)_{i\in I}$ are free with respect to ϕ.

The aim of this paper is to show the following result.

Theorem. *Let $\mathcal{E} = M_2(\mathbb{C}) * L^\infty([0,1],\nu)$ be endowed with the free product state ϕ where $M_2(\mathbb{C})$ is endowed with the state ϕ_0 which is subject to the condition*

$$\phi_0(e_{11})/\phi_0(e_{22}) = \lambda \in (0,1) \text{ and } \phi(e_{12}) = \phi(e_{21}) = 0,$$

while $L^\infty([0,1],\nu)$ has the state given by Lebesgue measure on [0,1]. With these hypothesis, $M_2(\mathbb{C}) * L^\infty([0,1],\nu)$ is a type III_λ factor and its core is isomorphic to $\mathcal{L}(F_\infty) \otimes B(H)$.

In the proof of the theorem we will also obtain a model for the core of $\mathcal{E} \otimes B(H)$ and for the corresponding (dual) action on the core, of the modular group of the weight $\phi \otimes tr$ (tr is the canonical semifinite trace on $B(H)$). This model will be a submodel of the one parameter action of $\mathbb{R}_+/\{0\}$ on $\mathcal{L}(F_\infty) \otimes B(H)$,that we have constructed in [Ra].

The model. *Model for the core of $(M_2(\mathbb{C}) * L^\infty([0,1],\nu)) \otimes B(H)$ and of the corresponding dual action on the core for the modular group of automorphism for the weight $\phi \otimes tr$:*

Let $\mathcal{A}_0$ be the subalgebra in the algebraic free product

$$L^\infty(\mathbb{R}) * (\mathbb{C}[X] * \mathbb{C}[Y])$$

generated by $\{pXp, pYp, p|\ p \text{ finite projection in } L^\infty(\mathbb{R})\}$ where $L^\infty(\mathbb{R})$ is endowed with the Lebesgue measure.

Let τ be the unique trace on $\mathcal{A}_0$ defined by the requirement that the restriction τ_p to the algebra generated in $p\mathcal{A}p$ by $pXp, pYp, pL^\infty(\mathbb{R})$ is subject to the following conditions:

(i) The three algebras generated respectively by pXp, pYp, $pL^\infty(\mathbb{R})$ are free with respect to τ_p

(ii) $\tau(p)^{-1/2}pXp, \tau(p)^{-1/2}pYp$ are semicircular (with respect τ_p)(see [Vo1] for the definition of a semicircular element).

Such a construction is possible because of the Theorem 1 in [Ra].

Assume that pXp, pYp are selfadjoint and let $\mathcal{A}$ be the weak completion of $\mathcal{A}_0$ in the G.N.S. representation for τ. Then (cf. [Ra]), $\mathcal{A}$ is a type II_∞ factor isomorphic to $\mathcal{L}(F_\infty) \otimes B(H)$ and the trace τ extends to a semifinite normal trace on $\mathcal{A}$ (which we also denote by τ).

Recall (by [Ra]) that in this case, there exists a one parameter group of automorphism $(\alpha_t)_{t \in \mathbb{R}_+ \setminus \{0\}}$ on $\mathcal{A}$, scaling trace by t, for each $t \in \mathbb{R}_+ \setminus \{0\}$, which is induced by $d_t * M_t$ on $L^\infty(\mathbb{R}) * (\mathbb{C}[X] * [Y])$ where d_t is dilation by t on $L^\infty(\mathbb{R})$, while $M_t(X) = t^{-1/2}X; M_t(Y) = t^{-1/2}Y$, $t > 0$.

Let $\mathcal{B}$ the von Neumann subalgebra of $\mathcal{A}$ generated by

$$q_n = \chi_{[\lambda^{n-1}, \lambda^n]}, n \in \mathbb{Z},$$

the characteristic functions of the intervals $[\lambda^{n-1}, \lambda^n]$ and by the following subsets of $\mathcal{A}$:

$$\tilde{X} = \{q_n X q_m | n, m \in \mathbb{Z}, |n - m| \leq 1\},$$

$$\tilde{Y} = \{q_n Y q_n | n \in \mathbb{Z}\}.$$

Clearly $\mathcal{B}$ is invariant under $\{\alpha_{\lambda^n}\}_{n \in \mathbb{Z}}$ and by Lemma 3 in [Ra], $\mathcal{B}$ is isomorphic to $\mathcal{L}(F_\infty) \otimes B(H)$. Let $\beta_n = \alpha_{\lambda^n} | \mathcal{B}$.

