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On isotype subgroups of abelian groups


<http://www.numdam.org/item?id=BSMF_1961__89__451_0>
ON ISOTYPE SUBGROUPS OF ABELIAN GROUPS;

BY

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In his book *Abelian groups*, L. Fuchs asks the following question. Let $G$ be a $p$-group and $H$ be a subgroup without elements of infinite height. Under what conditions can $H$ be embedded in a pure subgroup of the same power and again without elements of infinite height? (See [2], p. 96.) This question has been answered by Charles [1] and Irwin [3]. Irwin's solution was effected by showing that any subgroup maximal with respect to disjointness from the subgroup of elements of infinite height is pure. For $p$-groups, the subgroups of element of infinite height is $p^0 G$. Now for any Abelian group $G$, any prime $p$, and any ordinal $\alpha$, one may define $p^\alpha G$, and this suggests the following problem. Is any subgroup of $G$ maximal with respect to disjointness from $p^\alpha G$ pure in $G$? Or, more generally, does any such subgroup $H$ of $G$ have the property that $H \cap p^\delta G = p^\delta H$ for all ordinals $\delta$? That is to say, is $H$ a $p$-isotype in $G$? We will show that indeed any such $H$ is $p$-isotype, and we will give a partial solution to the problem of determining whether any two such $H$'s are isomorphic. The foregoing considerations will lead to the solution of a more general version of the above mentioned problem of L. Fuchs.

All groups considered in this paper will be Abelian.

**Definition 1.** — Let $G$ be a group and $p$ be a prime. Define $p^0 G = G$. If $p^\delta G$ is defined for all ordinals $\delta < \alpha$, then define $p^\alpha G = \bigcap_{\delta < \alpha} p^\delta G$ when $\alpha$ is a limit ordinal. If $\alpha = \delta + 1$ for some ordinal $\delta$, let $p^\alpha G = p(p^\delta G)$.

Thus we have defined $p^\alpha G$ for all ordinals $\alpha$, and clearly the $p^\alpha G$'s form a chain of fully invariant subgroups of $G$. 

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DEFINITION 2. — Let \( p \) be a prime and \( g \in G \). The \( p \)-height \( H_p(g) \) of \( g \) is the ordinal \( \alpha \) such that \( g \in p^\alpha G \) and \( g \notin p^{\alpha+1} G \). If no such ordinal \( \alpha \) exists, then \( H_p(g) = \infty \), where the symbol \( \infty \) is considered larger than any ordinal. Let \( \alpha \) be an ordinal or \( \infty \). Then a subgroup \( H \) of \( G \) is \( p^\alpha \)-pure in \( G \) if and only if \( H \cap p^\beta G = p^\beta H \) for all ordinals \( \beta \leq \alpha \); \( H \) is \( \alpha \)-pure in \( G \) if and only if \( H \) is \( p^\alpha \)-pure in \( G \) for all primes \( p \). A subgroup \( H \) is \( p \)-isotype in \( G \) if and only if \( H \) is \( p^\alpha \)-pure in \( G \). The subgroup \( H \) is isotype in \( G \) if and only if \( H \) is \( \alpha \)-isotype in \( G \) for all primes \( p \).

It follows easily from the definitions that the properties of being isotype, \( \alpha \)-pure, or \( p^\alpha \)-pure are transitive. Moreover, the union of an ascending chain of subgroups with one of these properties is a subgroup with that property.

It is easy to see that there are groups in which not every pure subgroup is isotype. In fact, there exist reduced \( p \)-groups \( G \) such that \( |p^\beta G| = S_\beta \) and \( |\beta| \geq 2^\aleph_0 \). (See [2], p. 131, Theorem 38.2 for the existence of such a \( G \).) Embed \( p^\beta G \) in a pure subgroup \( K \) of \( G \) with \( |K| = S_\beta \). Clearly \( K \) is not isotype since \( p^\beta K = 0 \) and \( K \cap p^\beta G = p^\beta G \neq 0 \).

We now state and prove a few facts which will be useful in what follows, and which illustrate the relation between the above definitions and the ordinary notions of purity and height.

