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VLASTIMIL PTÁK

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COMPLETENESS AND THE OPEN MAPPING THEOREM;

BY

VLASTIMIL PTÁK,

(Praha).

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1. Introduction. — In the present paper we intend to give an account of some investigations concerned with the open mapping theorem. The starting point of these investigations was the natural desire to understand what is at the back of this theorem which, undoubtedly, is one of the deepest in functional analysis. Their final aim is to clear this result of unnecessary assumptions — especially that of the metrizability of the spaces in question — and to extend it to as wide a class of topological linear spaces as possible.

These investigations have led quite naturally to the introduction of what we believe to be an interesting notion — namely the notion of B-completeness, which coincides with ordinary completeness in metrizable spaces and which is strong enough to ensure the validity of the open mapping theorem in the general case.

This has been done by the author in a paper entitled *On complete topological linear spaces* published in the *Czechoslovak Mathematical Journal* in 1953. The greater part of the present article is devoted to results obtained there. Of course, some proofs have been simplified and some results given a more general form. We reproduce here only those parts of our original work which have, at least for the present time, obtained their definitive form.

Let us try now to give a sketch of the ideas that have led the author to the notion of B-completeness and to make clear the meaning of the results obtained in the main text.

Let us recall first the classical open mapping theorem. Under some slight restrictions, it assumes the following form :

Let E be a complete normed linear space. Let φ be a continuous linear

mapping of E onto a normed space F. Let F be of the second category in itself. In these conditions, the mapping φ is open and the space F is complete.

A simple analysis of the proof shows that the essential point lies in the following property of complete normed linear spaces : the image of the unit sphere of E is either nondense in F or a neighbourhood of zero in F.

Let us examine more closely what is essential in these considerations. It is the following alternative. Let E be a complete normed linear space and φ a continuous linear mapping of E onto a normed space F. Let us denote by U the unit sphere of E and let us consider the set $\varphi(U)$. We have then the following situation :

Either not even the closure of $\varphi(U)$ is a neighbourhood of zero in F or the set $\varphi(U)$ itself is a neighbourhood of zero in F.

Or in other words : if the closure of $\varphi(U)$ is a neighbourhood of zero in F, then the set $\varphi(U)$ itself is a neighbourhood of zero.

It is thus natural to consider mappings with the following property.

Let T and V be two topological spaces and f a mapping of T onto V. We shall call the mapping f nearly open, if it has the following property. If G is a neighbourhood of some point t_0, then the closure of $f(G)$ is a neighbourhood of $f(t_0)$.

It will perhaps be useful to compare this notion with that of an open mapping. By means of the interior operator the definition of an open mapping may be given by the inclusion

$$f(\operatorname{int} M) \subset \operatorname{int} f(M)$$

which is to be fulfilled by every subset M of T. Nearly open mappings are characterised by almost the same inclusion, namely

$$f(\operatorname{int} M) \subset \operatorname{int} \overline{f(M)}$$

to be fulfilled by every subset M of T.

It is easy to see that the theorem of Banach may now be formulated as follows : let E be a complete normed linear space. Let φ be a continuous linear mapping of E onto a normed space F. Let φ be nearly open. Then φ is open.

Having thus found what we believe to be the substance of the open mapping theorem, we may now formulate our problem. Let us replace normed spaces by locally convex topological vector spaces. How is completeness to be taken in this case so as to obtain the same result? We shall investigate locally convex topological vector spaces E with the following

property :

(B). *Let f be a continuous and nearly open linear mapping of E onto some space F. Then f is open.*

Spaces with property (B) will be called B-complete.

Spaces which fulfill the same condition with the restriction to one-to-one mappings will be called B_r-complete.

The most essential point in the further investigations is a simple result which enables us to characterise the above condition by means of an interesting property of the dual space. We find that the following condition is necessary and sufficient for a space E to possess property (B) : *If Q is a subspace of E' such that, for every neighbourhood U of zero in E, the intersection $Q \cap U^0$ is closed, then Q itself is closed.*

Now this condition is surprisingly similar to a result of Banach in the *Théorie des opérations linéaires.* There is a well-known theorem which affirms that, if Q is a subspace of the adjoint of a Banach space such that Q is "transfiniment fermé", then Q is "régulièrement fermé". In 1942, KREIN and ŠMULYAN threw some more light on this matter by proving the following result;

Let Q be a subspace of the adjoint of a Banach space. Suppose that the intersection of Q with the closed unit sphere is weakly closed. Then Q itself is weakly closed.

If we compare this result with the condition for B-completeness obtained above we see that this result can be used as another proof of the open mapping theorem.

It is surprising that it has not been noticed before that these two results actually mean the same thing.

Clearly it is to be expected that the open mapping theorem will be closely connected with the notion of completeness even in the general case. It is thus natural to try to obtain a similar characterization of complete spaces by means of the structure of their duals. We find that the following condition is necessary and sufficient for a space E to be complete :

If Q is a hyperplane in E' such that, for every neighbourhood of zero U in E, the intersection $Q \cap U^0$ is closed, then Q itself is closed.

If we compare this condition with that for B-completeness, we see at once their deep connection. The same condition is imposed, first on every subspace, second on hyperplanes only.

In any case, we can say at once that completeness is necessary for a space to be B-complete. The theorem of Krein and Šmulyan shows the equivalence of these two properties in the case of a normed linear space. Their reasoning may be applied without essential changes to extend this result to spaces with a countable system of pseudonorms. Does it subsist in the

general case, too? The author expected for some time a positive answer to this question; the study of some concrete spaces, however, has revealed several rather surprising facts, and has led to the discovery of a space which is complete but not B-complete.

We are not going to reproduce the original example in the present paper, since we have obtained simple examples later. Nevertheless, we feel that the original ideas have not lost their interest and we intend to give a sketch of them here. We may return to them later.

If T is a completely regular topological space, we shall denote by $C(T)$ the linear space of all continuous functions on T in the compact-open topology. It is natural enough to examine the connection of the structure of T with the properties of $C(T)$, such as completeness or B-completeness. These considerations are not without interest even from the standpoint of general topology. First of all it is easy to see that $C(T)$ will be complete if and only if every function r defined on T and continuous on every compact subset of T is continuous. At the same time it is possible to show that for $C(T)$ to be B-complete, the following condition is necessary. If M is a dense subset of T such that its intersection with every compact subset of T is compact, then $M = T$. Now if B-completeness were a consequence of completeness, we should have the following implication.

Suppose that T is a space where a function is continuous whenever it is continuous on every compact subset of T. It would follow then that there can be no dense set different from the whole space and such that its intersection with every compact subset is compact.

This implication is not true, however, as may be shown by an example of a suitable T. The corresponding $C(T)$ is, consequently, complete but not B-complete.

The same method may be used to show another perversity of complete spaces. We shall see that a quotient of a complete space need not be complete. To see that, let us take a space T and a closed subset B of T. The mapping which assigns to every element of $C(T)$ its restriction to B as an element of $C(B)$ is easily seen to be both continuous and open. It maps $C(T)$ onto a part of $C(B)$ which consists exactly of those continuous functions on B wich have a continuous extension to the whole of T. Now it is not difficult to construct a space T with a closed subset B such that both $C(T)$ and $C(B)$ are complete but not every function continuous on B has a continuous extension to the whole of T. It follows that the complete space $C(T)$ is mapped on a dense subspace H of the complete space $C(B)$. Since H is different from $C(B)$, it cannot be complete.

Let us turn our attention now to the closed graph theorem. In the case of normed spaces, this theorem is an immediate consequence of the open mapping theorem. To understand this theorem in the general case it is necessary first to clear up the meaning of the assumption that the graph of the mapping is closed. This we owe to A. Robertson and W. Robertson [10].

The closed graph theorem may then be extended to the general case without difficulties if B-completeness of the space in question is assumed.

To sum up : we feel that some progress towards understanding the open mapping and closed graph theorems has been made and

(i) a connection between two classical theorems, namely the open mapping theorem and the theorem on subspaces of the adjoint space has been discovered;
(ii) a suitable notion of completeness, which generalizes the metric completeness, has be found;
(iii) the open mapping and closed graph theorems have been extended to what is, perhaps, the natural boundary of their validity.

It may safely be said that — if B-completeness is taken instead of metric completeness — all results concerned with the open mapping and closed graph theorems, known from the theory of Banach spaces are valid and some of them even in a strengthened form.

We conclude with a few remarks of "historical" character. The discusion of the open mapping theorem and of B-completeness is contained in the paper of V. Pták [8]. The notion of B-completeness has been also introduced independently by H. S. Collins [2] under the name of "full completeness". Collins, however, does not discuss the connection of this concept with the open mapping theorem which forms the natural starting point and justification of our definition. The paper of Collins has been published two years after our own. Our example of a complete space which is not B-complete is the first published. There is an example due to Grothendieck, which can be used to show that B-completeness is not a consequence of completeness. This example has been constructed to illustrate some properties of LF-spaces, and has been published in [4]. We use this opportunity to state that the first example of a complete space with a noncomplete quotient is due to G. Köthe [6]. His example, however, is not of the form $C(T)$.

In this paper, we try to follow the ideas as they occurred to the author. This way of publishing results — so carefully avoided by many authors — is, we feel, the best for an article of this kind. The reader will find that, in many cases, a shorter proof might have been given. Nevertheless, we feel that even a longer proof is justified if it provides further insight into the matter.

2. Terminology and notation. — Terminology, as a rule, coincides with that of Bourbaki. In some points, however, it will be necessary to introduce notations different from those in general use. We intend to explain them in this section. The reader will see for himself how far these changes are justified.

Let T be a nonvoid set, let u be a system of subsets of T with the following properties :

1° the empty set and the set T belong to u;

2° the union of an arbitrary system of sets belonging to u belongs to u as well;

3° the intersection of a finite number of sets belonging to u belongs to u as well.

Then u is called a topology on T and (T, u) a topological space. The topology is called a Hausdorff topology if, for every two distinct points $t_1 \in T$, $t_2 \in T$ there exist two disjoint sets $G_1 \in u$, $G_2 \in u$ such that $t_1 \in G_1$ and $t_2 \in G_2$. A set U is called a neighbourhood of t if there exists a $G \in u$ such that $t \in G \subset U$. A system u_0 of subsets of T is said to be a complete system of neighbourhoods of a point $t_0 \in T$ if the following conditions are fulfilled :

1° every member of u_0 is a neighbourhood of t_0;

2° if V is an arbitrary neighbourhood of t_0, there exists a $U_0 \in u_0$ such that $U_0 \subset V$.

In the sequel, we shall have to compare closures af a given set in different topologies. Since there must be something to remind us in which topology the closure has been taken, we shall denote by uM the closure of the set $M \subset T$ in the topology u.

