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TWO CARTESIAN PRODUCTS WHICH ARE EUCLIDEAN SPACES

BY

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WHITEHEAD has given an example of a three-dimensional manifold W which is not (homeomorphic to) E^3 , Euclidean 3-space [3]. We prove the following theorem about W , the first statement of which is due to A. SHAPIRO.

THEOREM. — *If W is the manifold described below then $W \times E^1$ is homeomorphic to E^3 . Also $W \times W$ is homeomorphic to $E^3 \times W$ (which is homeomorphic to E^6).*

That W is not homeomorphic to E^3 was proved in [1], [2]. In [1] it is shown that no cube in W contains W_0 (defined below), which implies W is not E^3 . The homeomorphism $W \times E^1 \approx E^3$ can be used to show the existence of a two element (and so compact) group of homeomorphisms of E^3 onto itself whose fixed point set is W . The problem of showing that $W \times W$ is homeomorphic to E^6 was suggested to the author by L. ZIPPIN.

Let W_0, W_1, R_0, R_1 be solid tori with W_0 simply self-linked in the interior of W_1 (see fig. 1) and R_0 trivially imbedded in the interior of R_1 . Let I_0 and I_1 be closed bounded intervals of E^1 with I_0 contained in the interior of I_1 . Let w (resp. r) be a 3-cell in the interior of W_0 (resp. R_0), let e (resp. f, g) be a homeomorphism of E^3 (resp. E^3, E^1) onto itself with $e(W_0) = W_1$ [resp. $f(R_0) = R_1, g(I_0) = I_1$] and $e|w$ (resp. $f|r$) the identity. Let

$$W_n = e^n(W_0), \quad R_n = f^n(R_0), \quad I_n = g^n(I_0).$$

Let $W = \bigcup_{n=1}^{\infty} W_n$, we suppose that

$$E^3 = \bigcup_{n=1}^{\infty} R_n, \quad E^1 = \bigcup_{n=1}^{\infty} I_n.$$

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Let $S = \{ h | A : A \subset E^3, h \text{ is a homeomorphism of } E^3 \text{ onto itself which is the identity outside a compact set} \}$; we further suppose $e \in S, f | R_i \in S$ and $\lambda'(R_0) = W_0$ for some $\lambda' \in S$.

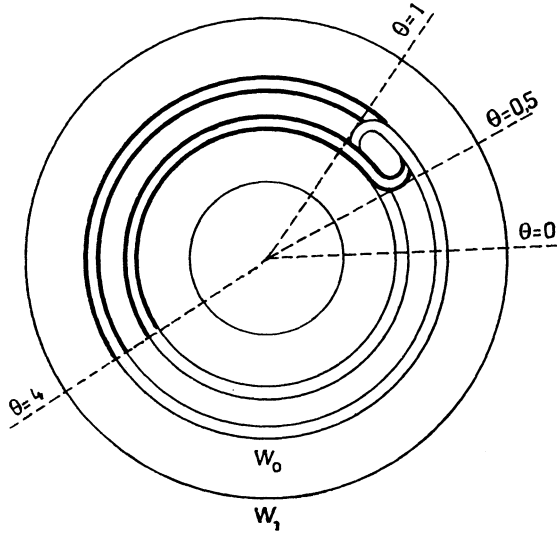


Fig. 1.

PROOF. — We prove both statements simultaneously. Let V_n denote I_n (resp. W_n), V denote E^1 (resp. W). For each positive integer n , we construct a homeomorphism $h_n : W_n \times V_n \rightarrow R_n \times V_n$ with the properties

- (1) $h_n(W_{n-1} \times V_{n-1}) = R_{n-1} \times V_{n-1}$;
- (2) $h_n | W_{n-2} \times V_{n-2} = h_{n-1} | W_{n-2} \times V_{n-2}$ ($n \geq 2$).