**LEMMA 1.** — For a positive integer \( n \), let \( n = \prod_{i=1}^{r} p_i^{n_i} \) be its prime decomposition. Then for any group \( G \), \( nG = \bigcap_{i=1}^{r} p_i^{n_i}G \).

**PROOF.** — Let \( T = \bigcap_{i=1}^{r} p_i^{n_i}G \). Clearly \( nG \subseteq T \). Now let \( g \in T \). For \( n_i = n/p_i^{n_i} \), there exist integers \( a_i \) with \( \sum a_i n_i = 1 \). But \( g \in T \) yields \( g = p_i^{n_i}g_i \), \( i = 1, \ldots, r \). Hence

\[
g = \sum a_i n_i g = \sum a_i n_i p_i^{n_i}g_i = \sum a_i n_i g_i = n \sum a_i g_i \in nG.
\]

Hence \( nG = T \), and the proof is complete.

**COROLLARY 1.** — A subgroup \( H \) of a group \( G \) is pure in \( G \) if and only if \( H \) is \( \omega \)-pure.

**PROOF.** — Suppose \( H \) is pure in \( G \). In particular, \( H \cap p^m G = p^m H \) for each prime \( p \) and non-negative integer \( m \). Now

\[
H \cap p^\omega G = H \cap \left( \bigcap_{k<\omega} p^k G \right) = \bigcap_{k<\omega} (H \cap p^k G) = \bigcap_{k<\omega} p^k H = p^\omega H.
\]
Hence $H$ is $\omega$-pure. Next suppose $H$ is $\omega$-pure, and $n$ is a positive integer. Then
\[ H \cap nG = H \cap \left( \prod p_i^{t_i} \right) G = H \cap \left( \bigcap p_i^{t_i} G \right) = \bigcap (H \cap p_i^{t_i} G) = \bigcap p_i^{t_i} H = nH \]
by Lemma 1.

The following definition is standard.

**Definition 3.** — The subgroup $G^1 = \bigcap nG$ is the subgroup of elements of infinite height in $G$.

We are now in a position to prove the following useful

**Corollary 2.** — Let $P$ be the set of all primes. Then $G^1 = \bigcap p^n G$.

**Proof.** — Set $T = \bigcap p^n G$. Then from $p^n G = \bigcap p^n G$ for each $p \in P$, it follows that $p^n G \supset \bigcap nG$ for each $p \in P$, and hence $T \supset G^1$. Now for each $n$ we have $nG = \bigcap p^n G \supset T$. Hence $G^1 \supset T$, whence $G^1 = T$.

This corollary shows that the subgroup $G^1$ of elements of infinite height in $G$ is the set of elements of infinite $p$-height for each prime $p$. The following theorem and corollary are generalizations of Kaplansky's Lemma 7 ([5], p. 20)

**Theorem 1.** — Let $H$ be a subgroup of a $p$-group $G$, and let $\alpha$ be a limit ordinal or $\infty$. Then $H$ is $p^\alpha$-pure in $G$ if and only if whenever $\beta < \alpha$, $h \in H[p]$, and the $p$-height in $G$ of $h$ is $\geq \beta$, then the $p$-height in $H$ of $h$ is $\geq \beta$.

**Proof.** — If $H$ is $p^\alpha$-pure, then clearly the elements in $H[p]$ have the desired property. To prove the converse, it must be established that $H \cap p^\delta G = p^\delta H$ for all $\delta < \alpha$. Obviously $H \cap p^\delta G \supset p^\delta H$. Let $P(\alpha)$ be the statement: For $\beta < \alpha$, the elements in $H$ of exponent $\leq n$ have $p$-height $\geq \beta$ in $H$ if they have $p$-height $\geq \beta$ in $G$. We will prove $P(\alpha)$ is true for all $n$ by induction and consequently have that $H \cap p^\delta G \subseteq p^\beta H$ for all $\delta < \alpha$. Now $P(1)$ is true by hypothesis. Assume $P(n)$ holds, and let $h \in H$ with $o(h) = p^{n+1}$, and suppose the $p$-height of $h$ is $\geq \beta$ in $G$. Then $ph$ has exponent $n$ and $p$-height $\geq \beta + 1$ in $G$. Since $\beta + 1 < \alpha$, our induction hypothesis yields $ph = p^\delta h$ with $h \in p^\delta H$. Hence $(h - h) \in H[p]$, has $p$-height $\geq \beta$ in $G$, and so $p$-height $\geq \beta$ in $H$. Therefore $H \cap p^\delta G \subseteq p^\beta H$ for all $\delta < \alpha$ and since $\alpha$ is a limit ordinal, this holds for all $\delta \leq \alpha$. Thus $H$ is $p^\alpha$-pure in $G$. 
Corollary 3. — Let $H$ be a subgroup of a $p$-group $G$. Then $H$ is isotype in $G$ if and only if the elements in $H[p]$ have the same $p$-height in $H$ as in $G$.