Let $\mathcal{D} = \mathcal{B} \rtimes_\beta \mathbb{Z}$ be the cross product of $\mathcal{B}$ by the action $\mathbb{Z}$ given by β. Then by [Co], $\mathcal{D}$ is a type III_λ factor. Let $u \in \mathcal{D}$ be the unitary implementing the cross product. Moreover let ψ be the normal semifinite faithful weight on $\mathcal{D}$ obtained as the composition expectation from $\mathcal{D}$ onto $\mathcal{B}$.

We will prove that $\mathcal{B}$, with the action of $\mathbb{Z}$ given by $(\beta_n)_{n\in\mathbb{Z}}$ is isomorphic to the core of $\mathcal{E}\otimes B(H)$, with the dual action (on the core) for the modular group of automorphisms of the weight $\phi\otimes tr$ on $\mathcal{E}\otimes B(H)$. Our main result will be a consequence of the following proposition:

Proposition.

Let $\mathcal{E}$ be the von Neumann algebra free product $M_2(\mathbb{C})*L^\infty([0,1],\nu)$, with the free product state $\phi=\phi_0*\nu$, where $M_2(\mathbb{C})=(e_{ij})_{i,j=1}^2$ is endowed with the normalized state ϕ_0 with $\phi(e_{11})/\phi(e_{22})=\lambda$ and $\phi(e_{12})=\phi(e_{21})=0$. Then, with the above notation $\mathcal{E}$ is isomorphic to $(q_o+q_1)\mathcal{D}(q_0+q_1)$.

Moreover the state ϕ on $\mathcal{E}$ is (via this identification) the (normalized) restriction of ψ to $(q_o+q_1)\mathcal{D}(q_0+q_1)$.

(Here $\mathcal{D}=\mathcal{B}\rtimes_\beta\mathbb{Z}$, where $\mathcal{B}$ is the von Neumann subalgebra in $\mathcal{A}$ generated by $\tilde{X}=\{q_nXq_m|n,m\in\mathbb{Z},|n-m|\le 1\}$, $\tilde{Y}=\{q_nYq_n|n\in\mathbb{Z}\}$ and the characteristic functions $q_n=\chi_{[\lambda^{n-1},\lambda^n]},n\in\mathbb{Z}$, $q_n\in L^\infty(\mathbb{R})\subseteq\mathcal{A}$. Moreover $\beta_n=\alpha_{\lambda^n},n\in\mathbb{Z}$.)

Recall from above that the von Neumann algebra $\mathcal{A}$ is a type II_∞ factor isomorphic to $\mathcal{L}(F_\infty)\otimes B(H)$ and $\mathcal{A}$ is generated by

$$\{pXp,pYp,p|\ p \text{ finite projection in } L^\infty(\mathbb{R})\}.$$

Here $\alpha_t,t>0$ acts as dilation by t on $L^\infty(\mathbb{R})$ and multiplies X,Y by $t^{-1/2}$. The trace on $\mathcal{A}$ is subject to the above conditions (i), (ii) and it is scaled by the automorphisms $\alpha_t,t>0$.

This proposition will be a consequence of the following two lemmas.

Lemma 1.

With $\mathcal{A},\mathcal{B},\mathcal{D},\psi,\tau$ and u as before let

$$e_{11}=q_1u=uq_0;e_{11}=q_0;e_{22}=q_1$$

Let $a=x+y$, where

$$x=(q_0+q_1)X(q_0+q_1)-q_0Xq_0$$

$$y=q_0Yq_0.$$

Then $M_2(\mathbb{C})=(e_{ij})_{i,j=1}^2$ is free with respect to

$$\psi_1=(\psi(q_0+q_1))^{-1}\psi|_{(q_o+q_1)\mathcal{D}(q_0+q_1)},$$

to the semicircular element a, *in the algebra* $(q_0 + q_1)\mathcal{D}(q_0 + q_1)$ *with unit* $q_0 + q_1$.