Let u_1 and u_2 be two topologies on T such that $u_1 \supset u_2$. The topology u_1 is said to be finer than u_2, the topology u_2 coarser than u_1. If M is a given subset of T, we have $u_1 M \subset u_2 M$. The finer topology gives smaller closures.

Suppose now that two topologies u and v on a set T are given. Let us define a system $v(u)$ of subsets of T in the following manner : a set $H \subset T$ will belong to $v(u)$ if and only if, for every $t \in H$, there exists a $U \in u$ such that $t \in U$ and $vU \subset H$. It is easy to see that $v(u)$ is a topology on T as well and that $u \supset v(u)$. We are not going to discuss the properties of this topology before the reader has seen the meaning of it. We shall return to it in the main text after having shown a quite natural way to its introduction.

Let X be a real vector space. We shall denote by X^* the linear space of all linear forms defined on X. If $f \in X^*$, the value of the form f at the point x will be denoted by $\langle x, f \rangle$.

Suppose now that u is a locally convex topology defined on X. The space of all linear forms on X continuous in the topology u is a subspace of X^* and will be denoted by $(X, u)'$.

If E is a locally convex topological vector space with topology u, we shall sometimes write simply E' for $(E, u)'$ in the case that only one topology on E is considered and there is no danger of misunderstanding.

If there are different locally convex topologies on a vector space X to be considered, there will be, in the general case, different dual spaces as well.

The usual notation for polarity is then not sufficient to distinguish in which duality the polar set has been taken. This leads to the following convention : if Q is a subspace of X^* and A a subset of X, we denote by A^Q the set

$$A^Q = \underset{y}{E}[y \in Q, \ |\langle A, y\rangle| \leqq 1].$$

Here, of course, $\langle A, y\rangle$ is the set of all real numbers $\langle a, y\rangle$ where a runs over A. If B is a set of real numbers, the inequality $B \leqq 1$ stands for the system of inequalities $b \leqq 1$ for every $b \in B$.

If E is a locally convex topological vector space, the term neighbourhood of zero is taken to mean, unless the contrary is specified, a closed absolutely convex neighbourhood of zero.

The topology of a locally convex topological linear space is always taken to be a Hausdorff topology. For the sake of brevity, we use the term " convex space " instead of " locally convex Hausdorff topological vector space over the real field ".

If X is a linear space, a topology u on X is said to be a convex topology on X if (X, u) is a convex space.

Suppose that u_1 and u_2 are two convex topologies on a linear space X. The topologies u_1 and u_2 are said to be equivalent if $(X, u_1)' = (X, u_2)'$. If u_1 and u_2 are equivalent, we shall write $u_1 \sim u_2$.

The terms linear functional or linear mapping are taken in the sense of algebra only with no condition of continuity whatsoever.

Let (E, u) and (F, v) be two convex spaces. The points of the cartesian product $E \times F$ will be denoted $[x, y]$, where $x \in E$, $y \in F$. The space $(E \times F)'$ consists of all couples $[x', y']$ where $x' \in (E, u)'$, $y' \in (F, v)'$ the scalar product being defined by

$$\langle [x, y], [x', y'] \rangle = \langle x, x' \rangle + \langle y, y' \rangle.$$

3. The open mapping theorem and the closed graph theorem for one-to-one mappings. — The reader is requested to look through the paragraph on terminology and notation. For the motivation of the definitions below, see the introduction.

DEFINITION 1. — *Let T and V be two topological spaces and f a mapping of T into V. The mapping f will be called nearly open if it has the following property. If G is a neighbourhood of some point $t_0 \in T$, then the closure of $f(G)$ is a neighbourhood of $f(t_0)$ in $f(T)$.*

In the present paragraph we shall confine our attention to one-to-one mappings. The general case may be obtained from this one by taking quotients, which is a purely technical matter. This restriction has the further advantage of presenting all the essential points of the theory without troublesome technical details. At the same time, we can simplify matters

by considering one space only with two topologies. Indeed, suppose we have two topological spaces (T, u) and (T_1, u_1) and a one-to-one mapping f of (T, u) onto (T_1, u_1). It is easy to see that the system of all sets $f^{-1}(G)$, where G runs over u_1, is a topology on T. We shall denote it by v. Clearly every topological property of the mapping f may be expressed in terms of the two topologies u and v. Thus, e. g., for f to be continuous it is sufficient and necessary that v be coarser than u. Instead of considering a continuous one-to-one mapping of (T, u) onto some other space it is thus sufficient to consider another topology v on T, coarser than u.

Let us suppose now that we have a set T with two topologies u and v, v coarser than u. The identical mapping of (T, u) onto (T, v) is thus continuous. Suppose now that it is nearly open. How can this be expressed by means of the two topologies?

If U is a neighbourhood of t_0 in (T, u), its image in (T, v) should be dense in some neighbourhood of t_0 in (T, v). It follows that the set vU should be a v-neighbourhood of t_0. If we recall the definition of the topology $v(u)$ in the preceding paragraph we may clearly reformulate this fact in the following manner : the topology $v(u)$ is coarser than v or $v(u) \subset v$.

We have the following proposition :

(3.1). — *Let u and v be two convex topologies on a linear space X and let $u \supset v$. We have then $u \supset v(u) \supset v$ and $v(u)$ is a convex topology as well.*

Proof. — The inclusion $u \supset v(u)$ is easily seen to hold for any two topologies without any particular assumptions. Now let V be a set open in the topology v and let $t \in V$. The topology v being completely regular there exists a $G \in v$ such that $t \in G$ and $vG \subset V$. Since $u \supset v$, we have $G \in u$. This proves the inclusion $v(u) \supset v$. The topology v being Hausdorff, it follows from this inclusion that $v(u)$ is Hausdorff as well. The rest is easy.

We have seen already that the identical mapping of (X, u) on (X, v) is nearly open if and only if $v(u) \subset v$. If $u \supset v$, we have $v(u) \supset v$ according to the preceding lemma. It follows that, in the case $u \supset v$, the identical mapping of (X, u) onto (X, v) is nearly open if and only if $v(u) = v$.

Our further considerations are based on the following proposition :

(3.2). — *Let (E, u) be a convex space and let $Y = (E, u)'$. Let Y be equipped vith the topology $\sigma(Y, E)$. Now let v be another convex topology on E, coarser than u. Let us write $Q = (E, v)'$ so that $Q \subset Y$. The following two conditions are equivalent :*

1° $v(u) \sim v$;

2° *for every neighbourhood of zero U in (E, u), the intersection $Q \cap U^Y$ is closed in Y.*

PROOF. — Suppose first that $v(u) \sim v$. It follows that, for every U, the set vU is a neighbourhood of zero in the finest convex topology equivalent to v. [In the notation of the French school, this topology would be denoted by $\tau(E, Q)$.] It follows that the set $(vU)^Q$ is compact in the topology $\sigma(Q, E)$. Since $\sigma(Q, E)$ and $\sigma(Y, E)$ coincide on Q, the set $(vU)^Q$, considered as a subset of Y, is weakly compact and, consequently, closed in Y.

The proof will be concluded if we show that, for every U, we have $Q \cap U^Y = (vU)^Q$. First of all, the set U being absolutely convex, its closure is equal to its bipolar set, so that $vU = U^{QE}$. Hence $(vU)^Q = U^{QEQ} = U^Q$. It follows immediately from the definition of polarity that $U^Q = Q \cap U^Y$. We have thus

$$(vU)^Q = (U^{QE})^Q = U^Q = Q \cap U^Y$$

which completes the proof.

On the other hand, suppose that $Q \cap U^Y$ is closed for every U. Since U^Y is compact in $\sigma(Y, E)$ and $Q \cap U^Y$ is closed in U^Y, it follows that $Q \cap U^Y$ is compact in $\sigma(Y, E)$ and, consequently, in $\sigma(Q, E)$. Hence $(Q \cap U^Y)^E$ is a neighbourhood of zero in the topology $\tau(E, Q)$. Similarly as in the preceding part of the proof we obtain

$$(Q \cap U^Y)^E = (U^Q)^E = vU.$$

We have thus obtained the inclusion $\tau(E, Q) \supset v(u)$. Since $v(u) \supset v$ and $v \supset \sigma(E, Q)$, we have

$$\tau(E, Q) \supset v(u) \supset \sigma(E, Q)$$

which is the same as $v(u) \sim v$.

(3.3). — *Let (E, u) be a convex space, let Y be the space $(E, u)'$ in the topology $\sigma(Y, E)$. The following properties of E are equivalent :*

1° *every continuous and nearly open one-to-one linear mapping of E into some convex space F is open;*

2° *let v be a convex topology on E such that $v \subset u$ and $v(u) \sim v$; then $v \sim u$;*

3° *let v be a convex topology on E such that $v \subset u$ and $v(u) = v$; then $v = u$;*

4° *let v be a convex topology on E such that $v \subset u$ and $v(u) \subset v$; then $v = u$;*

5° *let Q be a dense subspace of Y; if every intersection $Q \cap U^Y$ is closed in Y, then Q itself is closed in Y (and, consequently, $Q = Y$).*

PROOF. — The equivalence of properties 1°, 3° and 4° is an immediate consequence of the remarks preceding lemma (3.1) and of lemma (3.1) itself.

Suppose that 2° is fulfilled and let us have a convex topology v on E, coarser than u and such that $v(u) = v$. It follows that $v \sim u$. It is easy to

see that $v \sim u$ implies $v(u) = u$. This, together with $v(u) = v$, gives $v = u$ so that 3° is established. Suppose that 3° is fulfilled and let Q be a dense subspace of Y such that $Q \cap U^Y$ is closed in Y for every U. For every u-neighbourhood of zero U, let us consider the set $U_0 = (Q \cap U^Y)^E$. The system of the sets U_0 forms a complete system of neighbourhoods of zero in a convex topology v. This is easy to prove with the exception, perhaps, of the fact that v is Hausdorff. To see that, it is sufficient to show that the intersection of the system U_0 is the point zero. Now if a point x_0 belongs to every $(Q \cap U^Y)^E$, we have $\langle x_0, Q \cap U^Y \rangle \leqq 1$ for every U, so that $\langle x_0, Q \rangle = 0$. The space Q being dense in Y, we have $x_0 = 0$, so that v is separated.

