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ON A CERTAIN PURIFICATION PROBLEM FOR PRIMARY ABELIAN GROUPS

BY

FRED RICHMAN AND CAROL P. WALKER.

1. **Introduction.** — MITCHELL has shown in [4] that if G is an abelian p -group and K is a neat subgroup of $G' = \bigcap_n G$ then there exists a pure subgroup P of G such that $P \cap G' = K$. He then raises the question whether the converse holds, i. e. if P is pure in G is $P \cap G'$ neat in G' ? This question is one of the important family of questions dealing with purification. The general purification problem is to ascertain precisely which subgroups of a subgroup A of an abelian p -group G are the intersections of A with a pure subgroup of G . It is the purpose of this note to solve the purification problem for $A = G'$.

Terminology and notation will not deviate sharply from [1]. All groups are abelian p -groups. Cardinal numbers are identified with the least ordinal number of that cardinality.

2. **Quasi-neatness, high subgroups and the main theorem.** — A subgroup K of a group G is *neat* if $pG \cap K = pK$. In any event $pG \cap K \supseteq pK$. If K is not neat in G the quotient $(pG \cap K)/pK$ gives some measure as to how neat K is in G . If α is a cardinal number, we shall say that K is α -*quasi-neat* in G if $|(pG \cap K)/pK| \leq \alpha$.

Recall that a high subgroup of G is a subgroup which is maximal with respect to disjointness from G' [2]. Since two high subgroups of G are pure with the same socle in G/G' they have the same final rank. We can now state the main theorem of this note.

THEOREM. — *Let G be an abelian p -group, K a subgroup of G' and α the final rank of a high subgroup of G . There exists a pure subgroup P of G such that $P \cap G' = K \Leftrightarrow K$ is α -quasi-neat in G' .*

In the sequel K will be a subgroup of G' , H will be a high subgroup of G , and α will be the final rank of H . The phrase "can purify K " will signify that there exists a pure subgroup P of G such that $P \cap G' = K$.

3. The dirty work. — We make the first simplification.

LEMMA 1. — *Can purify $K \Leftrightarrow$ There exists a $P \subseteq G$ such that $P' = P \cap G' = K$.*

Proof. \Rightarrow Clear.

\Leftarrow Choose a maximal such P . We shall show that P is pure. Suppose $p^n g = x \in P$ for some $g \in G$ and some positive integer n . By induction on n , we show that $x \in p^n P$. If $g \notin P$, then $p^t g + y = g_1 \in G' - K$ for some $y \in P$ and non-negative integer $t < n$ by the maximality of P . Therefore $y = p^t z$ for some $z \in P$ by induction. Multiplying by p^{n-t} we get $x + p^n z \in G'$ and so $x + p^n z \in P'$ by hypothesis. Thus $x \in p^n P$ as claimed.

Bounded summands often make no difference. This is the case in our endeavors.

LEMMA 2. — *Let $G = A \oplus B$ where B is bounded. Can purify K in $G \Leftrightarrow$ can purify K in A .*

Proof. \Leftarrow Trivial.

\Rightarrow Let P purify K in G . Then $(P \cap A)' = K = (P \cap A) \cap G'$ and we are done by Lemma 1.

Half of the theorem is now relatively painless.

LEMMA 3. — *Can purify $K \Rightarrow |(pG' \cap K)/pK| \leq \alpha = \text{final rank of } H$.*

Proof. — Using Lemma 2 to chop off a bounded piece of G , we may assume that the final rank of H is the rank of H . Suppose that $|(pG' \cap K)/pK| = \delta > \alpha$ and P purifies K . Let $\{x_i\}$ be a set of elements of $pG' \cap K$ independent mod pK and indexed by a set I of cardinal δ . There exist $y_i \in P$ such that $py_i = x_i$. Now $x_i = pg_i$ for some $g_i \in G'$. Thus

$$y_i - g_i \in G[p] = G'[p] \oplus H[p].$$

By adjusting g_i , we may assume that $y_i - g_i \in H[p]$. Therefore there exist indices $i \neq j$ such that $y_i - g_i = y_j - g_j$ since $\text{rank } H < \delta$. Hence

$$p(y_i - y_j) = x_i - x_j \notin pK$$

and so $y_i - y_j \notin K$. But $y_i - y_j = g_i - g_j \in G'$ and $y_i - y_j \in P$ and so $y_i - y_j$ is in K , a contradiction.

For the other half of the theorem, it is convenient to reduce the problem to direct sums of cyclic groups.

LEMMA 4. — *Let B be a basic subgroup of K . Then*

$$(pG' \cap K)/pK \cong (pG' \cap B)/pB$$

and K can be purified if B can.