Proof. — Since $G$ is a $p$-group, we have $qH = H$ for all $q \neq p$, and hence $H$ is $q$-isotype for all $q \neq p$. To get $H$ $p$-isotype, let $x$ be $\infty$ in Theorem 1.

We proceed now to our main results and begin with the following definition:

Definition 4. — Let $K$ and $L$ be subgroups of $G$. Then $H$ is $L$-high in $K$ if and only if $H$ is a subgroup of $K$ maximal with respect to the property that $H \cap L = 0$. A high subgroup $H$ of $G$ is a subgroup maximal with respect to the property $H \cap G^1 = 0$. (See [3].)

The principal result of this paper is the following theorem:

Theorem 2. — Let $G$ be a group, let $p$ be a prime, let $x$ be an ordinal, let $K$ be a subgroup of $p^x G$, and let $H$ be $K$-high in $G$. Then $H$ is $p^{x+1}$-pure in $G$, and $p^\beta H$ is $K$-high in $p^\beta G$ for all ordinals $\beta \leq x$.

Proof. — To show that $H$ is $p^{x+1}$-pure in $G$ we need to establish that $H \cap p^\beta G = p^\beta H$ for all $\beta \leq x + 1$. We induct on $\beta$, and if $\beta = 0$, the equality is trivial. Now suppose $0 < \beta \leq x + 1$, and suppose the equality holds for all ordinals less than $\beta$. If $\beta$ is a limit ordinal, then

$$H \cap p^\beta G = H \cap \bigcap_{\delta < \beta} p^\delta G = \bigcup_{\delta < \beta} (H \cap p^\delta G) = \bigcup_{\delta < \beta} p^\delta H = p^\beta H.$$ 

Next suppose $\beta$ is not a limit ordinal. Then there is an ordinal $\gamma$ such that $\beta = \gamma + 1$. Then

$$p^\gamma H \subseteq H \cap p^\beta G = H \cap p(p^\beta G).$$

Let $h = pgz$ with $h \in H$ and $g \in p^\beta G$. If $g \in H$, then

$$g \in H \cap p^\beta G = p^\gamma H,$$

and

$$h = pgz \in p(p^\beta H) = p^\gamma H.$$ 

So suppose $g \notin H$. Since $H$ is $K$-high in $G$ and $K \notin p^x G$, we have

$$h_1 + ngz = k \neq 0,$$

where $h_1 \in H$, $k \in K$, and $n$ an integer. Clearly $(n, p) = 1$, and $k \in p^x G$. Since $\gamma \leq x$, we have $h_1 \in p^\gamma G$. The induction hypothesis yields $h_1 \in p^\beta H$. Now

$$ph_1 + npgz = ph_1 + nh = pk = 0.$$ 

Therefore

$$nh = ph_1 \in p(p^\beta H) = p^\beta H.$$
Also \( ph \in p^3 H \) since \( h \in p^3 G \subseteq p^3 G \), consequently \( h \in p^3 H \). There exist integers \( a \) and \( b \) such that \( an + bp = 1 \). Thus

\[
anh + bph = h \in p^3 H.
\]

Hence \( H \cap p^3 G = p^3 H \) and \( H \) is \( p^{x+1} \)-pure in \( G \) as stated.