Proof. We have to check freeness, which means that the value of ψ_1 on certain monomials in a, u, e_{11}, e_{22} is null. Since by definition , ψ_1 vanishes the monomials containing a different number of u 's and u^* 's, we have only to check this if the number of occurrences for u is equal to the one for u^*.

Let $p_n = q_n + q_{n+1} = \chi_{[\lambda^{n-1},\lambda^{n+1}]}$.

Using the fact that u implements β_1 on $\mathcal{D}$ it follows that we only have to check $\psi_1(m) = 0$ if

$$m = p_0 f_1 q_{i_1} f_2 q_{i_2} f_3 ... q_{i_n} f_{n+1} p_0$$

where the following conditions are fulfilled:

(a) i_{j+1} is either i_j or $i_j \pm 1$.

(b) Card $\{s | i_j = s, j = 1, 2, ..., n\}$ is even for every s.

(c) f_k is a product

$$f_1^k a_1^k ... f_{n_k-1}^k a_{n_k-1}^k f_{n_k}^k,\ n_k \geq 1$$

where f_s^k, $s = 1, 2, 3...n_k$, is an element of null value under the state ψ_1 in the algebra generated by $\alpha_j(a)$ while a_s^k is an element of null trace in the algebra generated by q_j, q_{j+1}. Here j is an integer which is completely determined, for each k. If $i_k \neq i_{k+1}$ then j is the minimum of the i_k and i_{k+1}. If $i_k = i_{k+1}$ then j is either i_k if $i_{k-1} \leq i_k$ or either $i_k - 1$ if $i_{k-1} > i_k$.

To see that those are all the monomials of null state that may appear in the algebra generated by $M_2(\mathbb{C})$ and a it is sufficient to note that any string

$$f_1 e_{21} f_2 e_{21} ... f_p e_{21} f_{p+1} e_{12} f_{p+2} e_{12} ... e_{12} f_{2p+1} =$$

$$= f_1(q_1 u) f_2 q_1 u ... f_p q_1 u f_{p+1} (u^* q_1) ... (u^* q_1) f_{2p+1},$$

after cancelation, is equal to

$$f_1(q_1 u) f_2 ... q_1 u f_p q_1 \beta_1(f_{p+1}) q_1 f_{p+2} (u^* q_1) ... (u^* q_1) f_{2p+1} =$$

$$= f_1(q_1 u) f_2 ... q_1 \beta_1(f_p) q_2 \beta_2(f_{p+1}) q_2 \alpha_1(f_{p+2}) q_1 ... (u^* q_1) f_{2p+1} =$$

$$= f_1 q_1 \alpha_1(f_2) q_2 ... \beta_{p-1}(f_p) q_p \beta_p(f_{p+1}) q_p \beta_{p-1}(f_{p+2}) ... q_1 f_{2p+1}$$

and similarly for a string in which each $q_1 u$ is replaced by $u^* q_1$ and conversely.

Here the f_i 's are products of the form $f_1^i a_1^i f_2^i a_2^i ... f_n^i$ where f_j^i are elements of null trace in the algebra generated by $a = (q_0 + q_1) a (q_0 + q_1)$, while a_j^i are elements of null trace in the algebra generated by q_0, q_1.

The monomials in the algebra generated by $M_2(\mathbb{C})$ and a that are to be checked for having zero value under ψ_1 are obtained by replacing certain f_j by other strings of this form, or by putting together such strings.

To show that the value of $\psi_1(m)$ is zero we will use the following observation which is a consequence of Lemma 3.1 in [Vo2]. This observation will be used to replace the elements $f_1, ..., f_{n+1}$ in the monomial m by elements of null trace.

Observation. *Let B be a W^*-algebra with trace τ, let X be a semicircular element and p a nontrivial projection that is free with X. Then any element of null trace in the algebra (with unit p) generated by pXp is a sum of monomials which are products either of elements of null trace in the algebra generated by pXp or either of the form $p - \tau(p)$, but no such monomial is $p - \tau(p)$ itself.*

Proof. Indeed if x is such an element then $pxp = x$, and moreover any other such monomial, which is different from $p - \tau(p)$, when multiplied with p, preserves the property of having null trace.