Proof.—The isomorphism is clear. Let P purify B . Then $G/P = D \oplus T$ where D , the image of K , is divisible. The inverse image of D purifies K .

To prove the next lemma, we use the high subgroup to escort elements out of G' .

LEMMA 5. — *Let K a direct sum of cyclic groups contained in G' such that $|K| \leq \alpha =$ final rank of H . Then there exists a subgroup P of G such that $|P| \leq \alpha$ and $P' = P \cap G' = K$.*

Proof. — Well order the cyclic generators of K by $\{k_\beta\}_{\beta < \alpha}$. Let $p^n k_\beta'' = k_\beta$, n a positive integer. *Claim* : There exist $h_\beta'' \in H$, $\beta < \alpha$ n a positive integer such that :

- (i) order of $(k_\beta'' + h_\beta'' + G') = p^n$;
- (ii) $\{k_\beta'' + h_\beta'' + G'\}$ are independent, $\beta < \alpha$, n a positive integer.

To see this, well order the pairs (β, n) by α , and use transfinite induction. There is clearly no trouble at limit ordinals. To advance one step, we note that there are α possible h_β'' at our disposal which will satisfy (i) and which yield distinct $p^{n-1}(k_\beta'' + h_\beta'' + G')$ since the final rank of H is α and $H \cap G' = 0$. But there are less than α things for $p^{n-1}(k_\beta'' + h_\beta'' + G')$ to avoid to insure (ii). Letting P be generated by $\{k_\beta'' + h_\beta''\}_{(\beta, n) < \alpha}$ brings us home.

We have reduced the problem to K a direct sum of cyclics. A further reduction allows us to assume that $K[p] = G'[p]$. This follows upon writing $G'[p] = K[p] \oplus L$ and replacing G by a subgroup S containing $H \oplus K$ and maximal with respect to disjointness from L . The subgroup S is pure in G ([3], Theorem 5) and so $K \subseteq S'$. Clearly $S'[p] = K[p]$ and H is high in S . Since purifying K in S will purify K in G , we have achieved the desired reduction.

We now take care of the elements that need no escort and so finish off the other half of the theorem.

LEMMA 6. — *Let K be a direct sum of cyclic groups contained in G' such that $K[p] = G'[p]$ and $|(pG' \cap K)/pK| \leq \alpha =$ final rank of H . Then there exists a P in G such that $P' = P \cap G' = K$.*

Proof. — Let $|K| = \gamma$. If $\gamma \leq \alpha$, we are done by Lemma 5. Let A be generated by those cyclic summands of K (relative to a given decomposition) for which some element of $pG' \cap K$ has a height-0 coordinate. From the hypothesis, it is easily seen that $|A| \leq \alpha$. Let B be generated by the remaining cyclic summands of K . By Lemma 5, we can find a subgroup Q of G such that $Q' = Q \cap G' = A$.

Claim : There exists a subgroup C of G' such that $A \subseteq C$, $|C| \leq \alpha$ and $C + B = G'$. It will suffice to show that $|(G'/B)[p]| = |A[p]|$ for then $|G'/B| \leq \alpha$, and we let C be generated by A and representatives

of G^1/B . But if $p(x+B) = 0$, $x \in G^1$, then $px \in B$, and so $px = pb$ for some $b \in B$ by the construction of B . Thus

$$x - b \in G^1[p] = A[p] \oplus B[p]$$

and hence $x + B = a + B$ for some $a \in A[p]$.

Now let the cyclic generators of B be $\{b_\beta\}_{\beta < \gamma}$. *Claim* : There exist $b_\beta'' \in G$, $\beta < \gamma$, n a positive integer such that :

- (1) $p^n b_\beta'' = b_\beta$;
- (2) $\left(Q + \sum \{b_\beta''\}\right) \cap C \subseteq K$.

We prove this by induction on (β, n) well ordered by γ . Again, there is no trouble at limit ordinals. To advance one step, we note that there exist γ elements which satisfy (1) with pairwise intersection $\{b_\beta\}$, e. g. alter an element z such that $p^n z = b_\beta$ by elements g such that $p^{n-1}g \in G^1[p]$. That the g yield the required elements is assured by the fact that $p^{n-1}z \notin G^1$ and that $|G^1[p]| = \gamma$. To show that (2) is preserved upon adjoining one of these elements z we need only worry about $p^j z$ where $j < n$, since $Q \cap G^1 = A$. But we can insure that for some such z , $p^j z \notin Q + C$ for all $j < n$ since we have γ such z with all $p^j z$ distinct, for $j < n$, and $|Q + C| \leq \alpha$.

Finally, let $P = \left(Q + \sum \{b_\beta''\}\right)$ and all is well.

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