It remains to show that \( p^3 H \) is \( K \)-high in \( p^3 G \) for \( \beta \leq \alpha \). Suppose this is not the case. Then there exists \( g_\beta \in p^3 G, g_\beta \notin p^3 H \) such that the subgroup generated by \( p^3 H \) and \( g_\beta \) is disjoint from \( K \). If \( g_\beta \notin H \), then since \( H \) is \( p^{x+1} \)-pure in \( G \) and \( \beta \leq \alpha \), \( g_\beta \in p^3 H \) contrary to the choice of \( g_\beta \).

Hence \( g_\beta \in H \). Since \( H \) is \( K \)-high in \( G \), we have \( h + ng_\beta = k \neq 0 \), where \( h \in H \) and \( k \in K \subseteq p^3 G \). From \( \beta \leq \alpha \) we have that \( h \in p^3 G \), and hence \( h \in p^3 H \) by \( p^{x+1} \)-purity of \( H \) in \( G \). But this together with the equation \( h + ng_\beta = k \neq 0 \) contradicts the fact that the subgroup generated by \( p^3 H \) and \( g_\beta \) is disjoint from \( K \). This concludes the proof.

As an easy consequence of Theorem 2 we obtain a generalization of Irwin's result mentioned above.

**Corollary 1.** — Let \( K \) be any subgroup of \( G \) and \( H \) be \( K \)-high in \( G \). Then \( H \) is \((\omega + 1)\)-pure (and hence pure) in \( G \). In particular, if \( H \) is high in \( G \), then \( H \) is pure in \( G \).

**Proof.** — Since \( K \subseteq p^{\omega} G \) for each prime \( p \), \( H \) is \( p^{\omega+1} \)-pure for each \( p \).

Hence \( H \) is \((\omega + 1)\)-pure.

Another result along these lines is

**Corollary 2.** — Let \( H \) be \( p^\times G \)-high in \( G \). Then \( H \) is \( p \)-isotype in \( G \), and \( p^\beta H \) is \( p^{\beta} G \)-high in \( p^\beta G \) for all \( \beta \).

**Proof.** — Since \( H \) is \( p^\times G \)-high in \( G \), then \( H \cap p^\beta G = p^\beta H = 0 \) for all \( \beta \geq \alpha \), and Theorem 2 yields \( H \) is \( p \)-isotype. For ordinals \( \beta \geq \alpha \), the only \( p^\times G \)-high subgroup in \( p^\beta G \) is 0 and \( p^\beta H = 0 \) for such \( \beta \). By Theorem 2, \( p^\beta H \) is \( p^\beta G \)-high in \( p^\beta G \) for all \( \beta \).

**Lemma 3.** — For any group \( G \) and any ordinals \( \alpha \) and \( \beta \), \( p^\times (p^\beta G) = p^{\beta + \alpha} G \).

**Proof.** — Induct on \( \alpha \). The assertion is true for \( \alpha = 0 \). Now assume \( \alpha > 0 \) and that the assertion is true for all ordinals \( \beta < \alpha \). Suppose \( \alpha \) is a limit ordinal. Then

\[
p^\times (p^\beta G) = \bigcap_{\beta < \alpha} p^\beta (p^\beta G)
\]

\[
= \bigcap_{\beta < \alpha} (p^{\beta + \alpha} G) = \bigcap_{\beta < \alpha} (p^\beta (p^\beta G)) = p^{\beta + \alpha} G.
\]
since $\beta + \alpha$ is a limit ordinal. Suppose $\alpha = \delta + 1$. Then
$$p^\alpha(p^\beta G) = p(p^{\beta + 1}(p^\delta G)) = p(p^{\beta + \delta + 1} G) = p^{\beta + (\delta + 1)} G = p^{\beta + \alpha} G.$$ 

As a simple application of Lemma 3 we have

**Corollary 6.** — *Let $H$ be $p^\beta G$-high in $G$. Then $p^\beta H$ is $p$-isotype in $p^\beta G$ for all $\beta$.*

**Proof.** — By Corollary 5, $p^\beta H$ is $p^\beta G$-high in $p^\beta G$ for all $\beta$. If $\alpha \leq \beta$, then $p^\beta H = 0$ and hence is isotype. If $\beta < \alpha$, then $\alpha = \beta + \delta$ for some $\delta$. By Lemma 3 we have that $p^\beta H$ is $p^\beta G = p^{\beta + \delta} G = p^\delta(p^\beta G)$-high in $p^\beta G$, and Corollary 5 completes the proof.

