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## DUALITY OF THE SPACES OF LINEAR FUNCTIONALS ON DUAL VECTOR SPACES;

By H. S. ALLEN.

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1. Dual linear vector spaces have been studied by Dieudonné [1] and Jacobson [2] and self-dual spaces by Rickart [3], [4]. In this paper it is established that the spaces of linear functionals on dual spaces are dual spaces and that the space of linear functionals on a self-dual space is self-dual and of the same type (symmetric or unitary), assuming that the characteristic of the field of scalars is not two.

2. Let  $E$  and  $F$  be left and right linear vector spaces over a field  $K$ . Suppose there is a bilinear functional  $(x, y)$  defined on  $E \times F$  to  $K$  which is non-degenerate, i. e.  $(x, y) = 0$  for all  $x$  (resp. all  $y$ ) implies  $y = 0$  (resp.  $x = 0$ ): then  $E$  and  $F$  are said to be dual spaces relative to  $(x, y)$ . Let  $F^*$  and  $E^*$  be the left and right  $K$ -spaces whose elements are the linear functionals on  $F$  and  $E$ , the algebraic operations of addition and scalar multiplication being defined as in Bourbaki [5]. If  $x \in E$  and  $f \in E^*$  we write  $f(x) = \langle x, f \rangle$ : the spaces  $E$  and  $E^*$  are dual spaces relative to  $\langle x, f \rangle$ . If  $y \in F$  and  $g \in F^*$  we write  $g(y) = \langle g, y \rangle$ : the spaces  $F^*$  and  $F$  are dual spaces relative to  $\langle g, y \rangle$ . We prove the following theorem.

**THEOREM 1.** — *If the characteristic of  $K \neq 2$ , then  $F^*$  and  $E^*$  are dual spaces.*

If  $y_1 \in F$ , the functional  $f_1$  defined on  $E$  by  $\langle x, f_1 \rangle = (x, y_1)$  belongs to  $E^*$  and the mapping  $y_1 \rightarrow f_1$  is an isomorphic mapping of  $F$  on a subspace  $M$  of  $E^*$ . We shall denote this correspondence by writing  $f_1^x = y_1$ . Similarly  $E$  is isomorphic to a subspace  $N$  of  $F^*$  under a mapping  $x_1 \rightarrow g_1$  where  $\langle g_1, y \rangle = (x_1, y)$  and we write  $g_1^y = x_1$ .

There is a subspace  $Q$  of  $E^*$  which is the algebraic complement of  $M$  (i. e.  $E^*$  is the direct sum  $M + Q$ ) and a subspace  $P$  of  $F^*$  which is the algebraic complement of  $N$ . Suppose  $g \in F^*$  and  $f \in E^*$ . Let  $g = g_1 + g_2$  where  $g_1 \in N$ ,  $g_2 \in P$  and  $f = f_1 + f_2$  where  $f_1 \in M$ ,  $f_2 \in Q$ . If  $g_1^y = x_1$  and  $f_1^x = y_1$  we define

$$\{g, f\} = \frac{1}{2}[\langle x_1, f \rangle + \langle g, y_1 \rangle].$$

It is easily proved that the functional  $\{g, f\}$  is bilinear. Suppose  $f$  is fixed and  $\{g, f\} = 0$  for every  $g$ . Taking  $g_2 = 0$  we obtain

$$\langle g, y_1 \rangle = \langle g_1, y_1 \rangle = (x_1, y_1) = \langle x_1, f_1 \rangle \quad \text{and} \quad 0 = \{g_1, f\} = \frac{1}{2} \langle x_1, f + f_1 \rangle.$$

This holds for every  $x_1 \in E$  and it follows that

$$f + f_1 = 2f_1 + f_2 = 0.$$

Hence  $2f_1 = -f_2 \in M \cap Q$  and therefore  $f_1 = 0, f_2 = 0$  and  $f = 0$ . Similarly  $\{g, f\} = 0$  for every  $f$  implies  $g = 0$ . It follows that  $F^*$  and  $E^*$  are dual spaces relative to  $\{g, f\}$ .