On the other hand

$$\tau(p(p - \tau(p))) = 1 - \tau(p) \neq 0.$$

This ends the proof of the observation.

To conclude the proof of Lemma 1 we let p a projection which is greater than the supremum of all the projections $\{q_i | i \in I_m\}$ that are involved in m.

We may then assume by construction that we are given a finite family of semicircular elements z^j so that $z^j = pz^jp$ and so that (modulo a multiplicative constant) $\alpha_j(a) = (q_j + q_{j+1})z^j(q_j + q_{j+1})$ for j in I_m.

Using the above observation we may express $f_k = f_1^k a_1^k ... f_{n_k-1}^k a_{n_k-1}^k f_{n_k}^k$ as a sum of products of null trace in the algebras generated by $\{q_j\}$ and $\{z_j\}$ (adjacent elements are allways in different algebras).

(Note that $(q_j + q_{j+1})(q_j - \tau(q_j)(\tau(q_j + q_{j+1}))^{-1}(q_j + q_{j+1})$ has always null trace).

Again the above observation shows that each of these monomials must contain at least on term in z^j. Since consecutive f^i involve different elements in the set $\{z^j\}$ it follows that $\psi_1(m) = 0$.

This ends the proof of Lemma 1.

Lemma 2. *With $\mathcal{B}, \mathcal{D}$ as before we have that $(q_0 + q_1)\mathcal{B}(q_0 + q_1)$ coincides with the von Neumann subalgebra $\mathcal{C} \subseteq (q_o + q_1)\mathcal{D}(q_0 + q_1)$ (with unit $q_0 + q_1$) that is generated by the monomials with an equal number of e_{12} 's and e_{21} 's.*

Proof. We have to show that the subalgebra $(q_0+q_1)\mathcal{B}(q_0+q_1)$ coincides with the subalgebra $\mathcal{C} \subseteq (q_0+q_1)(\mathcal{B} \rtimes_\beta \mathbb{Z})(q_0+q_1) = (q_0+q_1)\{\mathcal{B}, u\}''(q_0+q_1)$ that is generated by monomials in a and $(e_{ij})_{i,j=1}^2$ containing an equal number of e_{12} 's and e_{21} 's.

Clearly $\mathcal{C}$ is invariant under the action of $\mathbb{R}$ (or $\mathbb{T}$) on $\mathcal{D}$ given by the modular group of ψ which acts by $\sigma_t^\psi(u) = \lambda^{it}u, \sigma_t^\psi|_\mathcal{B} = Id_\mathcal{B}$ so that $\mathcal{C} \subseteq (q_0+q_1)\mathcal{D}^\mathbb{R}(q_0+q_1) = (q_0+q_1)\mathcal{B}(q_0+q_1)$.

Hence we have to only prove the reverse inclusion. But due to the specific form of the generators in $\mathcal{B}$, we obtain that $\mathcal{B}$ is generated by elements of the form

$$m = f^1 q_{i_1} f^2 q_{i_2} ... f^n q_{i_n} f^{n+1}$$

where the conditions on $i_1, ..., i_n$ are

$$a)\ i_{j+1} \in \{i_j, i_j-1, i_j+1\},\ j=1,2,...,n,\ i_0, i_n \in \{0,1\}$$

$$b)\ \text{card}\ \{j | i_j = s\} \text{ is even,}$$

while f is one of the elements

$$\alpha_s(q_0Xq_0);\ \alpha_s(q_0Xq_1); \alpha(q_1Xq_0) \text{ or } \alpha_s(q_1Yq_1),$$

where s is either i_j or i_{j+1} if $i_j \neq i_{j+1}$. If $i_j = i_{j+1}$, then either $s = i_j$ and $f^j = \alpha_s(q_0Xq_0)$ or either $s = i_{j-1}$ and $f^j = \alpha_s(q_1Yq_1)$.

