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Duality of the spaces of linear functionals on dual vector spaces


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DUALITY OF THE SPACES OF LINEAR FUNCTIONALS
ON DUAL VECTOR SPACES;

By H. S. ALLEN.

1. Dual linear vector spaces have been studied by Dieudonné [1] and
Jacobson [2] and self-dual spaces by Rickart [3], [4]. In this paper it is estab-
lished that the spaces of linear functionals on dual spaces are dual spaces and
that the space of linear functionals on a self-dual space is self-dual and of the
same type (symmetric or unitary), assuming that the characteristic of the field
of scalars is not two.

2. Let E and F be left and right linear vector spaces over a field K. Suppose
there is a bilinear functional \((a, y)\) defined on \(E \times F\) to \(R\) which is non-degen-
erate, i.e. \((a, y) = 0\) for all \(a\) (resp. all \(y\)) implies \(y = 0\) (resp. \(a = 0\)) : then \(E\)
and \(F\) are said to be dual spaces relative to \((a, y)\). Let \(F^*\) and \(E^*\) be the left
and right K-spaces whose elements are the linear functionals on \(F\) and \(E\), the algebraic
operations of addition and scalar multiplication being defined as in Bourbaki [5].
If \(a \in E\) and \(f \in E^*\) we write \(f(a) = \langle a, f \rangle\) : the spaces \(E\) and \(E^*\) are dual spaces
relative to \(\langle a, f \rangle\). If \(y \in F\) and \(g \in F^*\) we write \(g(y) = \langle g, y \rangle\) : the spaces \(F^*
and \(F\) are dual spaces relative to \(\langle g, y \rangle\). We prove the following theorem.

Theorem 1. — If the characteristic of \(K \neq 2\), then \(F^*\) and \(E^*\) are dual spaces.

If \(y_1 \in F\), the functional \(f_1\) defined on \(E\) by \(\langle x, f_1 \rangle = (x, y_1)\) belongs to \(E^*
and the mapping \(y_1 \mapsto f_1\) is an isomorphic mapping of \(F\) on a subspace \(M\) of \(E^*\).
We shall denote this correspondence by writing \(f_1^x = y_1\). Similarly \(E\) is isomor-
phic to a subspace \(N\) of \(F^*\) under a mapping \(x_1 \mapsto g_1\) where \(\langle g_1, y \rangle = (x_1, y)\)
and we write \(g_1^x = x_1\).

There is a subspace \(Q\) of \(E^*\) which is the algebraic complement of \(M\) (i.e. \(E^*\)
is the direct sum \(M + Q\)) and a subspace \(P\) of \(F^*\) which is the algebraic comple-
ment of \(N\). Suppose \(g \in F^*\) and \(f \in E^*\). Let \(g = g_1 + g_2\) where \(g_1 \in N, g_2 \in P\)
and \(f = f_1 + f_2\) where \(f_1 \in M, f_2 \in Q\). If \(g_1^x = x_1\) and \(f_1^y = y_1\) we define

\[ |g, f| = \frac{1}{2} [\langle x_1, f \rangle + \langle g, y_1 \rangle]. \]
It is easily proved that the functional \( \{ g, f \} \) is bilinear. Suppose \( f \) is fixed and \( \{ g, f \} = 0 \) for every \( g \). Taking \( g = 0 \) we obtain
\[
\langle g, y_1 \rangle = \langle g_1, y_1 \rangle = \langle x_1, y_1 \rangle = \langle x_1, f_1 \rangle \quad \text{and} \quad 0 = \{ g, f \} = \frac{1}{2} \langle x_1, f + f_1 \rangle.
\]
This holds for every \( x_1 \in E \) and it follows that
\[
f + f_1 = 2f + f_1 = 0.
\]
Hence \( 2f = f_1 + f_2 \in M \cap Q \) and therefore \( f_1 = 0 \), \( f_2 = 0 \) and \( f = 0 \). Similarly \( \{ g, f \} = 0 \) for every \( f \) implies \( g = 0 \). It follows that \( F^* \) and \( E^* \) are dual spaces relative to \( \{ g, f \} \).

