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ON ISOTYPE SUBGROUPS OF ABELIAN GROUPS ;

BY

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In his book *Abelian groups*, L. FUCHS asks the following question. Let G be a p -group and H be a subgroup without elements of infinite height. Under what conditions can H be embedded in a pure subgroup of the same power and again without elements of infinite height? (See [2], p. 96.) This question has been answered by Charles [1] and IRWIN [3]. Irwin's solution was effected by showing that any subgroup maximal with respect to disjointness from the subgroup of elements of infinite height is pure. For p -groups, the subgroup of elements of infinite height is $p^\omega G$. Now for any Abelian group G , any prime p , and any ordinal α , one may define $p^\alpha G$, and this suggests the following problem. Is any subgroup of G maximal with respect to disjointness from $p^\alpha G$ pure in G ? Or, more generally, does any such subgroup H of G have the property that $H \cap p^\beta G = p^\beta H$ for all ordinals β ? That is to say, is H p -isotype in G ? We will show that indeed any such H is p -isotype, and we will give a partial solution to the problem of determining whether any two such H 's are isomorphic. The foregoing considerations will lead to the solution of a more general version of the above mentioned problem of L. FUCHS.

All groups considered in this paper will be Abelian.

DEFINITION 1. — Let G be a group and p be a prime. Define $p^0 G = G$. If $p^\beta G$ is defined for all ordinals $\beta < \alpha$, then define $p^\alpha G = \bigcap_{\beta < \alpha} p^\beta G$ when α is a limit ordinal. If $\alpha = \delta + 1$ for some ordinal δ , let $p^\alpha G = p(p^\delta G)$.

Thus we have defined $p^\alpha G$ for all ordinals α , and clearly the $p^\alpha G$'s form a chain of fully invariant subgroups of G .

DEFINITION 2. — Let p be a prime and $g \in G$. The p -height $H_p(g)$ of g is the ordinal α such that $g \in p^\alpha G$ and $g \notin p^{\alpha+1} G$. If no such ordinal α exists, then $H_p(g) = \infty$, where the symbol ∞ is considered larger than any ordinal. Let α be an ordinal or ∞ . Then a subgroup H of G is p^α -pure in G if and only if $H \cap p^\beta G = p^\beta H$ for all ordinals $\beta \leq \alpha$; H is α -pure in G if and only if H is p^α -pure in G for all primes p . A subgroup H is p -isotype in G if and only if H is p^∞ -pure in G . The subgroup H is isotype in G if and only if H is p -isotype in G for all primes p .

It follows easily from the definitions that the properties of being isotype, α -pure, or p^α -pure are transitive. Moreover, the union of an ascending chain of subgroups with one of these properties is a subgroup with that property.

It is easy to see that there are groups in which not every pure subgroup is isotype. In fact, there exist reduced p -groups G such that $|p^\beta G| = \aleph_0$ and $|\beta| \geq 2^{\aleph_0}$. (See [2], p. 131, Theorem 38.2 for the existence of such a G .) Embed $p^\beta G$ in a pure subgroup K of G with $|K| = \aleph_0$. Clearly K is not isotype since $p^\beta K = 0$ and $K \cap p^\beta G = p^\beta G \neq 0$.

We now state and prove a few facts which will be useful in what follows, and which illustrate the relation between the above definitions and the ordinary notions of purity and height.

LEMMA 1. — For a positive integer n , let $n = \prod_{i=1}^r p_i^{s_i}$ be its prime decomposition. Then for any group G , $nG = \bigcap_{i=1}^r p_i^{s_i} G$.

PROOF. — Let $T = \bigcap_{i=1}^r p_i^{s_i} G$. Clearly $nG \subseteq T$. Now let $g \in T$. For $n_i = n/p_i^{s_i}$, there exist integers a_i with $\sum a_i n_i = 1$. But $g \in T$ yields $g = p_i^{s_i} g_i$, $i = 1, \dots, r$. Hence

$$g = \sum a_i n_i g = \sum a_i n_i p_i^{s_i} g_i = \sum a_i n g_i = n \sum a_i g_i \in nG.$$

Hence $nG = T$, and the proof is complete.

COROLLARY 1. — A subgroup H of a group G is pure in G if and only if H is ω -pure.