We have thus defined another convex topology v on E; v is coarser than u since $U_0 \supset U$ for every U. It follows that $(E, v)' \subset Y$. Clearly $Q \subset (E, v)'$. On the other hand, let $y \in (E, v)'$. There exists a U such that $|\langle U_0, y \rangle| \leqq 1$. This implies $y \in U_0^Y$. We have, however

$$U_0^Y = (Q \cap U^Y)^{EY}.$$

The set $Q \cap U^Y$ being closed, $(Q \cap U^Y)^{EY} = Q \cap U^Y$ whence $y \in Q \cap U^Y$, so that $(E, v)' = Q$. We assert now that, for every U, we have $U_0 = vU$. The set U being absolutely convex, the closure vU coincides with the bipolar set U^{QE}, whence

$$vU = U^{QE} = (Q \cap U^Y)^E = U_0.$$

It follows that $v(u) = v$. This, together with 3°, implies that $v = u$ so that $Q = (E, v)' = (E, u)' = Y$ and 5° is established.

Suppose now that 5° holds and let v be a convex topology on E, $v \subset u$ and $v(u) \sim v$. If we denote by Q the space $(E, v)'$, the intersection $Q \cap U^Y$ is closed in Y for every U according to (3.2). The space Q is dense in Y. Indeed, suppose there is an $x_0 \in E$ such that $x_0 \neq 0$ and $\langle x_0, Q \rangle = 0$. It follows from the Hahn-Banach theorem that there exists a linear functional f on E, continuous in the topology v, with $f(x_0) \neq 0$. Since f may be considered as an element of Q, we have a contradiction with $\langle x_0, Q \rangle = 0$. According to 5°, we have $Q = Y$ whence $v \sim u$. The last implication is thus established.

DEFINITION 2. — *A convex space E is said to be B_r-complete if it fulfills one of the conditions of the preceding proposition.*

We are not going now to investigate more closely the class of B_r-complete spaces. It will turn out later that it is properly contained in that of complete spaces. In the rest of this section we shall endeavour to obtain the most general form of the open mapping and closed graph theorems.

We have thus far restricted our attention to the behaviour of topologies coarser than the given topology u of E. It is interesting and — as we shall see later — useful to consider even the case where the assumption $u \supset v$ is

dropped. It turns out, however, that in this general case the topology $v(u)$ need not be Hausdorff.

[We recall that, in the preceding case, the fact that $v(u)$ is Hausdorff was a consequence of the inclusion $v(u) \supset v$ which, in its turn, follows from $u \supset v$.] We have the following proposition :

(3.4). — *Let* (E, u) *be a convex space. The following condition is necessary and sufficient for* (E, u) *to be* B_r*-complete : if* v *is a convex topology on E such that* $v(u)$ *is Hausdorff and* $v(u) \subset v$, *then* $v \supset u$.

PROOF. — Suppose first that (E, u) is B_r-complete. Let us denote by w the topology $v(u)$. We have thus $w \subset u$ and $w \subset v$ according to our assumption. Let us form now the topology $w(u)$. If U is a given u-neighbourhood of zero in E, we have $vU \subset wU$ so that $w = v(u) \supset w(u)$. The space (E, u) being B_r-complete, it follows from $w \subset u$ and $w(u) \subset w$ that $w = u$, whence $v \supset w = u$ and the first implication is established.

The second implication follows immediately from the fact that our condition reduces to condition 3° of (3.3) if $v \subset u$. Before giving this lemma an interpretation in familiar terms it will be necessary to examine conditions under which the topology $v(u)$ is Hausdorff. We have the following lemma :

(3.5). — *Let E be a linear space, u and* v *two convex topologies on E. The following conditions are equivalent* :

1° *the topology* $v(u)$ *is Hausdorff;*

2° *for any two points* x_1 *and* x_2 *of* E, *different from each other, there exist a u-neighbourhood of zero U and a v-neighbourhood of zero V such that* $x_1 + U$ *does not meet* $x_2 + V$;

3° *the diagonal in* $(E, u) \times (E, v)$ *is closed;*

4° *the set H of those* $x' \in (E, u)$ *which are continuous in the topology* v, *is dense in* $(E, u)'$.

PROOF. — Suppose that 1° holds and let $x_1 \neq x_2$. The topology $v(u)$ being Hausdorff, there exists a u-neighbourhood of zero U such that x_2 does not belong to $x_1 + vU = v(x_1 + U)$. It follows that there exists a v-neighbourhood of zero V such that $x_2 + V$ does not meet $x_1 + U$ and 2° is established. The equivalence of 2° and 3° is obvious. Suppose that 3° is fulfilled and that 4° fails. Then there exists an $x_0 \neq 0$ such that $\langle x_0, H \rangle = 0$. The diagonal D being closed in $(E, u) \times (E, v)$, there exist functionals $x' \in (E, u)$ and $y' \in (E, v)'$ such that

$$\langle D, [x', y'] \rangle = 0 \qquad \text{and} \qquad \langle [x_0, 0], [x', y'] \rangle \neq 0.$$

The first equation means that $\langle x, x' \rangle + \langle x, y' \rangle = 0$ for every $x \in E$ so that x' is continuous on E both in u and v. Hence $x' \in H$. We have,

however

$$\langle x_0, x' \rangle = \langle [x_0, 0], [x', y'] \rangle \neq 0$$

which is a contradiction, Let 4° be fulfilled and suppose that x_0 belongs to every vU where U runs over all u-neighbourhoods of zero. We are to show that $x_0 = 0$. If this were not the case, there would be an $x' \in H$ with $\langle x_0, x' \rangle > 0$. Let us denote by U the set of all $x \in E$ which fulfill the inequality

$$|\langle x, x' \rangle| \leq \frac{1}{2} \langle x_0, x' \rangle.$$

Clearly U is a u-neighbourhood of zero. Since x' is continuous in the topology v as well, we have $U = vU$ whence $x_0 \in U$ which is a contradiction. The proof is complete.

DEFINITION 3. — *Let f be a linear mapping of a convex space E into some convex space F. The subset of $E \times F$ consisting of all points of the form $[x, f(x)]$ is called the graph of f.*

Lemma (3.4) may now be given the following form :

(3.6). — *Let E be a convex space. Then the following condition is sufficient and necessary for E to be B_r-complete.*

Let f be a one-to-one linear mapping of E into some convex space F. If the graph of f is closed in $E \times F$ and if f is nearly open, then f is open.

We shall see later that this result is a special case of the closed graph theorem. We shall need first the following simple lemma, a strengthening of condition 4° of (3.5).

(3.7). — *Let f be a linear mapping of a convex space E into some convex space F. The graph G of f is closed in $E \times F$ if and only if the subspace H of F' consisting of those $z' \in F'$ for which $\langle f(x), z' \rangle$ is continuous on E, is dense in F'.*

PROOF. — Suppose that the graph of f is closed in $E \times F$ and let $\langle z_0, H \rangle = 0$ for some $z_0 \in F$. We are going to show that the point $[0, z_0]$ belongs to the closure of G. Suppose not. Then there is a $[x', z'] \in E' \times F'$ such that

$$\langle G, [x', z'] \rangle = 0 \qquad \text{and} \qquad \langle [0, z_0], [x', z'] \rangle \neq 0.$$

It follows that $\langle x, x' \rangle + \langle f(x), z' \rangle = 0$ for every $x \in E$ whence $z' \in H$. At the same time $\langle z_0, z' \rangle \neq 0$ which is a contradiction. Hence $[0, z_0] \in G$, so that $z_0 = 0$. The density of H is thus established.

On the other hand suppose that H is dense in F'. Suppose that the point $[x_0, y_0] \in E \times F$ does not belong to G. It follows that $y_0 - f(x_0) \neq 0$.

Since H is dense in F', there is a $y' \in H$ such that $\langle y_0 - f(x_0), y' \rangle \neq 0$. Since $y' \in H$, there exists an $x' \in E'$ such that $\langle x, x' \rangle + \langle f(x), y' \rangle = 0$ for every $x \in E$. We have thus $\langle G, [x', y'] \rangle = 0$ and

$$\begin{aligned}\langle [x_0, y_0], [x', y'] \rangle &= \langle x_0, x' \rangle + \langle y_0, y' \rangle \\ &= \langle x_0, x' \rangle + \langle f(x_0), y' \rangle + \langle y_0 - f(x_0), y' \rangle \neq 0\end{aligned}$$

so that $[x_0, y_0]$ does not belong even to the closure of G. The proof is concluded.

Definition 4. — *Let T and V be two topological spaces and f a mapping of T into V. The mapping f will be called nearly continuous if it has the following property. If $x_0 \in T$ and H is a neighbourhood of $f(x_0)$ in V, then the closure of $f^{-1}(H)$ is a neighbourhood of x_0.*

(3.8). Theorem. — *Let F be a convex space and E a B_r-complete convex space. Let f be a linear mapping of F into E the graph of which is closed in $F \times E$. If f is nearly continuous, then f is continuous.*

Proof. — The theorem will be proved if we show that f is weakly continuous. Indeed, we know that, for every neighbourhood of zero U in E, the closure of $f^{-1}(U)$ is a neighbourhood of zero in F. If f is weakly continuous, the set $f^{-1}(U)$ is closed for every U so that f is continuous.

To show that f is weakly continuous, let us consider the subspace Q of E' consisting of those $x' \in E'$ for which the functional $\langle f(z), x' \rangle$ is continuous on F. The subspace Q is dense in E' according to (3.7). Let us show now that, for every neighbourhood of zero U in E, the set $Q \cap U^0$ is closed in E'. We note first that, for every neighbourhood of zero V in F, the set $f(V)^0$ is contained in Q. Indeed, if $x' \in f(V)^0$ the function $\langle f(z), x' \rangle$ is continuous since it is bounded on a neighbourhood of zero in F. Now let a neighbourhood of zero U in E be given. According to our assumption, the set $\overline{f^{-1}(U)}$ is a neighbourhood of zero in F. Let us denote it by V. First of all, let us prove the inclusion $Q \cap U^0 \subset f(V)^0$. If $x' \in U^0 \cap Q$, the function $\langle f(z), x' \rangle$ is continuous and $\langle U, x' \rangle \leqq 1$. Now $\langle f(f^{-1}(U)), x' \rangle \leqq 1$ and, by continuity $\langle f(V), x' \rangle \leqq 1$ and our inclusion is established. We have now

$$Q \cap U^0 \subset f(V)^0 \subset Q$$

whence $Q \cap U^0 = f(V)^0 \cap U^0$ which is a closed set in E'. It follows that $Q = E'$ so that f is weakly continuous. The proof is complete.