Suppose we have constructed all the h' s. Then we define

$$\Phi : W \times V \rightarrow E^3 \times V$$

as follows. If $(x, y) \in W \times V$, then for some $n, (x, y) \in W_n \times V_n$. Let $\Phi(x, y) = h_{n+1}(x, y)$. By (2) we see that Φ is well-defined, by (1) we see that Φ is onto. Since h_n is a homeomorphism, Φ is also.

Suppose the following lemma is true. Using the lemma, we will construct the h_n .

LEMMA. — If we are given a homeomorphism $\beta' : w \times V_0 \rightarrow R_0 \times V_0$ (into), and if β' has the form $\lambda' | w \times I$ where λ' is a homeomorphism in S of W_0 onto R_0 , then there is a homeomorphic extension β of β' ,

$$\beta : W_1 \times V_1 \rightarrow R_1 \times V_1, \quad \beta(W_0 \times V_0) = R_0 \times V_0,$$

and $\beta | \text{Bdry}(W_1 \times V_1) = \lambda \times I$ for λ some homeomorphism in S of W_1 onto R_1 .

Let λ' be a homeomorphism in S mapping W_0 onto R_0 . Let $h_1 = \beta$, the extension of $\beta' = (\lambda' | \omega) \times I$ given by the lemma. We suppose inductively that for n a positive integer greater or equal to 2, h_{n-1} has been constructed, and $h_{n-1} | \text{Bdry}(W_{n-1} \times V_{n-1}) = \gamma \times I$, for γ some homeomorphism in S of W_{n-1} onto R_{n-1} . We note that h_1 has this property. Observe that $(\gamma^{-1} \times I) h_{n-1}$ is a homeomorphism of $W_{n-1} \times V_{n-1}$ onto itself leaving the boundary pointwise fixed. Let h be the extension of this map to $W_n \times V_n$ which is the identity on $W_n \times V_n$ -Interior $(W_{n-1} \times V_{n-1})$. Let r' be a 3-cell with Interior $R_{n-1} \supset r' \supset R_{n-2}$. Let $\omega' = \gamma^{-1}(r')$. Let $k : W_n \rightarrow W_n$ be a homeomorphism in S , $k | (W_n$ -Interior $W_{n-1}) = \text{identity}$, $k(\omega') \subset \omega$. Let β be the extension of $\gamma k^{-1} \times I | \omega \times V_{n-1}$ to a homeomorphism of $W_n \times V_n$ onto $R_n \times V_n$ as given by the lemma. Let $h_n = \beta(k \times I) h$. We check that h_n satisfies (1) and (2),

$$h_n(W_{n-1} \times V_{n-1}) = \beta(W_{n-1} \times V_{n-1}) = R_{n-1} \times V_{n-1}.$$

If $z \in W_{n-2} \times V_{n-2}$, then $(k \times I) h(z) \in \omega \times V_{n-1}$ and

$$\begin{aligned} h_n(z) &= \beta(k \times I) h(z) \\ &= (\gamma k^{-1} \times I)(k \times I)(\gamma^{-1} \times I) h_{n-1}(z) = h_{n-1}(z) \end{aligned}$$

as asserted. Also

$$\begin{aligned} h_n | \text{Bdry}(W_n \times V_n) &= \beta(k \times I) h | \text{Bdry}(W_n \times V_n) \\ &= \lambda k \times I | \text{Bdry}(W_n \times V_n), \end{aligned}$$

where the last equality arises from the form of β on $\text{Bdry}(W_n \times V_n)$ and the fact that $(k \times I)(\text{Bdry}(W_n \times V_n)) = \text{Bdry}(W_n \times V_n)$. Thus h_n satisfies the induction hypothesis and all the h_n can be defined, if we prove the lemma.