PROOF. — Suppose H is pure in G . In particular, $H \cap p^m G = p^m H$ for each prime p and non-negative integer m . Now

$$H \cap p^\omega G = H \cap \left(\bigcap_{k < \omega} p^k G \right) = \bigcap_{k < \omega} (H \cap p^k G) = \bigcap_{k < \omega} p^k H = p^\omega H.$$

Hence H is ω -pure. Next suppose H is ω -pure, and n is a positive integer. Then

$$\begin{aligned} H \cap nG &= H \cap \left(\left(\prod p_i^{s_i} \right) G \right) = H \cap \left(\bigcap p_i^{s_i} G \right) \\ &= \bigcap (H \cap p_i^{s_i} G) = \bigcap p_i^{s_i} H = nH \end{aligned}$$

by Lemma 1.

The following definition is standard.

DEFINITION 3. — The subgroup $G^1 = \bigcap_{n < \omega} nG$ is the *subgroup of elements of infinite height* in G .

We are now in a position to prove the following useful

COROLLARY 2. — Let P be the set of all primes. Then $G^1 = \bigcap_p p^\omega G$.

PROOF. — Set $T = \bigcap_p p^\omega G$. Then from $p^\omega G = \bigcap_n p^n G$ for each $p \in P$, it follows that $p^\omega G \supset \bigcap_n nG$ for each $p \in P$, and hence $T \supset G^1$. Now for each n we have $nG = \bigcap p_i^{s_i} G \supset T$. Hence $G^1 \supset T$, whence $G^1 = T$.

This corollary shows that the subgroup G^1 of elements of infinite height in G is the set of elements of infinite p -height for each prime p . The following theorem and corollary are generalizations of Kaplansky's Lemma 7 ([5], p. 20)

THEOREM 1. — Let H be a subgroup of a p -group G , and let α be a limit ordinal or ∞ . Then H is p^α -pure in G if and only if whenever $\beta < \alpha$, $h \in H[p]$, and the p -height in G of h is $\geq \beta$, then the p -height in H of h is $\geq \beta$.

PROOF. — If H is p^α -pure, then clearly the elements in $H[p]$ have the desired property. To prove the converse, it must be established that $H \cap p^\delta G = p^\delta H$ for all $\delta \leq \alpha$. Obviously $H \cap p^\delta G \supset p^\delta H$. Let $P(n)$ be the statement: For $\beta < \alpha$, the elements in H of exponent $\leq n$ have p -height $\geq \beta$ in H if they have p -height $\geq \beta$ in G . We will prove $P(n)$ is true for all n by induction and consequently have that $H \cap p^\delta G \subseteq p^\delta H$ for all $\delta < \alpha$. Now $P(1)$ is true by hypothesis. Assume $P(n)$ holds, and let $h \in H$ with $o(h) = p^{n+1}$, and suppose the p -height of h is $\geq \beta$ in G . Then ph has exponent n and p -height $\geq \beta + 1$ in G . Since $\beta + 1 < \alpha$, our induction hypothesis yields $ph = ph_\beta$ with $h_\beta \in p^\beta H$. Hence $(h - h_\beta) \in H[p]$, has p -height $\geq \beta$ in G , and so p -height $\geq \beta$ in H . Therefore $H \cap p^\delta G \subseteq p^\delta H$ for all $\delta < \alpha$ and since α is a limit ordinal, this holds for all $\delta \leq \alpha$. Thus H is p^α -pure in G .

COROLLARY 3. — *Let H be a subgroup of a p -group G . Then H is isotype in G if and only if the elements in $H[p]$ have the same p -height in H as in G .*

PROOF. — Since G is a p -group, we have $qH = H$ for all $q \neq p$, and hence H is q -isotype for all $q \neq p$. To get H p -isotype, let α be ∞ in Theorem 1.

We proceed now to our main results and begin with the following definition :

DEFINITION 4. — Let K and L be subgroups of G . Then H is L -high in K if and only if H is a subgroup of K maximal with respect to the property that $H \cap L = 0$. A *high subgroup* H of G is a subgroup maximal with respect to the property $H \cap G^1 = 0$. (See [3].)