(3.9) *Let E be a B_r-complete space, let E_0 be a closed subspace of E. Then E_0 is B_r-complete.*

Proof. — Let Q_0 be a dense subspace of E'_0 such that $Q_0 \cap U_0^0$ is $\sigma(E'_0, E_0)$ closed for every neighbourhood of zero U_0 in E_0. Let us denote

by Q the set of all functionals $x' \in E'$ such that their restriction $\varphi(x')$ to E_0 is a member of Q_0. Suppose that we have $\langle x_0, Q \rangle = 0$ for some $x_0 \in E$. If $x_0 \in E_0$, we have $0 = \langle x_0, q \rangle = \langle x_0, \varphi(q) \rangle$ for every $q \in Q$, so that $\langle x_0, Q_0 \rangle = 0$ whence $x_0 = 0$. If x_0 non $\in E_0$, there exists an x'_0 such that $\langle E_0, x'_0 \rangle = 0$ and $\langle x_0, x'_0 \rangle = 1$. Now $x'_0 \in Q$ since $\varphi(x'_0) = 0 \in Q_0$. This is, however, a contradiction with $\langle x_0, Q \rangle = 0$. It follows that Q is dense in E'.

Now let U be an arbitrary neighbourhood of zero in E. We have clearly

$$U^0 \cap \varphi^{-1}(Q_0) \subset \varphi^{-1}((U \cap E_0)^0 \cap Q_0) \subset \varphi^{-1}(Q_0)$$

whence

$$U^0 \cap \varphi^{-1}(Q_0) = U^0 \cap \varphi^{-1}((U \cap E_0)^0 \cap Q_0);$$

the set $(U \cap E_0)^0 \cap Q_0$ being closed in E'_0, it follows that $\varphi^{-1}((U \cap E_0)^0 \cap Q_0)$ is closed in E'; hence $U^0 \cap \varphi^{-1}(Q_0)$ is closed in E'. It follows that $\varphi^{-1}(Q_0) = E'$ so that $Q_0 = E'_0$. The proposition is established.

(3. 10). — *Let P be a B_r-complete space and Q a convex space. Let f be a one-to-one linear mapping of a subspace $P_0 \subset P$ into Q. Suppose that the graph of f is closed in $P \times Q$. If f is nearly open, then f is open.*

Proof. — Let us denote by F the space $f(P)$. For every $x \in F$, let us define $g(x) \in P_0$ by the postulate that $g(x) \in P_0$ and $f(g(x)) = x$. The graph of f being closed in $P \times Q$, the graph of g is closed in $F \times P$. Clearly g is nearly continuous on F. It follows from the preceding theorem that g is continuous and, consequently, f open.

4. The open mapping theorem. — We turn now to the general form of the open mapping theorem without the restriction to one-to-one mappings. We prove first a proposition analogous to (3.3).

(4. 1). — *Let (E, u) be a convex space, let Y be the space $(E, u)'$ in the topology $\sigma(Y, E)$. The following properties of E are equivalent :*

1° *every continuous and nearly open linear mapping of E into some convex space F is open;*

2° *let Q be a subspace of Y such that every intersection $Q \cap U^Y$ is closed in Y; then Q is closed in Y.*

Proof. — Suppose that 1° is fulfilled and let Q be a subspace of Y such that $Q \cap U^Y$ is closed in Y for every neighbourhood of zero U in E. Let us denote by f the canonical mapping of E onto $F = E/Q^0$. In F we define a convex topology v by means of the sets $f((Q \cap U^0)^0)$ where U runs over all neighbourhoods of zero in E. Since $f^{-1}(f((Q \cap U^0)^0)) \supset (Q \cap U^0)^0 \supset U$, the mapping f is continuous. Clearly $(E/Q^0, v)' = Q$. Since $f(U)^Q = Q \cap U^0$,

we have

$$v f(U) = f(U)^{QF} = f((Q \cap U^0)^0)$$

so that f is nearly open. It follows that f is open or, in other words, that v is equal to the quotient topology, whence $Q = Q^{00}$.

On the other hand, suppose that 2^0 is fulfilled and let f be a continuous and nearly open linear mapping of E onto some convex space (F, v). Put $Q = f'(F')$. It is easy to see that

$$Q \cap U^0 = f'(f(U)^0).$$

Since $f(U)^0 = (v f(U))^0$ and $v f(U)$ is a neighbourhood of zero in (F, v), the set $f(U)^0$ is compact in $\sigma(F', F)$. The mapping f' being continuous in the topologies $\sigma(F', F)$ and $\sigma(E', E)$ the set $f'(f(U)^0)$ is compact in $\sigma(E', E)$. It follows from our assumption that Q is closed in E'. Let us denote by w the convex topology on F defined by the sets $f(U)$. Since f is continuous, w is finer than v. We intend to show now that $w \sim v$. It is sufficient to show that $(F, w)' \subset (F, v)'$. To see that, take a linear form g on F such that g is bounded on some set $f(U)$. The linear form on E defined by $g(f(x))$ is thus continuous on E so that there exists an $x' \in E'$ such that $g(f(x)) = \langle x, x' \rangle$ for every $x \in E$.

Suppose now that x' non $\in Q$. Since Q is closed in E', there exists an $x_0 \subset E$ such that

$$\langle x_0, Q \rangle = 0 \qquad \text{and} \qquad \langle x_0, x' \rangle \neq 0.$$

We have thus

$$\langle x_0, f'(F') \rangle = 0 \qquad \text{whence} \quad \langle f(x_0), F' \rangle = 0$$

so that $f(x_0) = 0$. On the other hand

$$g(f(x_0)) = \langle x_0, x' \rangle \neq 0$$

which is a contradiction. It follows that $x' \in Q$, so that $x' = f'(z')$ for a suitable $z' \in F'$. Hence

$$g(f(x)) = \langle x, f'(z') \rangle = \langle f(x), z' \rangle$$

for every $x \in E$ so that $g(z) = \langle z, z' \rangle$ for every $z \in F$. The equivalence $w \sim v$ is thus established. This equivalence implies $v(w) = w$. It follows from our assumption that $v(w) = v$. Hence $v = w$ so that the mapping f is open.

Definition 5. — *A convex space E is said to be B-complete if it fulfills one of the conditions of the preceding proposition.*

(4. 2). — *Let E be a B-complete convex space. Then E is B_r-complete.*

Proof. — This is an immediate consequence of (3.3) and (4.1). It is hardly to be expected that the converse is true.

(4. 3). — *Let (E, u) be a convex space and let u_1 be a finer convex topology on E such that $u_1 \sim u$. If (E, u) is B-complete, then (E, u_1) is B-complete. If (E, u) is B_r-complete, then (E, u_1) is B_r-complete.*

Proof. — This is an immediate consequence of the dual characterizations in (3.3) and (4.1).

The proof of the second part of (4.1) may be adapted to obtain a simple property of permanence for B-completeness. Although we have now several different ways of proving it more directly, we choose the following one on account of the further information it provides of the structure of the dual space.

(4. 4). — *Let f be a continuous and nearly open linear mapping of a convex space E onto a convex space F. If E is B-complete then F is B-complete.*

Proof. — Let Q be a subspace of F' such that $Q \cap V^0$ is $\sigma(F', F)$ closed for every neighbourhood of zero V in F. Let us examine the subspace $f'(Q)$ in E'. We have

$$f'(Q) \cap U^0 = f'(f(U)^0 \cap Q)$$

for every neighbourhood of zero U in E. Now $f(U)^0$ is the polar of a neighbourhood of zero in F, so that $f(U)^0 \cap Q$ is compact in $\sigma(F', F)$. Hence $f'(Q) \cap U^0$ is weakly compact for every U so that $f'(Q)$ is closed in E'. It follows that Q is closed in F'.

(4. 5). — *Let f be a one-to-one linear mapping of a convex space F onto a closed subspace of a convex space E. Suppose that f is open and nearly continuous. If E is B_r-complete then F is B_r-complete.*

Proof. — Let us denote by E_1 the space $f(F)$. According to our assumption E_1 is closed in E so that E_1 is B_r-complete according to (3.9), Let us denote by g the linear mapping from E_1 onto F which is inverse to f. Clearly g is continuous and nearly open. It follows that g is open so that f is both open and continuous. Hence F is both algebraically and topologically isomorphic to the B_r-complete space E_1.

We are going now to prove the general form of the open mapping theorem. We shall need first a simple lemma.

(4. 6). — *Let E and F be two convex spaces. Let E_0 be a dense subspace of E. Let f be a linear mapping of E_0 into F, the graph of which is closed in $E \times F$. Then the subspace H of F' consisting of those $z' \in F'$ for which $\langle f(x), z' \rangle$ is continuous on E_0, is dense in F'.*

PROOF. — Suppose that $\langle z_0, H \rangle = 0$ for some $z_0 \in F$. We are going to show that the point $[0, z_0]$ belongs to the closure of G. Suppose not. Then there is a point $[x', z'] \in E' \times F'$ such that

$$\langle G, [x', z'] \rangle = 0 \quad \text{and} \quad \langle [0, z_0], [x', z'] \rangle \neq 0.$$

It follows that

$$\langle x, x' \rangle + \langle f(x), z' \rangle = 0$$

for every $x \in E_0$ whence $z' \in H$. At the same time $\langle z_0, z' \rangle \neq 0$ which is a contradiction. Hence $[0, z_0] \in G$ so that $z_0 = 0$. The density of H is thus established.

(4.7). — THEOREM. — *Let E be a B-complete convex space and F a convex space. Let f be a linear mapping of a subspace $E_0 \subset E$ into F, the graph of which is closed in $E \times F$. If f is nearly open, then f is open.*

PROOF. — Let us denote by E_1 the closure of E_0 in E and by F_1 the space $f(E_0)$. Clearly the graph of f will be closed in $E_1 \times F_1$, The space E_1 is B-complete according to (3.9). Hence it is sufficient to prove our theorem under the additional assumption that E_0 is dense in E and that the mapping f is onto. Let us denote by H the set of all $z' \in F'$ such that $\langle f(x), z' \rangle$ is continuous on E_0. If $z' \in H$, there is an $x' \in E'$ such that $\langle f(x), z' \rangle = \langle x, x' \rangle$ for every $x \in E_0$. The space E_0 being dense in E, there is exactly one x' of that property. We shall write $x' = f'(z')$. We have thus a mapping f' of H into E'. Let us denote by Q the subspace $f'(H) \subset E'$. Clearly f' is a continuous mapping of $(H, \sigma(H, F))$ onto $(Q, \sigma(Q, E_0))$. Let U be an arbitrary neighbourhood of zero in E. The mapping f being nearly open, the set $\overline{f(U \cap E_0)}$ is a neighbourhood of zero in F so that the set $f(U \cap E_0)^0$ is compact in $\sigma(F', F)$. If $z' \in f(U \cap E_0)^0$, the linear form $\langle f(x), z' \rangle$ is bounded on $U \cap E_0$ so that $z' \in H$. Hence $f(U \cap E_0)^0 \subset H$ for every neighbourhood of zero in E. Further it is easy to see that $Q \cap U^0 = f'(f(U \cap E_0)^0)$, so that $Q \cap U^0$ is $\sigma(E', E_0)$ compact. The topologies $\sigma(E', E_0)$ and $\sigma(E', E)$ coïncide, however, on U^0 so that $Q \cap U^0$ is $\sigma(E', E)$ closed in E'. The space E being B-complete, the space Q is closed in E'. Let us denote by t the topology on F defined by the sets $\overline{f(U \cap E_0)}$ where U runs over all neighbourhoods of zero in E. Let us denote by w the topology on F defined by the sets $f(U \cap E_0)$ where U runs over all neighbourhoods of zero in E. Clearly $t \subset w$. Since H is dense in F', the topology t is Hausdorff so that both t and w are convex topologies on F. Since $t \subset w$, we have $(F, w)' \supset (F, t)'$. We have seen already that $(F, t)' = H$.