Making certain provisions about $\delta$, we are able to say when $p^\beta G$-high subgroups are $q$-isotype for any prime $q$. In this connection we have

**Theorem 3.** — *Let $H$ be $p^\beta G$-high in $G$, and suppose $p^\alpha G$ has no elements of order $q$, where $q$ is a prime. Then $H$ is $q$-isotype in $G$.*

**Proof.** — If $q = p$, the assertion follows from Corollary 5. Now assume $q \neq p$. We show that $H \cap q^\beta G = q^\beta H$ for all ordinals $\beta$. For this purpose it suffices to verify that $H \cap q^\beta G \subseteq q^\beta H$. For $\beta = 0$ this is trivial. Let $\beta > 0$, and suppose the inequality holds for all ordinals $\delta < \beta$. If $\beta$ is a limit ordinal, then

$$H \cap q^\beta G = H \cap \bigcap_{\delta < \beta} (q^\delta G) = \bigcap_{\delta < \beta} (H \cap q^\delta G) = \bigcap_{\delta < \beta} (q^\delta H) = q^\beta H.$$ 

Next suppose $\beta = \delta + 1$. Let $h \in H \cap q^\beta G = H \cap q(q^\delta G)$. Then $h = qg_\delta$, where $g_\delta \in q^\delta G$. By the induction hypothesis, if $g_\delta \in H$, then $g_\delta \in q^\delta H$ and $h = qg_\delta \in q(q^\delta H) = q^\beta H$. Now assume $g_\delta \notin H$. Then since $H$ is $p^\beta G$-high in $G$, we have $h_1 + ng_\delta = g_2 \neq 0$, where $h_1 \in H$, $g_2 \in p^\alpha G$, and $n$ is an integer. Thus $qh_1 + nqg_\delta = qh_1 + nh = qg_2 \in H$. Therefore $qg_2 = 0$, and since $p^\alpha G$ has no elements of order $q$, $g_2 = 0$. This contradiction establishes the theorem.

The following two corollaries follow immediately from Theorem 3.

**Corollary 7.** — *Let $H$ be $p^\beta G$-high in $G$, and suppose $p^\alpha G$ is torsion-free. Then $H$ is isotype in $G$, and in particular $H$ is pure in $G$.*

**Corollary 8.** — *Let $H$ be $p^\beta G$-high in $G$, and suppose $p^\alpha G$ is a $p$-group. Then $H$ is isotype in $G$. In particular, $H$ is pure in $G$.*

If $G$ is a $p$-group, then the subgroup $G^1$ of elements of infinite height in $G$ is $p^{\alpha G}$. Thus Corollary 8 implies that a high subgroup $H$ of a $p$-group is isotype, and consequently pure. The answer to Fuchs' question is readily obtained from the purity of $H$. (See [3].) However, we proceed now to derive more general results.
Theorem 4. — Let $A$ be a subgroup of $G$, and let $S$ be a non-void set of primes. For each $p \in S$, let $x_p$ be an ordinal. Suppose that for each $a \in A$, $a \neq 0$, there exists $p \in S$ such that $H_p(a) < x_p$. Then $A$ is contained in a subgroup $H$ of $G$ such that $H$ is $p^{x_p+1}$-pure in $G$ for each $p \in S$, and for each $h \in H$, $h \neq 0$, there exists $p \in S$ such that $H_p(h) < x_p$.

Proof. — Since $A \cap \left( \bigcap_{p \in S} p^{x_p} G \right) = 0$, $A$ is contained in a $\bigcap_{p \in S} p^{x_p} G$-high subgroup $H$ of $G$. Now the proof follows immediately from Theorem 2.

The following result generalizes a theorem of Erdélyi ([2], p. 81).

Corollary 9. — Let $H$ be a subgroup of $G$, let $p$ be a prime, and let $\alpha$ be an ordinal. Suppose that for each nonzero $h \in H_p$, $H_p(h) < \alpha$. Then $H$ is contained in a $p$-isotype subgroup $A$ of $G$ such that for each nonzero $a \in A$, $H_p(a) < \alpha$.