The principal result of this paper is the following theorem :

THEOREM 2. — *Let G be a group, let p be a prime, let α be an ordinal, let K be a subgroup of $p^\alpha G$, and let H be K -high in G . Then H is $p^{\alpha+1}$ -pure in G , and $p^\beta H$ is K -high in $p^\beta G$ for all ordinals $\beta \leq \alpha$.*

PROOF. — To show that H is $p^{\alpha+1}$ -pure in G we need to establish that $H \cap p^\beta G = p^\beta H$ for all $\beta \leq \alpha + 1$. We induct on β , and if $\beta = 0$, the equality is trivial. Now suppose $0 < \beta \leq \alpha + 1$, and suppose the equality holds for all ordinals less than β . If β is a limit ordinal, then

$$H \cap p^\beta G = H \cap \left(\bigcap_{\delta < \beta} p^\delta G \right) = \bigcap_{\delta < \beta} (H \cap p^\delta G) = \bigcap_{\delta < \beta} p^\delta H = p^\beta H.$$

Next suppose β is not a limit ordinal. Then there is an ordinal δ such that $\beta = \delta + 1$. Then

$$p^\beta H \subseteq H \cap p^\beta G = H \cap p(p^\delta G).$$

Let $h = pg_\delta$ with $h \in H$ and $g_\delta \in p^\delta G$. If $g_\delta \in H$, then

$$g_\delta \in H \cap p^\delta G = p^\delta H,$$

and

$$h = pg_\delta \in p(p^\delta H) = p^\beta H.$$

So suppose $g_\delta \notin H$. Since H is K -high in G and $K \not\subseteq p^\alpha G$, we have

$$h_1 + ng_\delta = k \neq 0,$$

where $h_1 \in H$, $k \in K$, and n an integer. Clearly $(n, p) = 1$, and $k \in p^\alpha G$. Since $\delta \leq \alpha$, we have $h_1 \in p^\delta G$. The induction hypothesis yields $h_1 \in p^\delta H$. Now

$$ph_1 + npg_\delta = ph_1 + nh = pk = 0.$$

Therefore

$$nh = -ph_1 \in p(p^\delta H) = p^\beta H.$$

Also $ph \in p^\beta H$ since $h \in p^\beta G \subseteq p^\delta G$, consequently $h \in p^\delta H$. There exist integers a and b such that $an + bp = 1$. Thus

$$anh + bph = h \in p^\beta H.$$

Hence $H \cap p^\beta G = p^\beta H$ and H is $p^{\alpha+1}$ -pure in G as stated.

It remains to show that $p^\beta H$ is K -high in $p^\beta G$ for $\beta \leq \alpha$. Suppose this is not the case. Then there exists $g_\beta \in p^\beta G, g_\beta \notin p^\beta H$ such that the subgroup generated by $p^\beta H$ and g_β is disjoint from K . If $g_\beta \in H$, then since H is $p^{\alpha+1}$ -pure in G and $\beta \leq \alpha$, $g_\beta \in p^\beta H$ contrary to the choice of g_β . Hence $g_\beta \notin H$. Since H is K -high in G , we have $h + ng_\beta = k \neq 0$, where $h \in H$ and $k \in K \subseteq p^\alpha G$. From $\beta \leq \alpha$ we have that $h \in p^\beta G$, and hence $h \in p^\beta H$ by $p^{\alpha+1}$ -purity of H in G . But this together with the equation $h + ng_\beta = k \neq 0$ contradicts the fact that the subgroup generated by $p^\beta H$ and g_β is disjoint from K . This concludes the proof.

As an easy consequence of Theorem 2 we obtain a generalization of Irwin's result mentioned above.

COROLLARY 4. — *Let K be any subgroup of G^1 and H be K -high in G . Then H is $(\omega + 1)$ -pure (and hence pure) in G . In particular, if H is high in G , then H is pure in G .*

PROOF.— Since $K \subseteq p^\omega G$ for each prime p , H is $p^{\omega+1}$ -pure for each p . Hence H is $(\omega + 1)$ -pure.

Another result along these lines is

COROLLARY 5. — *Let H be $p^\alpha G$ -high in G . Then H is p -isotype in G , and $p^\beta H$ is $p^\alpha G$ -high in $p^\beta G$ for all β .*

PROOF. — Since H is $p^\alpha G$ -high in G , then $H \cap p^\beta G = p^\beta H = 0$ for all $\beta \geq \alpha$, and Theorem 2 yields H is p -isotype. For ordinals $\beta \geq \alpha$, the only $p^\alpha G$ -high subgroup in $p^\beta G$ is 0 and $p^\beta H = 0$ for such β . By Theorem 2, $p^\beta H$ is $p^\alpha G$ -high in $p^\beta G$ for all β .

LEMMA 3. — For any group G and any ordinals α and $\beta, p^\alpha(p^\beta G) = p^{\beta+\alpha} G$.