We are going to show now that $w \sim t$. To see that, let us take a linear form g on F, bounded on some set $f(U \cap E_0)$. It follows that the function

$g(f(x))$ is continuous on E_0 so that there exists a point $x'_0 \in E'$ such that $g(f(x)) = \langle x, x'_0 \rangle$ for every $x \in E_0$. Suppose that x'_0 does not belong to $f'(H)$. We know already that this subspace is closed in E'. It follows that there exists an $x_0 \in E$ such that

$$\langle x_0, f'(H) \rangle = 0 \qquad \text{and} \qquad \langle x_0, x'_0 \rangle \neq 0.$$

Suppose that $x_0 \in E_0$, We have then

$$\langle f(x_0), H \rangle = \langle x_0, f'(H) \rangle = 0$$

so that $f(x_0) = 0$, On the other hand,

$$g(f(x_0)) = \langle x_0, x'_0 \rangle \neq 0$$

which is a contradiction. It follows that x_0 non $\in E_0$. The point $[x_0, 0]$ does not belong to the graph of f. It follows that there exists a point $[x'_1, z'_1] \in E' \times F'$ such that

$$\langle x, x'_1 \rangle + \langle f(x), z'_1 \rangle = 0$$

for every $x \in E_0$ and $\langle x_0, x'_1 \rangle \neq 0$. Hence $z' \in H$ and $x'_1 \in f'(H)$ so that $\langle x_0, x'_1 \rangle = 0$. The contradiction obtained proves that $x'_0 \in f'(H)$. The inclusion $(F, w)' \subset (E, t)'$ is thus established.

We have thus, on F, three convex topologies : the original topology v and the two topologies w and t. We have $t \subset w$ and $t \sim w$. The mapping f being nearly open, we have $t \subset v$. Since $t \subset v$, we have $t(w) \subset v(w)$. Since $t \sim w$, we have $t(w) = w$. We have, accordingly,

$$w = t(w) \subset v(w) = t \subset v \qquad \text{so that} \quad w \subset v.$$

This inclusion shows that the mapping f is open. The proof is complete.

We may remark here that another proof of the preceding theorem may be obtained in the following manner. We prove first the following lemma. Let E and F be two convex spaces, let f be a linear mapping of a subspace $E_0 \subset E$ into F. Suppose that the graph of f is closed in $E \times F$. Let us denote by Z the set of all $x_0 \in E_0$ with $f(x_0) = 0$. Then Z is closed in E. To prove this lemma, take a point $z_0 \in E$ which belongs to the closure of Z. We are going to show that the point $[z_0, 0]$ belongs to the closure of the graph of f. Indeed, if U is an arbitrary neighbourhood of zero in E and V an arbitrary neighbourhood of zero in F, there exists a point $x_0 \in Z$ such that $x_0 \in z_0 + U$. We have thus

$$x_0 \in z_0 + U \qquad \text{and} \qquad f(x_0) = 0 \in V.$$

Hence

$$z_0 \in E_0 \qquad \text{and} \qquad f(z_0) = 0.$$

We may form now the quotient E/Z which is B-complete according

to (4.4). Let us denote by k the canonical mapping of E onto E/Z. There exists a one-to-one linear mapping g of E_0/Z onto $f(E_0)$ such that $f = g \circ k$. We show next that g is nearly open and that its graph is closed in $E/Z \times F$. The rest is a consequence of (3.10). The resulting proof is, perhaps, more simple than the preceding one; it is, however, less direct and certainly longer.

We shall need the following lemma.

(4.8). — *Let E and F be two convex spaces. Let F be a t-space (espace tonnelé). Let f be a linear mapping of F into E, let g be a linear mapping of E onto F. Then f is nearly continuous and g nearly open.*

PROOF. — This is an immediate consequence of the definition of a t-space.

Many of the preceding results have interesting corollaries based on this lemma. We formulate the following two consequences of (3.8) and (4.7).

(4.9). — THEOREM. — *Let F be a convex t-space, let E be a B_r-complete convex space. Let f be a linear mapping of F into E the graph of which is closed in $F \times E$. Then f is continuous.*

(4.10). — THEOREM. — *Let E be a B-complete convex space, let F be a convex t-space. Let be a linear mapping of a subspace $E_0 \subset E$ onto F. Suppose that the graph of f is closed in $E \times F$. Then f is open.*

5. **Complete convex spaces.** — One cannot help noticing a striking similarity between the dual condition for B-completeness and a property of normed spaces discovered by Banach. Indeed, we find, on page 129 of the *Théorie des opérations linéaires*, essentially the following result :

Let E be a complete normed linear space. Let Q be a subspace of E' such that the intersection of Q with the closed unit sphere of E is weakly closed. Then Q itself is weakly closed.

If we compare this result with the condition for B-completeness obtained above we see at once this result can be used as another proof of the open mapping theorem. These two results have always been treated separately and it has not been noticed before that they actually mean the same thing.

The above result of Banach shows that, in the case of normed spaces, B-completeness is a consequence of completeness. It is to be expected that B-completeness will be closely connected with the notion of completeness even in the general case. This suggests the following way of clearing up the connection between these two properties : Let us try to give a similar characterization of complete convex spaces by means of the structure of their duals.

We begin by recapitulating some definitions.

Definition 6. — *Let (E, u) be a convex space. A system $\mathfrak{A}$ of subsets of (E, u) is said to be a Cauchy system in (E, u) if it fulfills the following conditions :*

1° *every $A \in \mathfrak{A}$ is closed in (E, u);*

2° *the system $\mathfrak{A}$ possesses the finite intersection property;*

3° *for every neighbourhood of zero U in (E, u) there exists an $A \in \mathfrak{A}$ such that $A \subset a + U$ for every $a \in A$.*

Definition 7. — *A convex space (E, u) is called complete if every Cauchy system in (E, u) has a nonvoid intersection.*

Definition 8. — *A convex space (E, u) is said to be absolutely closed if it is closed in every convex space in which it is contained.*

(5.1). — *Let (E, u) be a complete convex space. Then (E, u) is absolutely closed.*

Proof : Obvious.

We know from the theory of general uniform structures that even the converse of (5.1) is true. We are not going to use the theory of uniform structures here since we need a much more precise result for which the convex structure of E is essential. The equivalence of completeness and the property of being absolutely closed will then follow as a simple corollary. We may thus expect the following result : a convex space (E, u) will be noncomplete if and only if there exists a bigger convex space in which E is contained as a dense subset. Let us adopt this equivalence as a heuristic principle to guide us in our further work.

Let (E, u) be a convex space, let (R, u_R) be a convex space, such that $(E, u) \subset (R, u_R)$ and that E is dense in R. Let x' be a continuous linear functional on (E, u). Then x' may be extended in a unique way to a continuous linear functional $\varepsilon(x')$ on (R, u_R). On the other hand, let f be a continuous linear functional on (R, u_R). The restriction of f to E is easily seen to be a member of $(E, u)'$. Let us denote it by x'; we have thus $f = \varepsilon(x')$. The mapping ε is thus seen to be a one-to-one linear correspondence between E' and R'. The bilinear functional $\langle x, x' \rangle$ on $E \times E'$ may thus be extended to $R \times E'$ if we write simply $\langle r, x' \rangle$ for $\langle r, \varepsilon(x') \rangle$. In this scalar product, the space E' may serve as dual for both E and R. Especially, we may consider on E' the topology $\sigma(E', R)$.

(5.2). — *Let (E, u) be a convex space, let (R, u_R) be a convex space such that $(E, u) \subset (R, u_R)$ and that E is dense in R. Let U be a neighbourhood of zero in E. Then the topologies $\sigma(E', E)$ and $\sigma(E', R)$ coincide on the set U^0.*

Proof. — Let U be an arbitrary neighbourhood of zero in E. Since E is dense in R, the closure $u_R U$ of U in R will be a neighbourhood of zero in R. It follows that the set $(u_R U)^0$ is $\sigma(E', R)$ compact and the set U^0 compact in $\sigma(E', E)$. Since clearly $U^0 = (u_R U)^0$, it follows that U^0 is compact both in $\sigma(E', R)$ and $\sigma(E', E)$. We have thus a set with two Hausdorff compact topologies on it, one finer than the other. It follows from a general theorem that these two topologies are identical. The proof is concluded.

Suppose now that we have a convex space (E, u). Suppose further that there exists a convex space (R, u_R) such that E is dense in (R, u_R) but different from R. Let us consider on E' the two topologies $\sigma(E', E)$ and $\sigma(E', R)$. Take now a point $r \in R$ which does not belong to E and consider the hyperplane $Z(r)$ of all $x' \in E'$ for which $\langle r, x' \rangle = 0$. Since $r \in R$ but not $r \in E$, the hyperplane $Z(r)$ is closed in $\sigma(E', R)$, but not in $\sigma(E', E)$. If U is an arbitrary neighbourhood of zero in E, it follows from the preceding lemma that $Z(r) \cap U^0$ is $\sigma(E', R)$ compact and, consequently $\sigma(E', E)$ compact. It follows that $Z(r)$ is a hyperplane in E which is not closed in E' but which has a closed intersection with every U^0. Let us examine more closely hyperplanes with this property. We have the following lemma.

(5.3). — *Let (E, u) be a convex space. Let r be a linear form defined on $Y = (E, u)'$. The following properties of r are equivalent :*

1° *for every neighbourhood of zero U, the intersection $Z(r) \cap U^0$ is closed in Y in the topology $\sigma(Y, E)$;*

2° *for every neighbourhood of zero U the function r is continuous on U^0 in the topology $\sigma(Y, E)$.*

Proof. — Suppose that 1° holds. For every real number α, let us denote by $Q(\alpha)$ the set of those $x' \in Y$ for which $\langle r, x' \rangle = \alpha$ so that $Q(0) = Z(r)$. We show first that $Q(\alpha) \cap U^0$ is $\sigma(Y, E)$ closed in Y for every α and every U. This is clear if $Q(\alpha) \cap U^0$ is empty. If there is an $y \in Q(\alpha) \cap U^0$, we find easily that

$$Q(\alpha) \cap U^0 = (y + 2(Q(0) \cap U^0)) \cap U^0$$

which is an intersection of two closed sets.