3. A left vector space  $E$  over a sfield  $K$  is said to be *self-dual* if there is an involution  $\alpha \rightarrow \alpha^j$  in  $K$  and a scalar product  $(x, y)$  defined on  $E \times E$  to  $K$  with the properties (i)  $(x, y)$  is linear in  $x$  for every  $y$ , (ii)  $(x, y) = 0$  for all  $y$  implies  $x = 0$ , (iii)  $(y, x) = e(x, y)^j$  where  $e = \pm 1$  is a constant independent of  $x$  and  $y$ . A self-dual space is said to be *symplectic* if every vector is isotropic, i. e.  $(x, x) = 0$ . If there exist non-isotropic vectors in the space, the space is said to be *unitary*, (Rickart [3], [4]). As before  $E^*$  will denote the space of linear functionals on  $E$ . We prove the following result.

**THEOREM 2.** — *If the left vector space  $E$  over a sfield  $K$  of characteristic  $\neq 2$  is self-dual, then the right  $K$ -space  $E^*$  is self-dual. The space  $E^*$  is symplectic or unitary according as  $E$  is symplectic or unitary.*

Let  $E_r$  be the right  $K$ -space whose elements are the elements of  $E$  with addition defined as on  $E$  and scalar multiplication defined by  $xa = a^j x (x \in E, a \in K)$ . Then  $E$  and  $E_r$  are dual spaces relative to  $(x, y)$ . The space  $E_r$  is isomorphic to a subspace  $M$  of  $E^*$ : we have  $x_1 \rightarrow X_1$  where  $\langle x, X_1 \rangle = (x, x_1)$  and we write  $x_1 = X_1^j$ . There is a subspace  $Q$  of  $E^*$  which is the algebraic complement of  $M$ . Suppose  $X$  and  $Y$  in  $E^*$ . Let  $X = X_1 + X_2$ , where  $X_1 \in M, X_2 \in Q$  and  $Y = Y_1 + Y_2$  where  $Y_1 \in M, Y_2 \in Q$ . Let  $X_1^j = x_1$  and  $Y_1^j = y_1$ . We define the functional

$$[X, Y] = \frac{1}{2} [e \langle x_1, Y \rangle + \langle y_1, X \rangle^j]$$

on  $E^* \times E^*$  to  $K$ . It is easily verified that  $[X, Y]$  is linear in  $Y$  for every fixed  $X$ . Suppose  $[X, Y] = 0$  for every  $X$ . Take  $X_2 = 0$  and we obtain

$$\begin{aligned} 0 = [X_1, Y] &= \frac{1}{2} [e \langle x_1, Y \rangle + \langle y_1, X_1 \rangle^j] \\ &= \frac{1}{2} [e \langle x_1, Y \rangle + (y_1, x_1)^j] = \frac{1}{2} e [\langle x_1, Y \rangle + (x_1, y_1)] \\ &= \frac{1}{2} e [\langle x_1, Y \rangle + \langle x_1, Y_1 \rangle] = \frac{1}{2} e \langle x_1, Y + Y_1 \rangle. \end{aligned}$$

This holds for every  $x_1 \in E$  and it follows as in theorem 1 that  $Y = 0$ . Evidently  $[X, Y] = e[Y, X]^j$  and hence  $E^*$  is self-dual with respect to  $[X, Y]$ .

If  $E$  is unitary we may suppose that  $(x, y)$  is hermitian, i. e.  $e = 1$  as indi-

cated by Rickart [3], [4]. There is an element  $x_1 \in E$  such that  $(x_1, x_1) \neq 0$ . If  $X_1^2 = x_1$  we have  $[X_1, X_1] = (x_1, x_1)$  and  $E^*$  is unitary.

If  $E$  is symplectic the form  $(x, y)$  is skew-hermitian, i. e.  $e = -1$  and  $K$  is a field (Rickart). In this case  $a^j = a$  for every  $a \in K$  and  $E^*$  is symplectic.

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