PROOF. — Induct on α . The assertion is true for $\alpha = 0$. Now assume $\alpha > 0$ and that the assertion is true for all ordinals $\delta < \alpha$. Suppose α is a limit ordinal. Then

$$\begin{aligned} p^\alpha(p^\beta G) &= \bigcap_{\delta < \alpha} p^\delta(p^\beta G) \\ &= \bigcap_{\delta < \alpha} (p^{\beta+\delta} G) = \bigcap_{\beta \leq \lambda < \beta+\alpha} (p^\lambda G) = \bigcap_{\lambda < \beta+\alpha} (p^\lambda G) = p^{\beta+\alpha} G \end{aligned}$$

since $\beta + \alpha$ is a limit ordinal. Suppose $\alpha = \delta + 1$. Then

$$p^\alpha(p^\beta G) = p(p^{\delta}(p^\beta G)) = p(p^{\beta+\delta} G) = p^{(\beta+\delta)+1} G = p^{\beta+(\delta+1)} G = p^{\beta+\alpha} G.$$

As a simple application of Lemma 3 we have

COROLLARY 6. — *Let H be $p^\alpha G$ -high in G . Then $p^\beta H$ is p -isotype in $p^\beta G$ for all β .*

PROOF. — By Corollary 5, $p^\beta H$ is $p^\alpha G$ -high in $p^\beta G$ for all β . If $\alpha \leq \beta$, then $p^\beta H = 0$ and hence is isotype. If $\beta < \alpha$, then $\alpha = \beta + \delta$ for some δ . By Lemma 3 we have that $p^\beta H$ is $p^\alpha G = p^{\beta+\delta} G = p^\delta(p^\beta G)$ -high in $p^\beta G$, and Corollary 5 completes the proof.

Making certain provisions about G , we are able to say when $p^\alpha G$ -high subgroups are q -isotype for any prime q . In this connection we have

THEOREM 3. — *Let H be $p^\alpha G$ -high in G , and suppose $p^\alpha G$ has no elements of order q , where q is a prime. Then H is q -isotype in G .*

PROOF. — If $q = p$, the assertion follows from Corollary 5. Now assume $q \neq p$. We show that $H \cap q^\beta G = q^\beta H$ for all ordinals β . For this purpose it suffices to verify that $H \cap q^\beta G \subseteq q^\beta H$. For $\beta = 0$ this is trivial. Let $\beta > 0$, and suppose the inequality holds for all ordinals $\delta < \beta$. If β is a limit ordinal, then

$$H \cap q^\beta G = H \cap \left(\bigcap_{\delta < \beta} (q^\delta G) \right) = \bigcap_{\delta < \beta} (H \cap q^\delta G) = \bigcap_{\delta < \beta} (q^\delta H) = q^\beta H.$$

Next suppose $\beta = \delta + 1$. Let $h \in H \cap q^\beta G = H \cap q(q^\delta G)$. Then $h = qg_\delta$, where $g_\delta \in q^\delta G$. By the induction hypothesis, if $g_\delta \in H$, then $g_\delta \in q^\delta H$ and $h = qg_\delta \in q(q^\delta H) = q^\beta H$. Now assume $g_\delta \notin H$. Then since H is $p^\alpha G$ -high in G , we have $h_1 + ng_\delta = g_\alpha \neq 0$, where $h_1 \in H$, $g_\alpha \in p^\alpha G$, and n is an integer. Thus $qh_1 + nqg_\delta = qh_1 + nh = qg_\alpha \in H$. Therefore $qg_\alpha = 0$, and since $p^\alpha G$ has no elements of order q , $g_\alpha = 0$. This contradiction establishes the theorem.

The following two corollaries follow immediately from Theorem 3.

COROLLARY 7. — *Let H be $p^\alpha G$ -high in G , and suppose $p^\alpha G$ is torsion-free. Then H is isotype in G , and in particular H is pure in G .*

COROLLARY 8. — *Let H be $p^\alpha G$ -high in G , and suppose $p^\alpha G$ is a p -group. Then H is isotype in G . In particular, H is pure in G .*

If G is a p -group, then the subgroup G^1 of elements of infinite height in G is $p^\omega G$. Thus Corollary 8 implies that a high subgroup H of a p -group is isotype, and consequently pure. The answer to Fuchs' question is readily obtained from the purity of H . (See [3].) However, we proceed now to derive more general results.