Now let $y_0 \in U^0$ and let ε be an arbitrary positive number. The set

$$W = (Q(\langle r, y_0 \rangle - \varepsilon) \cap U^0) \cup (Q(\langle r, y_0 \rangle + \varepsilon) \cap U^0)$$

is closed in Y and y_0 does not belong to W. Hence there exists a neighbourhood of zero V in the topology $\sigma(Y, E)$ such that $y_0 + V$ does not meet W. We are going to show that $y \in U^0$ and $y \in y_0 + V$ implies

$$|\langle r, y \rangle - \langle r, y_0 \rangle| \leqq \varepsilon.$$

Indeed, suppose that there exists a

$$y \in U^0 \cap (y_0 + V) \qquad \text{such that} \quad |\langle r, y - y_0 \rangle| > \varepsilon.$$

Let

$$\lambda = \frac{\varepsilon}{|\langle r, y - y_0 \rangle|} \qquad \text{so that} \quad 0 < \lambda < 1.$$

The set $U^0 \cap (y_0 + V)$ being convex, we have

$$y_0 + \lambda(y - y_0) \in U^0 \cap (y_0 + V).$$

The value of r in this point is

$$\langle r, y_0 + \lambda(y - y_0) \rangle = \langle r, y_0 \rangle + \varepsilon \frac{\langle r, y - y_0 \rangle}{|\langle r, y - y_0 \rangle|}.$$

It follows that $y_0 + \lambda(y - y_0) \in W$ which is a contradiction. The continuity of r is thus established.

The other implication being obvious, the proof is concluded.

Now let (E, u) be a given convex space. Let us consider the linear space R consisting of all linear forms r defined on E' such that $Z(r) \cap U^0$ is $\sigma(E', E)$ closed for every neighbourhood of zero U in E. If $x \in E$, the hyperplane $Z(x)$ is $\sigma(E', E)$ closed in E'. It is thus possible to consider E as a subspace of R. The inclusion $E \subset R$ is, however, not only a set-theoretical one, as may be seen from the following theorem.

(5.4). — *Let (E, u) be a convex space. Let $R \supset E$ be the space described above. Then there exists a convex topology u_R defined on R with the following properties :*

1° *the space (R, u_R) is complete;*
2° *the topology u_R induces the original topology u on E;*
3° *E is dense in (R, u_R).*

PROOF. — For every neighbourhood of zero U in E, consider the set $U_R = U^{YR}$. We have

$$U_R \cap E = U^{YR} \cap E = U^{YE} = U$$

for every U. Suppose that $r_0 \in R$ belongs to every U_R and that $r_0 \neq 0$. Then there exists an $x'_0 \in Y$ with

$$\beta = \langle r_0, x'_0 \rangle > 0.$$

Let us denote by U the set of those $x \in E$ for which

$$|\langle x, x'_0 \rangle| \leq \frac{1}{2} \beta.$$

Clearly U is a neighbourhood of zero in E. Now $r_0 \in U_R$ so that

$$\langle r_0, U^Y \rangle \leq 1.$$

Since $\frac{2}{\beta} x'_0 \in U^Y$, we have

$$\left\langle r_0, \frac{2}{\beta} x'_0 \right\rangle \leqq 1,$$

whence

$$\beta = \langle r_0, x'_0 \rangle \leqq \frac{1}{2} \beta$$

which is a contradiction. It follows that $r_0 = 0$ so that u_R is a convex topology on R. Since $U_R \cap E = U$ for every U, the topology u_R induces the topology u on E.

Let us denote now by S the space $(R, u_R)'$. Let $x' \in (E, u)'$ and let U be a neighbourhood of zero such that $x' \in U^Y$. It follows that $|\langle U_R, x' \rangle| \leqq 1$ so that x' may be considered as an element of S. We have thus $(E, u)' \subset (R, u_R)'$. We intend to show now that $(E, u)' = (R, u_R)'$. To see that, take an arbitrary $z' \in S$. There exists a U_R such that $\langle U_R, z' \rangle \leqq 1$. Let us consider the set U^Y. This set is compact in the weak topology corresponding to E. It follows easily from lemma (5.3) that it is also compact in the weak topology corresponding to R. Suppose now that $z' \text{ non} \in U^Y$. Since U^Y is absolutely convex and $\sigma(S, R)$ closed in S, there exists an $r_0 \in R$ such that

$$\langle r_0, U^Y \rangle \leqq 1 \quad \text{and} \quad \langle r_0, z' \rangle > 1.$$

According to the first inequality we have $r_0 \in U_R$. We obtain thus a contradiction with $\langle U_R, z' \rangle \leqq 1$. Hence $z' \in U^Y \subset (E, u)'$, so that $(E, u)' = (R, u_R)'$. It follows that E is dense in (R, u_R). To show that (R, u_R) is complete, let us take an arbitrary Cauchy system $\mathfrak{A}$ in (R, u_R). For every $x' \in Y$, consider the system $\mathfrak{A}(x')$ of subsets of the real line consisting of the closures of the sets $\langle A, x' \rangle$, where A runs over $\mathfrak{A}$. Clearly $\mathfrak{A}(x')$ is a Cauchy system on the real line and has, consequently, a one-point intersection which will be denoted by $\langle r_0, x' \rangle$. Clearly r_0 is a linear form defined on Y.

Let us prove now the following proposition : Let U be an arbitrary neighbourhood of zero aud let ε be an arbitrary positive number. Then there exists an $A \in \mathfrak{A}$ such that

$$|\langle r - r_0, U^0 \rangle| \leqq \varepsilon \quad \text{for every } r \in A.$$

To see that, it is sufficient to take an $A \in \mathfrak{A}$ such that for every $r \in A$ we have $A \subset r + \varepsilon U_R$. It follows that, for every $x' \in U^0$, the diameter of the set $\langle A, x' \rangle$ is at most ε. Take an arbitrary $r \in A$. The number $\langle r, x' \rangle$ belongs to the set $\langle A, x' \rangle$, the value $\langle r_0, x' \rangle$ to its closure. It follows that

$$|\langle r - r_0, x' \rangle| \leqq \varepsilon.$$

Since x' was arbitrary, we have

$$|\langle r - r_0, U^0 \rangle| \leqq \varepsilon \quad \text{for every } r \in A.$$

First of all, it follows from this proposition that $r_0 \in R$. Indeed, if we consider a fixed U^0, the function r_0 may be approximated arbitrarily well on U^0 by the continuous functions r. The function r_0 is, consequently, continuous on U^0 so that $r_0 \in R$.

We are going to show now that r_0 belongs to every $A \in \mathfrak{A}$. Take an arbitrary $A \in \mathfrak{A}$ and a arbitrary U. According to the preceding proposition, there exists an $A_U \in \mathfrak{A}$ such that

$$|\langle r - r_0, U^0 \rangle| \leqq 1 \quad \text{for every } r \in A_U.$$

Now there exists an $r \in A \cap A_U$. It follows that $r \in r_0 + U_R$. Since U was arbitrary, it follows that r_0 belongs to the closure of A, so that $r_0 \in A$. The proof is complete.

(5.5). — *Let (E, u) be a convex space. Then (E, u) is complete if and only if it is absolutely closed.*

Proof. — The "only if" part is obvious. To prove the "if" part, take an absolutely closed convex space (E, u) and construct the corresponding complete space R. Since E is both dense and closed in R, we have $E = R$ so that E is complete.

(5.6). — *Let (E, u) be a convex space. Then (E, u) is complete if and only if the following condition is fulfilled :*

Let Q be a hyperplane in E' such that $Q \cap U^0$ is $\sigma(E', E)$ closed for every neighbourhood of zero U in E; then Q is closed in E'.

Proof. — First of all, let (E, u) be complete. Then (E, u) is closed in every convex space in which it is contained. Since E is dense in R, we have $E = R$ so that our condition is fulfilled. If (E, u) is not complete, then there exists an $r \in R$ which does not belong to E. If $Q = Z(r)$, we have $Q \cap U^0$ closed in E' for every U but Q is not closed in E' since r non $\in E$.

(5.7). — *Let (E, u) be a B_r-complete convex space. Then (E, u) is complete.*

Proof. — Let Q be a hyperplane in E' such that $Q \cap U^0$ is $\sigma(E', E)$ closed for every U. Suppose that Q is dense in E'. It would follow then from (3.3) that $Q = E'$ which is a contradiction. Hence Q is closed in E' and the proposition is proved.

We shall see later that the converse is not true.

Let (E, u) be a convex space and let (R, u_R) be the convex space described in (5.3). Let us consider the space E' in the topology $\sigma(E', E)$. We

have seen already that there is, on E', a finer topology $\sigma(E', R)$ which coïncides with $\sigma(E', E)$ on all sets U^0. It thus natural to try to describe the finest convex topology w on E' which coincides with $\sigma(E', E)$ on all sets U^0. It is not difficult to give a complete description of the topology w. We begin with a simple lemma.

(5.8). — *Let (E, u) be a convex space and U a neighbourhood of zero in E. Let F be a finite subset of E and let M be the union of the sets $x + U$, where $x \in F$. Then there exists a finite set $A \subset E$ such that the absolutely convex envelope of M is contained in the union of the sets $a + 2U$, where $a \in A$.*

PROOF. — Let $x_1, x_2, \ldots, x_n$ be the points of F. There exist positive numbers α_i such that $\alpha_i x_i \in \frac{1}{n} U$. For every i, choose a natural number n_i such that $n_i \alpha_i > 1$. Let us denote by A the set consisting of all sums of the form $\sum_{i=1}^{n} k_i \alpha_i x_i$ where k_i are integers fulfilling

$$|k_i| \leq n_i + 1.$$

Suppose now that x belongs to the absolutely convex envelope of M. It follows that

$$x = \sum^{n} \lambda_i x_i + u$$

where

$$\sum_{i=1}^{n} |\lambda_i| \leq 1 \qquad \text{and} \qquad u \in U.$$

For every i, let

$$k_i = \left[\frac{\lambda_i}{\alpha_i}\right] \qquad \text{so that} \quad |k_i| \leq n_i + 1.$$

Let

$$a = \sum_{i=1}^{n} k_i \alpha_i x_i \qquad \text{so that} \quad a \in A.$$

We have then

$$x - a = u + \sum_{i=1}^{n} (\lambda_i - k_i \alpha_i) x_i.$$

Clearly

$$|\lambda_i - k_i \alpha_i| \leq \alpha_i, \qquad \text{whence} \qquad \sum_{i=1}^{n} (\lambda_i - k_i \alpha_i) x_i \in U.$$

Hence $x - a \in 2U$ and the proof is complete.