3. A left vector space \( E \) over a field \( K \) is said to be self-dual if there is an involution \( a \to a' \) in \( K \) and a scalar product \( (x, y) \) defined on \( E \times E \) to \( K \) with the properties (i) \( (x, y) \) is linear in \( x \) for every \( y \), (ii) \( (x, y) = 0 \) for all \( y \) implies \( x = 0 \), (iii) \( (y, x) = e(x, y)' \) where \( e = \pm 1 \) is a constant independent of \( x \) and \( y \). A self-dual space is said to be symplectic if every vector is isotropic, i.e. \( (x, x) = 0 \). If there exist non-isotropic vectors in the space, the space is said to be unitary. (Rickart [3], [4]). As before \( E' \) will denote the space of linear functionals on \( E \). We prove the following result.

**Theorem 2.** — If the left vector space \( E \) over a field \( K \) of characteristic \( \neq 2 \) is self-dual, then the right \( K \)-space \( E^* \) is self-dual. The space \( E^* \) is symplectic or unitary according as \( E \) is symplectic or unitary.

Let \( E_r \) be the right \( K \)-space whose elements are the elements of \( E \) with addition defined as on \( E \) and scalar multiplication defined by \( xa = a'x (x \in E, a \in K) \). Then \( E \) and \( E_r \) are dual spaces relative to \( (x, y) \). The space \( E_r \) is isomorphic to a subspace \( M \) of \( E^* \) : we have \( x_1 \to X_1 \) where \( \langle x, X_1 \rangle = (x, x_1) \) and we write \( X_1 = X_1' \). There is a subspace \( Q \) of \( E^* \) which is the algebraic complement of \( M \). Suppose \( X \) and \( Y \) in \( E^* \). Let \( X = X_1 + X_2 \), where \( X_1 \in M \), \( X_2 \in Q \) and \( Y = Y_1 + Y_2 \) where \( Y_1 \in M \), \( Y_2 \in Q \). Let \( X_1' = x_1 \) and \( Y_1' = y_1 \). We define the functional
\[
[X, Y] = \frac{1}{2} \left[ e \langle x_1, Y \rangle + \langle y_1, X \rangle \right]
\]
on \( E^* \times E^* \) to \( K \). It is easily verified that \( [X, Y] \) is linear in \( Y \) for every fixed \( X \). Suppose \( [X, Y] = 0 \) for every \( X \). Take \( X_2 = 0 \) and we obtain
\[
o = [X_1, Y] = \frac{1}{2} \left[ e \langle x_1, Y \rangle + \langle y_1, X_1 \rangle \right]
= \frac{1}{2} \left[ e \langle x_1, Y \rangle + (y_1, x_1) \right] = \frac{1}{2} e [\langle x_1, Y \rangle + (x_1, y_1)]
= \frac{1}{2} e [\langle x_1, Y \rangle + \langle x_1, Y \rangle] = \frac{1}{2} e \langle x_1, Y + Y \rangle.
\]
This holds for every \( x_1 \in E \) and it follows as in theorem 1 that \( Y = 0 \). Evidently \( [X, Y] = e [Y, X]' \) and hence \( E^* \) is self-dual with respect to \( [X, Y] \).

If \( E \) is unitary we may suppose that \( (x, y) \) is hermitian, i.e. \( e = 1 \) as indi-
cated by Rickart [3], [4]. There is an element \( x_1 \in E \) such that \((x_1, x_1) \neq 0\). If \( x_1^2 = x_1 \) we have \([X_1, X_1] = (x_1, x_1)\) and \( E^* \) is unitary.

If \( E \) is symplectic the form \((x, y)\) is skew-hermitian, i.e. \( e = -i \) and \( K \) is a field (Rickart). In this case \( \alpha^1 = \alpha \) for every \( \alpha \in K \) and \( E^* \) is symplectic.

**BIBLIOGRAPHIE.**