THEOREM 4. — *Let A be a subgroup of G , and let S be a non-void set of primes. For each $p \in S$, let α_p be an ordinal. Suppose that for each $a \in A$, $a \neq 0$, there exists $p \in S$ such that $H_p(a) < \alpha_p$. Then A is contained in a subgroup H of G such that H is p^{α_p+1} -pure in G for each $p \in S$, and for each $h \in H$, $h \neq 0$, there exists $p \in S$ such that $H_p(h) < \alpha_p$.*

PROOF. — Since $A \cap \left(\bigcap_{p \in S} p^{\alpha_p+1} G \right) = 0$, A is contained in a $\bigcap_{p \in S} p^{\alpha_p}$ G -high

subgroup H of G . Now the proof follows immediately from Theorem 2.

The following result generalizes a theorem of Erdélyi ([2], p. 81).

COROLLARY 9. — *Let H be a subgroup of G , let p be a prime, and let α be an ordinal. Suppose that for each nonzero $h \in H$, $H_p(h) < \alpha$. Then H is contained in a p -isotype subgroup A of G such that for each nonzero $a \in A$, $H_p(a) < \alpha$.*

PROOF. — This proof is analogous to the proof of Theorem 4, using Corollary 3.

COROLLARY 10. — *Let G be a p -group, and let A be a subgroup of G such that A has no nonzero elements of infinite height. Then A is contained in an isotype subgroup H of G such that H has no nonzero elements of infinite height.*

PROOF. — The proof is similar to the proof of Corollary 9, using Corollary 8.

COROLLARY 11. — *Let A be a subgroup of G with no elements of infinite height; i. e., $A \cap G' = 0$. Then A is contained in a pure subgroup K of G such that K has no elements of infinite height and such that $|K| \leq \aleph_0 |A|$.*

PROOF. — The subgroup A is contained in a high subgroup H of G , and H is pure in G by Corollary 4. Now A can be embedded in a pure subgroup K of H such that $|K| \leq \aleph_0 |A|$. (See [2], p. 78.) Clearly K has no elements of infinite height and is pure in G .

We will now discuss the question of how isomorphic the $p^\alpha G$ -high subgroups are. In particular we will show that if G is a countable p -group, then any two $p^\alpha G$ -high subgroups of G are isomorphic. When any two such subgroups of an arbitrary group G are isomorphic is not known. However, we will state and prove an interesting theorem concerning the relationship of the Ulm invariants of these subgroups to those of G when G is a p -group.

LEMMA 4. — *Let L be a subgroup of a group G with H and K both L -high subgroups of G . Then*

$$((H \oplus L)/L)[p] = ((K \oplus L)/L)[p]$$

for each prime p .

PROOF. — For $h \in H$ we have that $o(h+L) = p$ if and only if $o(h) = p$. If $h \in (H \cap K)[p]$, then $h+L$ is in $((K \oplus L)/L)[p]$. Suppose $h \in H[p] \setminus K \cap H$. Then there exists $k \in K$, $x \in L$ with $h - k = x$, whence $o(k) = p$. Thus

$$h + L = k + L \in ((K \oplus L)/L)[p];$$

and since h was arbitrary, we have by symmetry that

$$((H \oplus L)/L)[p] = ((K \oplus L)/L)[p]$$

as stated.

LEMMA 5. — Let H and K be $p^\beta G$ -high in a reduced p -group G . Then $|H| = |K|$.

PROOF. — If $p^\beta G = 0$, $H = K$. When β is finite, then $H \cong K$. (See [2], p. 99 and 104). When β is infinite and $p^\beta G \neq 0$, embed G in a divisible hull E of G . (A divisible hull of G is a minimal divisible group containing G .) Then $r(H) = r(E/D) = r(K)$, where D is a divisible hull of $p^\beta G$ in E . That $|H| = |K|$ follows now from easy set theoretic considerations.