PROOF OF LEMMA. — Given $\beta' = \lambda' | \omega \times I : \omega \times V_0 \rightarrow R_0 \times V_0$, we can extend $\lambda' | \omega$ to a homeomorphism in S λ of W_1 onto R_1 . In fact let j be a homeomorphism in S of R_1 onto itself which maps R_0 onto R_0 and $\lambda'(\omega)$ into r . Let

$$\lambda = j^{-1} f j \lambda' e^{-1}.$$

Then λ is a homeomorphism in S of W_1 onto R_1 and $\lambda | \omega = j^{-1} f j \lambda' | \omega = \lambda' | \omega$ so λ is the desired extension of $\lambda' | \omega$. It is now sufficient to construct a homeomorphism h of $W_1 \times V_1$ onto itself which leaves $\omega \times V_0$ pointwise fixed with $h | \text{Bdry}(W_1 \times V_1) = \mu \times I$ for some μ in S which maps W_1 onto W_1 , and with $h(W_0 \times V_0) = \lambda^{-1}(R_0) \times V_0$. In fact $(\lambda \times I) h = \beta$ is a homeomorphism of $W_1 \times V_1$ onto $R_1 \times V_1$, β extends β' , and

$$\begin{aligned} \beta(W_0 \times V_0) &= \lambda \lambda^{-1}(R_0) \times V_0 = R_0 \times V_0, \\ \beta | \text{Bdry}(W_1 \times V_1) &= \lambda \mu \times I | \text{Bdry}(W_1 \times V_1). \end{aligned}$$

The homeomorphism h will be given as the product of four homeomorphisms Λ , Σ , Δ and P of $W_1 \times V_1$ onto itself. Λ , Σ and Δ will each leave $\text{Bdry}(W_1 \times V_1) \cup (\omega \times V_0)$ pointwise fixed. Λ will lift the dark portion of W_0 , Σ will slide this lifted part away from the link, and Δ will drop the image under $\Sigma\Lambda$ of the dark part of W_0 back into its original plane. We suppose W_1 is $D \times C$ where D is the square $\{(u, v) : 0 \leq u, v \leq 20\}$ and C is the circle $\{\theta : 0 \leq \theta < 2\pi\}$. We suppose that

$$W_0 \subset \{(u, v) : 9 \leq u, v \leq 10\} \times C, \quad \omega \subset D \times \{\theta : 6 \leq \theta < 2\pi\},$$

the link in $W_0 \subset D \times \{\theta : .5 \leq \theta \leq 1\}$. Let $\alpha, \beta, \gamma, \delta$ be functions on C , let a, b, c be functions on $[0, 20]$, defined as follows. Let

$$\begin{aligned} \alpha([0, 2]) &= 1, & \alpha([4, 2\pi]) &= 0, & \beta(0) &= 0, \\ \beta([.5, 4]) &= 1, & \beta([6, 2\pi]) &= 0, \\ \gamma([0, 1]) &= 0, & \gamma([2, 2\pi]) &= 1, & \delta([0, 1]) &= 0, \\ \delta([1.5, 3]) &= 1, & \delta([5, 2\pi]) &= 0, \end{aligned}$$

and let $\alpha, \beta, \gamma, \delta$ be linear on intervals for which they are not defined above. Let

$$\begin{aligned} \alpha(0) &= 0, & a([9, 10]) &= 1, & a(20) &= 0, \\ b([0, 10]) &= 0, & b([11, 12]) &= 1, & b(20) &= 0, \\ c(0) &= 0, & c([9, 12]) &= 1, & c(20) &= 0, \end{aligned}$$

and let a, b, c be linear on intervals for which they are not defined above. Let ε be a continuous map of W_1 into $[0, 1]$ such that $\varepsilon(u, v, \theta) = \alpha(\theta)$ for (u, v, θ) in the dark part of W_0 , $\varepsilon = 0$ on the rest of W_0 and on $\text{Bdry } W_1$. If $(u, v), (x, y) \in D, \theta, \psi \in C$, let

$$\begin{aligned} \Lambda(u, v, \theta, x, y, \psi) &= (u, v, \theta, x, y + 2\varepsilon(u, v, \theta) a(x) a(y), \psi), \\ \Sigma(u, v, \theta, x, y, \psi) &= (u, v, \theta + \beta(\theta) a(x) \\ &\quad \times [(1 - \gamma(\theta)) b(y) + \gamma(\theta) c(y)] a(u) a(v), x, y), \\ \Delta(u, v, \theta, x, y, \psi) &= (u, v, \theta, x, y - 2\delta(\theta) c(y) a(x) a(u) a(v), \psi). \end{aligned}$$