We may return now to the topology $\mathfrak{w}$. A complete description of it is contained in the following theorem.

(5.9). — *Let* (E, u) *be a convex space. Let us denote by* $\mathfrak{W}$ *the set of all convex topologies on* E' *which coincide with* $\sigma(E', E)$ *on every* U^0. *Then there exists a convex topology* $\mathfrak{w} \in \mathfrak{W}$ *which is finer than any* $\mathfrak{w}' \in \mathfrak{W}$. *The topology* $\mathfrak{w}$ *may be described as follows : Let* W *be an absolutely convex subset of* E' *such that* W *generates the whole of* E'. *Then* W *is a neighbourhood of zero in the topology* $\mathfrak{w}$ *if and only if the following conditions are fulfilled* :

1° *the intersection* $W \cap U^0$ *is* $\sigma(E', E)$ *closed for every* U;
2° *the polar* W^E *is a praecompact subset of* (E, u).

Proof. — 1° Let $\mathfrak{w}' \in \mathfrak{W}$ and let W be a $\mathfrak{w}'$-neighbourhood of zero in E'. The set $W \cap U^0$ is $\mathfrak{w}'$ closed in U^0. It follows that it is $\sigma(E', E)$ compact and therefore $\sigma(E' E)$ closed in E' so that condition 1° of our theorem is fulfilled.

2° Let $\mathfrak{w}' \in \mathfrak{W}$ and let W be a neighbourhood of zero in the topology $\mathfrak{w}'$. Let us show that W^0 is a praecompact subset of (E, u). Let U be a given neighbourhood of zero in E. Since W is a neighbourhood of zero in the topology $\mathfrak{w}'$, the set $W \cap U^0$ is $\sigma(E', E)$ compact and there exists a finite set $F \subset E$ consisting of the points $x_1, \ldots, x_n$ such that $F^0 \cap U^0 \subset W$. It follows that W^0 is contained in the closed absolutely convex envelope of $F \cup U$. Let us denote by M the set $U \cup (x_1 + U) \cup \ldots \cup (x_n + U)$. According to (5.8), there exists a finite set A such that the absolutely convex envelope of M is a subset of the union $S = \bigcup_{a \in A} (a + 2U)$. It follows that the absolutely convex envelope of $F \cup U$ is contained in S. Since S is a closed set, we have $W^0 \subset S$. Since U was arbitrary, it follows that W^0 is praecompact in (E, u).

3° Now let W be a set fulfilling the conditions of our theorem. We intend to show that there exists, for every U, a $\sigma(E', E)$ neighbourhood of zero V such that $V \cap U^0 \subset W$. To see that, take a fixed neighbourhood of zero U in E. The set W^0 being praecompact in (E, u), there exist n points $x_1, \ldots, x_n \in E$ such that

$$2W^0 \subset (x_1 + U) \cup \ldots \cup (x_n + U).$$

Let us denote by F the set consisting of the points x_i. The set F^{00} consists of all linear combinations

$$\sum_{i=1}^{n} \lambda_i x_i, \qquad \text{where} \quad \sum_{i=1}^{n} |\lambda_i| \leq 1.$$

Let us denote by P the convex envelope of the union $F^{00} \cup U$. If $x \in W^0$, we have

$$x = \frac{1}{2}(x_j + u)$$

for some j and some $u \in U$. It follows that $x \in P$ so that $W^0 \subset P$. Since $U \subset P$, we have $W^0 \cup U \subset P$.

Now we have

$$(W \cap U^0)^0 = \text{closed convex envelope of } W^0 \cup U,$$

whence

$$(W \cap U^0)^0 \subset P^{00}.$$

It follows that

$$P^0 \subset (W \cap U^0)^{00} = W \cap U^0.$$

But

$$P^0 = (F^{00} \cup U)^0 = F^0 \cap U^0$$

It follows that $F^0 \cap U^0 \subset W$ and the proof of our assertion is complete.

4° To complete the proof it is sufficient to show that a set W fulfilling the conditions of our theorem is w-closed. To see that, take a point $y_0 \in E'$ which does not belong to W. Let us denote by U the set of all $x \in E$ for which $|\langle x, y_0 \rangle| \leqq 1$ so that U is a neighbourhood of zero in E. The set U^0 clearly consists of all points of the form λy_0, where $-1 \leqq \lambda \leqq 1$. The intersection $W \cap U^0$ being $\sigma(E', E)$ closed it follows that there exists a number $0 < \omega < 1$ such that $\lambda y_0 \in W$ implies $|\lambda| \leqq \omega$. Suppose that $y_0 \in \frac{1+\omega}{2\omega} W$. It follows that $\frac{2\omega}{1+\omega} y_0 \in W$ which is a contradiction since $\frac{2\omega}{1+\omega} > \omega$. Hence y_0 does not belong to $\tilde{W} = \frac{1+\omega}{2\omega} W$. According to the Hahn-Banach theorem there exists a nonzero linear form f defined on E' such that $f(\tilde{W}) \leqq f(y_0)$. Since $\tilde{W}$ is absolutely convex and generates E', the value $\langle f, y_0 \rangle$ must be positive. We have now

$$\sup\langle f, W \rangle = \frac{2\omega}{1+\omega} \sup\langle f, \tilde{W} \rangle \leqq \frac{2\omega}{1+\omega} \langle f, y_0 \rangle < \langle f, y_0 \rangle.$$

If we show that f belongs to R, the set W will be $\sigma(E', R)$ closed and, consequently, w-closed. We may clearly assume that $\sup\langle f, W \rangle = 1$. Let U be a given neighbourhood of zero in E and ε an arbitrary positive number. According to the preceding section of the proof, there exists a $\sigma(E', E)$ neighbourhood of zero V such that $V \cap \frac{2}{\varepsilon} U^0 \subset W$. Suppose now that

$$y_1, y_2 \in U^0 \qquad \text{and} \qquad y_1 - y_2 \in \varepsilon V.$$

We have then

$$\frac{1}{\varepsilon}(y_1 - y_2) \in V \cap \frac{2}{\varepsilon} U^0 \subset W, \qquad \text{whence} \qquad \left| \left\langle f, \frac{1}{\varepsilon}(y_1 - y_2) \right\rangle \right| \leq 1$$

so that

$$|\langle f, y_1 \rangle - \langle f, y_2 \rangle| \leq \varepsilon.$$

It follows that f is continuous on U^0 in the topology $\sigma(E' E)$ so that $f \in R$. The proof is complete.

(5.10). — *Let (E, u) be a convex space. Let us consider on E' the following three convex topologies :*

(1) *the finest convex topology t_R coinciding with $\sigma(E', E)$ on every set U^0;*

(2) *the convex topology t_C defined by the sets polar to absolutely convex compact subsets of (E, u);*

(3) *the convex topology t_P defined by the sets polar to praecompact subsets of (E, u).*

We have

$$t_R \supset t_P \supset t_C \qquad \textit{and} \qquad t_C(t_R) = t_P.$$

Proof. — Let V be a t_P neighbourhood of zero. Then $V = P^0$, where P is a praecompact subset of (E, u). Now $t_C \sim \sigma(E', E)$ so that $t_C V = V$. Since V is a t_R neighbourhood of zero and $V = t_C V$, we have $t_P \subset t_C(t_R)$. Now let W be a t_R neighbourhood of zero. Since $t_C \sim \sigma(E', E)$, we have $t_C W = W^{00}$ so that $t_C W$ is a t_P neighbourhood of zero. Hence $t_C(t_R) \subset t_P$ and the proof is complete.

According to the above inclusion, we have

$$E = (E', t_C)' \subset (E', t_P)' \subset (E', t_R)' = R.$$

Clearly $(E', t_P)'$ coincides with the union of all sets $\overline{P}$ where P is a praecompact subset of (E, u), the closure being taken in R. It may be shown on examples that $(E', t_P)'$ may be different both from E and R.

(5.11). — *Let (E, u) be a convex space and P a praecompact subset of (E, u). Then P^{00} is praecompact as well.*

Proof. — Let U be a given neighbourhood of zero in (E, u). There exists a finite set F such that P is contained in the union M of the sets $x + \frac{1}{2} U$, where $x \in F$. It follows from (5.8) that the absolutely convex envelope of M is contained in a set $S = \bigcup_{a \in A} (a + U)$, where A is a finite subset of E. The absolutely convex envelope of P is thus contained in S. Since S is closed, we have $P^{00} \subset S$ and the proof is complete.

(5.12). — *Let (E, u) be a complete convex space. Then the topologies t_P and t_C of the space E' are identical.*

PROOF. — If P is a praecompact subset of (E, u), the set P^{00} is both praecompact and closed in the complete space (E, u). Hence P^{00} is compact. It follows that $t_P \subset t_C$, whence $t_P = t_C$ the inclusion $t_C \subset t_P$ being obvious.

Let us conclude this section with another property of complete spaces which, at the same time, illustrates the usefulness of the topology $v(u)$.

(5.13). — *Let (E, v) be a complete convex space. Let u be another convex topology on E, finer then v. Then E is complete in the topology $v(u)$.*

PROOF. — According to lemma (3.1), we have the following inclusions $u \supset v(u) \supset v$. Now let $\mathfrak{A}$ be a Cauchy system in the topology $v(u)$. Since $v(u) \supset v$, the system $\mathfrak{B}$ consisting of the sets vA where A runs over $\mathfrak{A}$, is a Cauchy system in the topology v. Hence there exists a point x_0 which belongs to every vA. Choose an arbitrary $A \in \mathfrak{A}$ and a neighbourhood of zero U in E. There exists a set $A_0 \in \mathfrak{A}$ such that $A_0 \subset a_0 + vU$ for every $a_0 \in A_0$. Choose $a_0 \in A \cap A_0$. It follows that

$$x_0 \in vA_0 \subset a_0 + vU, \qquad \text{whence} \qquad a_0 \in A \cap (x_0 + vU).$$

It follows that x_0 belongs to the $v(u)$-closure of every $A \in \mathfrak{A}$. The sets A being closed in $v(u)$, we have $x_0 \in A$ for every A and the proof is complete.

6. Completeness and B-completeness. — In this section we shall endeavour to describe more closely the class of B-complete convex spaces. We know already from (5.7) that it is contained in the class of complete convex spaces. We shall give an example which shows that the class of B_r-complete convex spaces is a proper subclass of that of complete convex spaces. We shall also describe two important classes of convex spaces which enjoy the property of being B-complete.