Proof. — This proof is analogous to the proof of Theorem 4, using Corollary 5.

Corollary 10. — Let $G$ be a $p$-group, and let $A$ be a subgroup of $G$ such that $A$ has no nonzero elements of infinite height. Then $A$ is contained in an isotype subgroup $H$ of $G$ such that $H$ has no nonzero elements of infinite height.

Proof. — The proof is similar to the proof of Corollary 9, using Corollary 8.

Corollary 11. — Let $A$ be a subgroup of $G$ with no elements of infinite height; i.e., $A \cap G^1 = 0$. Then $A$ is contained in a pure subgroup $K$ of $G$ such that $K$ has no elements of infinite height and such that $|K| \leq |S_0| |A|$

Proof. — The subgroup $A$ is contained in a high subgroup $H$ of $G$, and $H$ is pure in $G$ by Corollary 4. Now $A$ can be embedded in a pure subgroup $K$ of $H$ such that $|K| \leq |S_0| |A|$. (See [2], p. 78.) Clearly $K$ has no elements of infinite height and is pure in $G$.

We will now discuss the question of how isomorphic the $p^a G$-high subgroups are. In particular we will show that if $G$ is a countable $p$-group, then any two $p^a G$-high subgroups of $G$ are isomorphic. When any two such subgroups of an arbitrary group $G$ are isomorphic is not known. However, we will state and prove an interesting theorem concerning the relationship of the Ulm invariants of these subgroups to those of $G$ when $G$ is a $p$-group.

Lemma 4. — Let $L$ be a subgroup of a group $G$ with $H$ and $K$ both $L$-high subgroups of $G$. Then

$$((H \oplus L)/L)[p] = ((K \oplus L)/L)[p]$$

for each prime $p$. 
PROOF. — For \( h \in H \) we have that \( o (h + L) = p \) if and only if \( o(h) = p \). If \( h \in (H \cap K) \{ p \} \), then \( h + L \) is in \( (K \oplus L)/L \{ p \} \). Suppose \( h \in H[p] \setminus K \cap H \). Then there exists \( k \in K, x \in L \) with \( h - k = x \), whence \( o(k) = p \). Thus

\[
h + L = k + L \in ((K \oplus L)/L) \{ p \};
\]

and since \( h \) was arbitrary, we have by symmetry that

\[
((H \oplus L)/L) \{ p \} = ((K \oplus L)/L) \{ p \}
\]
as stated.

**Lemma 5.** — Let \( H \) and \( K \) be \( p^3 \)-high in a reduced \( p \)-group \( G \). Then \( |H| = |K| \).

**Proof.** — If \( p^3 \subseteq G = 0, H = K \). When \( \beta \) is finite, then \( H \cong K \). (See [2], p. 99 and 104). When \( \beta \) is infinite and \( p^3 \subseteq G \), embed \( G \) in a divisible hull \( E \) of \( G \). (A divisible hull of \( G \) is a minimal divisible group containing \( G \).) Then \( r(H) = r(E/D) = r(K) \), where \( D \) is a divisible hull of \( p^3 \) in \( E \). That \( |H| = |K| \) follows now from easy set theoretic considerations.

**Lemma 6.** — Let \( H \) be \( p^3 \)-high in \( G \). Then for each ordinal \( \alpha \) we have

\[
(p^2 H \oplus p^3 G)/p^3 G = p^2 ((H \oplus p^3 G)/p^3 G).
\]

**Proof.** — If \( \alpha \geq \beta \), then both sides are zero. We prove the assertion for \( \alpha < \beta \) by induction on \( \alpha \). So assume the equation holds for all ordinals \( \delta < \alpha \). (If \( \alpha = 0 \), then the equality is trivial.) If \( \alpha = \delta + 1 \), then

\[
(p^2 H \oplus p^3 G)/p^3 G = (p(p^\delta H) \oplus p^3 G)/p^3 G
\]

\[
= p ((p^\delta H \oplus p^3 G)/p^3 G)
\]

\[
= p ((H \oplus p^3 G)/p^3 G)) = p^2 ((H \oplus p^3 G)/p^3 G).
\]