LEMMA 6. — Let H be $p^\beta G$ -high in G . Then for each ordinal α we have

$$(p^\alpha H \oplus p^\beta G)/p^\beta G = p^\alpha((H \oplus p^\beta G)/p^\beta G).$$

PROOF. — If $\alpha \geq \beta$, then both sides are zero. We prove the assertion for $\alpha < \beta$ by induction on α . So assume the equation holds for all ordinals $\delta < \alpha$. (If $\alpha = 0$, then the equality is trivial.) If $\alpha = \delta + 1$, then

$$\begin{aligned} (p^\alpha H \oplus p^\beta G)/p^\beta G &= (p(p^\delta H) \oplus p^\beta G)/p^\beta G \\ &= p((p^\delta H \oplus p^\beta G)/p^\beta G) \\ &= p(p^\delta((H \oplus p^\beta G)/p^\beta G)) = p^\alpha((H \oplus p^\beta G)/p^\beta G). \end{aligned}$$

Now assume α is a limit ordinal. Set

$$L = \left(\left(\bigcap_{\delta < \alpha} p^\delta H \right) \oplus p^\beta G \right) / p^\beta G \quad \text{and} \quad R = \bigcap_{\delta < \alpha} p^\delta((H \oplus p^\beta G)/p^\beta G).$$

Since α is limit ordinal it suffices to prove $L = R$. Clearly $L \subseteq R$. Now let $h + p^\beta G \in R$. Then there exists $h_\delta \in p^\delta H$ such that $h + p^\beta G = h_\delta + p^\beta G$ for each $\delta < \alpha$. This means that for each $\delta < \alpha$ we have $h = h_\delta + g_{\beta\delta}$ for some $g_{\beta\delta} \in p^\beta G$. Thus since $\alpha < \beta$ and H is isotype, we have $h \in p^\delta H$ for each $\delta < \alpha$. Hence $h \in \bigcap_{\delta < \alpha} p^\delta H$, and $h + p^\beta G \in L$. This concludes the proof.

COROLLARY 12. — *Let H and K be $p^\beta G$ -high in G . Then for each ordinal α we have*

$$(p^\alpha((H \oplus p^\beta G)/p^\beta G)[p]) = (p^\alpha((K \oplus p^\beta G)/p^\beta G)[p]).$$

PROOF. — This follows from Lemma 6, the fact that $p^\alpha H$ and $p^\alpha K$ are $p^\beta G$ high in $p^\alpha G$, and Lemma 4.

THEOREM 5. — *Let H and K be $p^\beta G$ -high in a p -group G . Then H and K have the same Ulm invariants (as defined by KAPLANSKY in [5]). Moreover for all $\alpha < \beta$, the α -th Ulm invariant of H is the same as the α -th Ulm invariant of G .*

PROOF. — First observe that $H \cong (H \oplus p^\beta G/p^\beta G) = \tilde{H}$, and similarly $K \cong \tilde{K}$. We will show that \tilde{H} and \tilde{K} have the same Ulm invariants. From Corollary 12 we have for each ordinal α that

$$(p^\alpha((H \oplus p^\beta G)/p^\beta G)[p]) = (p^\alpha((K \oplus p^\beta G)/p^\beta G)[p])$$

so that

$$((p^\alpha \tilde{H})[p]) / (p^{\alpha+1} \tilde{H})[p] = ((p^\alpha \tilde{K})[p]) / (p^{\alpha+1} \tilde{K})[p].$$

This shows that H and K have the same Ulm invariants. To prove the second part of the theorem notice that for $\alpha < \beta$ we have

$$\begin{aligned} ((p^\alpha G)[p]) / (p^{\alpha+1} G)[p] &= ((p^\alpha H)[p] \oplus (p^\beta G)[p]) / ((p^{\alpha+1} H)[p] \oplus (p^\beta G)[p]) \\ &\cong (p^\alpha H)[p] / (p^{\alpha+1} H)[p]. \end{aligned}$$

The equality follows from Corollary 5 and the fact that $\alpha < \beta$. The isomorphism is the natural one.

As an easy application of Theorem 5 we have

THEOREM 6. *Let H and K be $p^\beta G$ -high in G , and let G be a p -group. If H is countable, then $H \cong K$. Moreover if H and K are both direct sums of countable groups, then $H \cong K$.*

PROOF. — Clearly H and K are reduced. For the first part, $|H| = |K| = \aleph_0$ by Lemma 5. Hence by Theorem 5 and Ulm's theorem, $H \cong K$. If H and K are both direct sums of countable groups, we have by a theorem of Kolettis (see [6]) that $H \cong K$.

We conclude with a corollary to Theorem 5.

THEOREM 7. — *Let G be a group of type β . (G is a p -group.) Then for each ordinal $\alpha \leq \beta$, there exists an isotype subgroup H of G such that the first α Ulm invariants of G coincide with the Ulm invariants of H .*

PROOF. — Let H be $p^\alpha G$ -high in G and apply Theorem 5

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