We shall need first the following simple result :

(6.1). — *Let E be a linear space, let E^* be the linear space of all linear forms defined on E. Then E is complete in the topology $\tau(E, E^*)$.*

PROOF. — Let $b_t (t \in T)$ be an algebraic basis of E. Let $y_v (v \in T)$ be the system of $y_v \in E^*$ defined by the postulate $\langle b_t, y_v \rangle = \delta_{tv}$. Suppose now that r_0 belongs to the completion of $(E, \tau(E, E^*))$. Let us denote by K the subset of T consisting of those $t \in T$ for which $\langle r_0, y_t \rangle \neq 0$. If $K \neq 0$, let us put, for every $x \in E$

$$p(x) = \sup_{t \in K} \frac{|\langle x, y_t \rangle|}{|\langle r_0, y_t \rangle|}.$$

Clearly p is a pseudonorm on E. Since r_0 belongs to the completion of E,

there exists an $x \in E$ such that

$$\sup_{t \in K} \frac{|\langle x, y_t \rangle - \langle r_0, y_t \rangle|}{|\langle r_0, y_t \rangle|} \leq \frac{1}{2}.$$

It follows that

$$|\langle x, y_t \rangle - \langle r_0, y_t \rangle| \leq \frac{1}{2} |\langle r_0, y_t \rangle|$$

for every $t \in K$. Hence $\langle x, y_t \rangle \neq 0$ for every $t \in K$. It follows that K is finite.

We have thus shown the existence of a point $x_0 \in E$ such that

$$\langle r_0, y_\nu \rangle = \langle x_0, y_\nu \rangle \qquad \text{for every} \quad \nu \in T.$$

Let us denote by Q the subspace of E^* spanned by the y_t.

We have thus $\langle r_0, y \rangle = \langle x_0, y \rangle$ for every $y \in Q$. Now let $y_0 \in E^*$ be given. The set consisting of those $y \in E^*$ which fulfill the inequality

$$|\langle b_t, y \rangle| \leq |\langle b_t, y_0 \rangle|$$

for every $t \in T$ is clearly a U^0. Clearly Q is dense in U^0. Since r is continuous on U^0 and equals x_0 on a dense subset of U^0, we have $r = x_0$ on U^0 so that, in particular, $\langle r, y_0 \rangle = \langle x_0, y_0 \rangle$. Since y_0 was arbitrary, we have $r = x_0$ and the proof is complete. The preceding result has been proved first by S. Kaplan [5].

Definition 9. — *We say that a convex space E is an F_0-space if its topology may be defined by a countable system of pseudonorms.*

(6.2). — *Let E be an F_0-space and f a linear mapping of E onto some convex space F. Suppose that f is both open and continuous. If E is complete then F is complete.*

Proof. — Let U_n be a countable complete system of neighbourhoods of zero in E. We may clearly suppose that $U_n \supset U_{n+1}$ for every natural n. Let z_n be a Cauchy sequence of points $z_n \in F$. There exists a subsequence z'_n such that

$$z'_{n+1} - z'_n \in \frac{1}{2^n} f(U_n)$$

for every natural n. There exist $r_n \in \frac{1}{2^n} U_n$ such that

$$f(r_n) = z'_{n+1} - z'_n.$$

Take a point $x_1 \in E$ such that $f(x_1) = z'_1$. We define now, for $n > 1$, the points

$$x_n = x_1 + r_1 + \dots + r_{n-1} \qquad \text{so that} \quad f(x_n) = z'_n$$

for every natural n. If $m > n$, we have

$$x_m - x_n = r_n + r_{n+1} + \ldots + r_{m-1} \in \frac{1}{2^n} U_n + \frac{1}{2^{n+1}} U_{n+1} + \ldots \subset \frac{1}{2^{n-1}} U_n.$$

It follows that x_n is a Cauchy sequence in E. Hence there exists an $x_0 \in E$ such that

$$\lim x_n = x_0, \qquad \text{whence} \qquad \lim z'_n = \lim f(x_n) = f(x_0).$$

It follows that the sequence z_n is convergent and the proof is complete.

(6.3). — *Let (E, u) be an F_0-space. If (E, u) is complete, then (E, u) is B_r-complete.*

PROOF. — Let v be another convex topology on E coarser than u and such that $v(u) = v$. Let U be an arbitrary neighbourhood of zero in E. Let $x \in vU$. Since E is an F_0-space, there exists a countable complete system U_n of neighbourhoods of zero in E. We may clearly suppose that $U \supset U_n$ for every n and that $U_n \supset U_{n+1}$ for every n. We shall put $U_0 = U$. Since $\frac{1}{2} v U_1$ is a v-neighbourhood of zero, there exists an

$$x_1 \in U_0 \cap \left(x + \frac{1}{2} v U_1 \right) \qquad \text{so that} \quad x - x_1 \in \frac{1}{2} v U_1.$$

There exists an

$$x_2 \in \frac{1}{2} U_1 \cap \left(x - x_1 + \frac{1}{2^2} v U_2 \right) \qquad \text{so that} \quad x - x_1 - x_2 \in \frac{1}{2^2} v U_2.$$

Suppose now that we have already defined the points $x_1, x_2, \ldots, x_n$ so that

$$\text{(i)} \qquad x_i \in \frac{1}{2^{i-1}} U_{i-1}$$

and

$$\text{(ii)} \qquad x - \sum_{i=1}^{n} x_i \in \frac{1}{2^n} v U_n;$$

then there exists an

$$x_{n+1} \in \frac{1}{2^n} U_n \cap \left(x - \sum_{i=1}^{n} x_i + \frac{1}{2^{n+1}} v U_{n+1} \right).$$

It follows that

$$x - \sum_{i=1}^{n+1} x_i \in \frac{1}{2^{n+1}} v U_{n+1}.$$

Let now

$$s_n = \sum_{i=1}^{n} x_i \qquad \text{so that} \quad x - s_n \in \frac{1}{2^n} v\, U_n.$$

It follows that $x = v \lim s_n$. If $p < q$, we have

$$s_q - s_p = x_{p+1} + \ldots + x_q \in \frac{1}{2^p} U_p + \frac{1}{2^{p+1}} U_{p+1} + \ldots \subset \frac{1}{2^{p-1}} U_p.$$

It follows that s_n is a Cauchy sequence in the topology u. Hence

$$x = u \lim s_n.$$

We have, for every natural n,

$$s_n = x_1 + \ldots + x_n \in U + \frac{1}{2} U_1 + \frac{1}{2^2} U_2 + \ldots \subset 2U.$$

It follows that

$$x = u \lim s_n \in 2U.$$

Since x was an arbitrary point of vU, we have $U \subset vU \subset 2U$ so that $v = u$. The proof is complete.

(6.4). — *Let E be an F_0-space. Then E is B-complete if and only if it is complete.*

PROOF. — If E is complete, it follows from (6.2) and (6.3) that every quotient of E is B_r-complete, so that E is B-complete. If E is B-complete, it is complete according to (5.7) and (4.2).

(6.5). — *Let E be a Fréchet space. Let us denote by t_C the convex topology on E' defined by the polars to absolutely convex and compact subsets of E. Then E' is B-complete under any convex topology finer then t_C and coarser than $\tau(E', E)$.*

PROOF. — With wiew to (4.3) it is sufficient to show that (E', t_C) is B-complete. Let Q be a subspace of E such that, for every t_C-neighbourhood of zero V in E', the intersection $Q \cap V^0$ is closed in E. Let $x_0 \in E$ belong to the closure of Q. Then there exists a sequence $q_n \in Q$ such that $q_n \to x_0$. The sequence q_n being convergent, the set P consisting of the points q_n is praecompact. It follows that P^{00} is compact and, consequently, a V^0. Since $q_n \in Q \cap P^{00}$ and $q_n \to x_0$, we have $x_0 \in Q \cap P^{00} \subset Q$. Hence Q is closed and the proof is complete.

We conclude this section with the discussion of an example. Let us denote by E the linear space of all sequences $x = \{x_k\}$ of real numbers such that $\sum_{k=1}^{\infty} |x_k| < \infty$. Let us denote by A the set of all sequences $\alpha = \{\alpha_k\}$

of real numbers with the following properties :

1° $1 \geq \alpha_k > 0$ for every k;
2° $\alpha_k \geq \alpha_{k+1}$ for every k;
3° $\lim \alpha_k = 0$.

For every $\alpha \in A$, let us define a pseudonorm p_α on E in the following manner :

$$p_\alpha(x) = \sum_{k=1}^{\infty} \alpha_k |x_k|.$$

Let us denote by w the convex topology on E defined by the family of pseudonorms p_α, where α runs over A. Let v be the topology on E defined by the norm $|x| = \sum_{k=1}^{\infty} |x_k|$. Let us denote by u the topology $\tau(E, E^\star)$. Clearly v is coarser than u. We have, further, $p_\alpha(x) \leq |x|$ for every α. It follows that $v \supset w$. We have thus $u \supset v \supset w$. Now let us denote by V the subspace of $E^\star$ consisting of those $f \in E^\star$ for which there exist a bounded sequence y_k of real numbers such that

$$\langle x, f \rangle = \sum_{k=1}^{\infty} x_k y_k$$

for every $x \in E$. Let W be the subspace of $E^\star$ consisting of those $f \in E^\star$ for which exists a sequence y_k of real numbers converging to zero such that

$$\langle x, f \rangle = \sum_{k=1}^{\infty} x_k y_k$$

for every $x \in E$. Clearly $(E, v)' = V$ and $(E, w)' = W$. The inclusions $E^\star \supset V \supset W$ being proper ones, it follows that w is properly coarser than v and that v is properly coarser than u. If W is equipped with the norm $|y| = \max |y_k|$, it becomes a complete normed space. It is easy to see that (E, w) may be identified with (W', t_C). It follows from (6.5) that (E, w) is B-complete. Since (E, v) is a complete normed space, it follows from (6.4) that (E, v) is B-complete and an espace tonnelé. According to (6.1) the space (E, u) is complete. The identical mapping of (E, u) onto (E, v) is a continuous mapping onto an espace tonnelé. If (E, u) were B_r-complete, this mapping would have to be open, or the topologies u and v identical. Since v is properly coarser than u, the space (E, u) cannot be B_r-complete. Hence (E, u) is an example of a complete convex space which is not B_r-complete. The identical mapping of (E, v) onto (E, w) is an example of a continuous on-to-one linear mapping of a

B-complete convex space onto another B-complete convex space the inverse of which is not continuous.

We intend to return later to further interesting questions connected with the notion of B-completeness.

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Vlastimil PTÁK,
Československá Akademie věd,
Matematicky ustav,
Praha, Tchécoslovaquie.
(Prague)