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A note on a paper of Bernard Charles (« Étude sur les sous-groupes d'un groupe abélien »)

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A NOTE ON A PAPER OF BERNARD CHARLES (ÉTUDE SUR LES SOUS-GROUPES D'UN GROUPE ABÉLIEN) ;

BY

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All groups considered in this note are assumed to be p-primary abelian groups. All statements of a topological nature will refer to the usual p-adic topology, that is, the topology on the group G obtained by taking as neighborhoods of O the subgroups $p^i G$ where i is a natural number. $G[p]$ will denote the socle of G and G^1 the subgroup of elements of infinite height.

Bernard CHARLES [1] has published an incorrect proof of the following conjecture : if G is a p-primary abelian group without elements of infinite height and if H is a subgroup of G then there is a minimal pure subgroup of G containing H provided that either

(1) $H \subseteq G[p]$; or

(2) there is a subgroup D of H which is dense in H (relative to the topology of G) and pure in G.

The conjecture is indeed false, and Thomas HEAD [2] has recently given a counter-example to case (2). We wish to show that condition (1) is not sufficient either for the existence of such a minimal pure subgroup.

PROPOSITION. — *If K is a minimal pure subgroup of G containing the subgroup H of G then $K[p] = H[p]$ provided either conditions* (1) *or* (2) *hold.*

PROOF. — Assume first that D is pure in G and dense in H. If K is a minimal pure subgroup of G containing H then (*see* [2]) K/D is a minimal divisible subgroup of G/D containing H/D. Therefore K/D and H/D have the same socle in G/D, and we conclude that $K[p] = H[p]$.

The case where $H \subseteq G[p]$ easily reduces to the case just considered. Indeed if K is a minimal pure subgroup of G containing H, choose a direct sum D of cyclic groups which is pure in K and such that $D[p]$ is a dense subgroup of H. Then D is dense in $\{H, D\}$ and $\{H, D\}[p] = H$.

Clearly then if $G^1 = 0$, the existence of a subgroup $P \subseteq G[p]$ such that G contains no pure subgroup having P as its socle will yield a counter-example to case (1) of Charles' proposition. We observe that such a $P \subseteq G[p]$ will also contradict case (2) as the construction used in the proof of the second half of the above proposition indicates. Finding such an example however seems to be fairly difficult. For example, G cannot be a direct sum of cyclic groups, nor can it be a closed p-group. Moreover P can be neither countable nor dense in $G[p]$.

We proceed now to construct such an example. Let $\{a_i\}$ be a cyclic group of order p^{2i} for $i = 1, 2, 3, \ldots$. Set $B_1 = \sum_1 \{a_i\}$ and $B_2 = \sum_2 \{a_i\}$, where the subscripts 1 and 2 will denote summation respectively over the odd and even natural numbers. Let H_1 and H_2 be the torsion subgroups of $\prod_1 \{a_i\}$ and $\prod_2 \{a_i\}$ respectively. Set $H = H_1 + H_2$ and $K = B_1 + H_2$. Let G_2 be the torsion subgroup of $\prod_2 \{a_{i-1} + pa_i\}$ and set $G = \{G_2, B_2\}$. Paul Hill [3] has shown that G and K are non-isomorphic pure subgroups of H having the same socle.

Suppose that there is a pure subgroup S of G such that $S[p] = H_2[p]$. Since $H = H_1 + H_2$ and since S is pure in H, $H = H_1 + S$. In particular $S \simeq H_2$. Since $S \subseteq G$, $G = (H_1 \cap G) + S = B_1 + S$. This however is a contradiction to the fact that G and K are not isomorphic.

Finally, as it may be of some interest, we mention that — although G and K have the same socle in H — each subgroup of $K[p]$ serves as the socle of some pure subgroup of K.

REFERENCES.

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[2] Head (Thomas J.). — Remarks on a problem in primary abelian groups, *Bull. Soc. math. France*, t. 91, 1963, p. 109-112.

[3] Hill (Paul). — Certain pure subgroups of primary groups, *Report of the New Mexico State University Conference on Abelian Groups*, 1963.

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