The assumptions we made are sufficient to show that in such a monomial we have some symbols corresponding to $\alpha_s(a)$ which are then necessary followed by symbols corresponding to $\alpha_{s+1}(a)$ (or to $\alpha_{s-1}(a)$). Moreover in m this sets of symbols are always separated by one of the projections q_p ($p \in \{s, s \pm 1\}$).

If we replace in m any such q_p by q_1u (or respectively by u^*q_1) and we replace the symbols from $\alpha_s(a)$ by the corresponding symbols in a we get the same m, but this time expressed as an element in the subalgebra of $\mathcal{C}$, generated by monomials with equal occurrence number of e_{12} 's and e_{21} 's. This ends the proof of Lemma 2.

To conclude the proof, we note the following observation:

Remark.

Let $\mathcal{B}, \mathcal{D} = \mathcal{B} \rtimes_\beta \mathbb{Z}$ and $u, \{q_i\}_{i\in\mathbb{Z}}$ be as before . Then $(q_0+q_1)\mathcal{D}(q_0+q_1)$ coincides with the algebra generated by $(q_0+q_1)\mathcal{B}(q_0+q_1)$ and $e_{12} = q_1u = uq_0$.

Proof. With q_0, q_1 as before we have to show that $q_0(u^*)^n bq_0 = q_0(u^*)^n q_n bq_0$ is contained in the algebra generated by $(q_0+q_1)\mathcal{B}(q_0+q_1)$. Assume $n > 1$; we may express q_nbq_0 as

$$q_n b_n q_{n-1} b_{n-1} q_{n-2} ... q_1 b_1 q_0.$$

Then

$$q_0(u^*)^n b q_0 = q_0(u^*)^n q_n b_n q_{n-1} \dots q_1 b_1 q_0 =$$
$$= q_0 u^* \alpha_{n-1}(b_n) q_0 u^* \alpha_{n-2}(b_{n-1}) q_0 \dots q_0 u^* b_1 q_0$$

which is an element in the algebra generated by $(q_0 + q_1)\mathcal{B}(q_0 + q_1)$ and uq_0.

Proof of the theorem.

Clearly the subalgebra generated by e_{12} and all the elements in

$$M_2(\mathbb{C}) * L^\infty([0,1],\nu)$$

with equal occurrence number of e_{12} 's and e_{21} 's coincides with the algebra itself. Thus $M_2(\mathbb{C}) * L^\infty([0,1],\nu)$ with the free product state ϕ is identified with $(q_0 + q_1)\mathcal{D}(q_0 + q_1)$ with the restriction of ψ (which is generated by $uq_0 = q_1 u$, and a). In particular the modular group of ϕ is $\sigma_t^\phi(e_{ij}) = \lambda^{it} e_{ij}$ and σ_t^ϕ is the identity on $L^\infty([0,1],\nu)$.

REFERENCES

[Co] A. Connes,, *Une classification des facteurs de type III,*, Ann. Scient. Ecole Norm. Sup. , 4eme Serie , tome 6, 133-252..

[Vo1] D.Voiculescu,, *Circular and semicircular systems and free product factors, Operator Algebras, Unitary Representations, Enveloping Algebras,*, Progress in Math., **vol 92, Birkhauser,** (1990,), 45-60.

[Vo2] D.Voiculescu,, *Limit laws for random matrices and free product factors,*, Invent. Math., **104** (1991,), 201-220..

[Ra] F. Rădulescu,, *A one parameter group of automorphisms of $\mathcal{L}(F_\infty) \otimes B(H)$ scaling the trace by t,*, C. R. Acad. Sci. Paris, t.314, Serie I, (1992), 1027-1032..

[Ta] M. Takesaki,, *Duality in cross products and the structure of von Neumann algebras of type III,*, Acta Math. 1312 (1973,), 249-310..

This work was elaborated during the time the author was a Miller Research Fellow at U.C. Berkeley.

Florin Rădulescu
Present Address:
Department of Mathematics, U.C. Berkeley
Berkeley, CA. 94709, U.S.A.
Permanent Address:
Institute of Mathematics of the Romanian Academy
Str. Academiei 14 Bucharest, Romania.