If $V_i = I_i$, we identify I_0 with $\{10\} \times [9, 10] \times \{0\} \subset W_1$ and I_1 with $\{10\} \times [0, 20] \times \{0\} \subset W_1$. Then Λ, Σ , and Δ map $W_1 \times I_1$ onto itself and $h' = \Delta\Sigma\Lambda|_{W_1 \times I_1}$ (resp. $h' = \Delta\Sigma\Lambda$) is a homeomorphism of $W_1 \times V_1$ onto itself which leaves $(\text{Bdry}(W_1 \times V_1)) \cup (\omega \times V_0)$ pointwise fixed. For $(x, y, \psi) \in V_0$, $\Delta\Sigma\Lambda(W_0 \times (x, y, \psi))$ is trivially imbedded in $W_1 \times (x, y, \psi)$ and the projection W'_0 on W_1 of $\Delta\Sigma\Lambda(W_0 \times (x, y, \psi))$ is independent of x, y, ψ in V_0 . To see this it is sufficient to compute $\Delta\Sigma\Lambda(u, v, \theta, x, y, \psi)$ for (u, v, θ) in W_0 , x, y in $[9, 10]$ and θ a point of non-linearity of α, β, γ or δ . Suppose we have a homeomorphism ρ' of W_1 onto W_1 which leaves $\text{Bdry } W_1 \cup \omega$ pointwise fixed, and with $\rho'(W'_0) = \lambda^{-1}(R_0)$. Define $P = \rho' \times I : W_1 \times V_1 \rightarrow W_1 \times V_1$, define $h = P h'$. Then h has the necessary properties.

Since $\lambda^{-1}(R_0)$ is trivially imbedded in W_1 , it is in a 3-cell in the interior of W_1 . There is a homeomorphism g' of E^3 onto itself leaving $E^3 - W_1$ pointwise fixed and such that $g'(W'_0)$ and $\lambda^{-1}(R_0)$ both lie in a 3-cell u in the interior of W_1 . It is evident that there is a homeomorphism in S mapping W_0 onto W'_0 and so there is a homeomorphism g'' in S of E^3 onto itself mapping $g'(W'_0)$ onto $\lambda^{-1}(R_0)$. We can find a 3-cell U outside of which g'' is the identity and a homeomorphism φ mapping U onto u which is the identity on $\lambda^{-1}(R_0) \cup g'(W'_0)$. Define $g = \text{identity outside } u, g = \varphi g'' \varphi^{-1}$ on u . Then $h = gg'$ is a homeomorphism leaving boundary W_1 fixed and mapping W'_0 onto $\lambda^{-1}(R_0)$. Since $\omega \subset \text{Interior } W'_0, h(\omega) \subset \text{Interior } \lambda^{-1}(R_0)$ and since $\omega \subset \text{Interior } \lambda^{-1}(R_0)$ there is a homeomorphism i of E^3 onto itself leaving $E^3 - \lambda^{-1}(R_0)$ fixed and mapping $h(\omega)$ into ω . Let U_0, u_0 be 3-cells, with $U_0 \supset W_1, \lambda^{-1}(R_0) \supset u_0, \text{Interior } u_0 \supset \omega$ and let φ_0 be a homeomorphism of U_0 onto u_0 leaving ω pointwise fixed. Let $j = \varphi_0(ih)^{-1} \varphi_0^{-1}$ on $u_0, j = \text{identity on } W_1 - u_0$. Then $\rho' = jih$ is a homeomorphism of W_1 onto W_1 ,

$$\rho'(W'_0) = ji\lambda^{-1}(R_0) = \lambda^{-1}(R_0),$$

$\rho' | \text{Bdry } W_1 = \text{identity}$ and $\rho' | \omega = \varphi_0(ih)^{-1} \varphi_0^{-1} ih | \omega = \varphi_0 | \omega = \text{identity}$. This completes the proof.

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