Now assume \( \alpha \) is a limit ordinal. Set

\[
L = \left( \bigcap_{\delta < \alpha} p^\delta H \right) \oplus p^\beta G / p^\beta G \quad \text{and} \quad R = \bigcup_{\delta < \alpha} p^\delta ((H \oplus p^3 G)/p^3 G).
\]

Since \( \alpha \) is limit ordinal it suffices to prove \( L = R \). Clearly \( L \subseteq R \). Now let \( h + p^3 G \subseteq R \). Then there exists \( h_2 \in p^2 H \) such that \( h + p^3 G = h_2 + p^3 G \) for each \( \delta < \alpha \). This means that for each \( \delta < \alpha \) we have \( h = h_2 + g_{\beta \delta} \) for some \( g_{\beta \delta} \in p^3 G \). Thus since \( \alpha < \beta \) and \( H \) is isotype, we have \( h \in p^2 H \) for each \( \delta < \alpha \). Hence \( h \in \bigcap_{\delta < \alpha} p^\delta H, \) and \( h + p^3 G \subseteq L \). This concludes the proof.
Corollary 12. — Let $H$ and $K$ be $p^3G$-high in $G$. Then for each ordinal $\alpha$ we have
\[
(p^2((H \oplus p^3 G)/p^3 G)[p]) = (p^2((K \oplus p^3 G)/p^3 G)[p]).
\]

Proof. — This follows from Lemma 6, the fact that $p^2H$ and $p^2K$ are $p^3G$-high in $p^2G$, and Lemma 4.

Theorem 5. — Let $H$ and $K$ be $p^3G$-high in a $p$-group $G$. Then $H$ and $K$ have the same Ulm invariants (as defined by Kaplansky in [5]). Moreover for all $\alpha < \beta$, the $\alpha$-th Ulm invariant of $H$ is the same as the $\alpha$-th Ulm invariant of $G$.

Proof. — First observe that $H \cong (H \oplus p^3 G/p^3 G) = \tilde{H}$, and similarly $K \cong \tilde{K}$. We will show that $\tilde{H}$ and $\tilde{K}$ have the same Ulm invariants. From Corollary 12 we have for each ordinal $\alpha$ that
\[
(p^2((H \oplus p^3 G)/p^3 G)[p]) = (p^2((K \oplus p^3 G)/p^3 G))[p]
\]
so that
\[
((p^2\tilde{H})[p])/(p^{\alpha+1}\tilde{H})[p] = ((p^2\tilde{K})[p])/(p^{\alpha+1}\tilde{K})[p].
\]
This shows that $H$ and $K$ have the same Ulm invariants. To prove the second part of the theorem notice that for $\alpha < \beta$ we have
\[
((p^2G)[p])/(p^{\alpha+1}G)[p] = ((p^2H)[p] \oplus (p^2G)[p])/(p^{\alpha+1}H)[p] \oplus (p^2G)[p]
\]
\[
\cong (p^2H)[p]/(p^{\alpha+1}H)[p].
\]
The equality follows from Corollary 3 and the fact that $\alpha < \beta$. The isomorphism is the natural one.

As an easy application of Theorem 5 we have

Theorem 6. Let $H$ and $K$ be $p^3G$-high in $G$, and let $G$ be a $p$-group. If $H$ is countable, then $H \cong K$. Moreover if $H$ and $K$ are both direct sums of countable groups, then $H \cong K$.

Proof. — Clearly $H$ and $K$ are reduced. For the first part, $|H| = |K| = \aleph_0$ by Lemma 5. Hence by Theorem 5 and Ulm's theorem, $H \cong K$. If $H$ and $K$ are both direct sums of countable groups, we have by a theorem of Kolettis (see [6]) that $H \cong K$.

We conclude with a corollary to Theorem 5.

Theorem 7. — Let $G$ be a group of type $\beta$. ($G$ is a $p$-group.) Then for each ordinal $\alpha \leq \beta$, there exists an isotype subgroup $H$ of $G$ such that the first $\alpha Ulm$ invariants of $G$ coincide with the Ulm invariants of $H$.

Proof. — Let $H$ be $p^2G$-high in $G$ and apply Theorem 3.
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(Manuscrit reçu le 10 mai 1961.)

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