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Remarks on a problem in primary abelian groups

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REMARKS ON A PROBLEM IN PRIMARY ABELIAN GROUPS ;

BY

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1. All groups considered in this note are assumed to be p-primary abelian groups. If A is a subgroup of G then $\overline{A}$ will denote the closure of A in the usual topology of G ([2], page 114). The closure of a subgroup is a subgroup, but the closure of a pure subgroup need not be pure. It is a consequence of lemma 20 of [3] that if G is a closed p-group (for definition *see* [2], page 114) then the closure of each pure subgroup of G is pure.

PROBLEM. — If G is a primary abelian group without elements of infinite height in which the closure of each pure subgroup is pure does it follow that G is a closed p-group ?

We do not know the answer to this question, but we can give an affirmative answer in the case of direct sums of cyclic groups :

THEOREM. — If G is a direct sum of cyclic p-groups and the closure of each pure subgroup of G is pure in G then G is a bounded p-group.

An outline of the proof of this theorem is given in paragraph 3 below.

2. **The relation of the problem to minimal pure embeddings.** — Following B. CHARLES [1], when a subgroup S of a group G is contained in a pure subgroup P of G which has the property that no proper pure subgroup of P contains S we say that P is *minimal pure containing* S.

When such a P exists we say that S *has a minimal pure embedding in* G. We will denote the subgroup of elements of infinite height in a group G by G^1.

In the proofs below we use the following two observations :

If A is a subgroup of G then $\overline{A}$ is that subgroup of G containing A for which $\overline{A}/A = (G/A)^1$. If a subgroup S of a group G is contained in G^1 and if P is minimal pure containing S then P is divisible.

The latter observation follows from the fact that is P were not divisible P would contain a finite cyclic direct summand $\{x\}$ and if $P = \{x\} \oplus C$ then S would be contained in C where C, being a direct summand of P, would be pure in G.

For a subgroup S of G we denote by S' the subgroup of G containing S for which S'/S is the maximal divisible subgroup of G/S.

Proposition. — Let P be a pure subgroup of a primary abelian group G. Let H be a subgroup of G for which $P \subset H \subset \overline{P}$. Then H has a minimal pure embedding in G if and only if $H \subset P'$.

Proof. — Suppose P_1 is minimal pure containing H. Then P_1/P s minimal pure containing H/P in G/P. Since

$$H/P \subset \overline{P}/P = (G/P)^1,$$

P_1/P is a divisible subgroup of G/P. Then $P_1 \subset P'$ and $H \subset P'$. Conversely, if $H \subset P'$ then H/P is contained in the maximal divisible subgroup of G/P. Then there exists a subgroup P_1 of G containing P such that P_1/P is minimal divisible containing H/P in G/P. Then P_1 is minimal pure containing H in G.

It has been suggested ([1], page 224) that if G is a primary abelian group without elements of infinite height and S is a subgroup of G which s the union of an ascending chain of discrete subgroups of G then S has a minimal pure embedding in G. The proposition and theorem above are sufficient to show that this is not true even if the discrete subgroups are finite :

Let G be a countable unbounded direct sum of cyclic p-groups. Let P be a pure subgroup of G for which $\overline{P}$ is not pure. $\overline{P}$ is the union of an ascending chain of finite (hence discrete) subgroups of G. Since P' is pure, $\overline{P} \neq P'$. Consequently $\overline{P}$ is not contained in any subgroup of G which is minimal pure containing $\overline{P}$.

This same example is a counter-example to part 2 of theorem 6 of [1] because P is a pure subgroup of G which is dense in $\overline{P}$ and yet $\overline{P}$ has no minimal pure embedding in G. Along this line we have :

Corollary. — For a primary abelian group G the following two conditions are equivalent :

(1) Each subgroup H of G that contains a subgroup P which is pure in G and dense in H (relative to the topology of G) has a minimal pure embedding in G.

(2) For each pure subgroup P of G, $\bar{P}$ is pure in G.

Proof. — Assume (1) and let P be pure in G. Then $\bar{P}$ has a minimal pure embedding in G. By the proposition $P = P'$ and $\bar{P}$ is pure in G.

Assume (2) and let H be a subgroup of G which contains a subgroup P which is pure in G and dense in H. We have $P \subset H \subset \bar{P}$. Since $\bar{P}$ is pure in G it follows from the proposition that $\bar{P} = P'$. The proposition then gives the conclusion that H has a minimal pure embedding in G.

3. **Outline of the proof of the theorem stated in paragraph 1.** — It is sufficient to show that if $G = \sum_{n=1}^{\infty} Z(p^{i(n)})$ where $i(n)$ is a strictly increasing sequence of positive integers, $i(1) \geqq 2$, and $Z(p^{i(n)})$ is a cyclic group of order $p^{i(n)}$ then G contains a pure subgroup P for which $\bar{P}$ is not pure. For each positive integer n let $g(n)$ be a generator of $Z(p^{i(n)})$. Then it may be verified that the following sequence of elements of G is a linearly independent set and that the subgroup, P, generated by this set is pure in G :

$$s(n) = g(2n-1) + p^{i(2n)-i(2n-1)+1} g(2n) + p^{i(2n+1)-i(2n-1)} g(2n+1), \quad (1 \leqq n < \infty).$$

Let

$$x = p^{i(1)-1} g(1).$$

Then $x \in \bar{P}$ since modulo P we have :

$$x = p^{i(1)-1} g(1) \equiv -p^{i(3)-1} g(3) \equiv \ldots \equiv (-1)^n p^{i(2n+1)-1} g(2n+1) \equiv \ldots.$$

Let y be any element of G for which $p^{i(1)-1} y = x$. There is an integer N such that the component of y in $Z(p^{i(N)})$ is different from 0 and the component of y in $Z(p^{i(n)})$ is 0 for each $n > N$. By proceeding from the fact the component of y in $Z(p^{i(1)})$ is the unique component of y which is not annihilated by $p^{i(1)-1}$, it can be verified that the neighborhood $y + p^{N+2} G$ of y is disjoint from P. Then $y \notin \bar{P}$ and the equation $p^{i(1)-1} z = x$, which has the solution $z = g(1)$ in G, is not solvable for z in $\bar{P}$. Thus $\bar{P}$ is not pure in G.

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