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A PROPERTY OF A-SEQUENCES

BY

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Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$, containing a field k (not necessarily its residue field). Recall ([1]; [7]) that an A-sequence is a finite set $x_1, \ldots, x_r$ of elements of A, contained in the maximal ideal $\mathfrak{m}$, such that x_1 is not a zero-divisor in A, and for each $i = 2, \ldots, r$, x_i is not a zero-divisor in $A/(x_1, \ldots, x_{i-1})$. We will show that for many purposes, the elements of an A-sequence behave just like the variables in a polynomial ring over a field. In particular, the sum, product, intersection and quotient of ideals generated by monomials in a given A-sequence are just what one would expect (*see* Corollary 1 below for a precise statement).

PROPOSITION 1. — *Let A be a noetherian local ring containing a field k, and let $x_1, \ldots, x_r$ be an A-sequence. Then the natural map*

$$\varphi : \quad T = k[X_1, \ldots, X_r] \to A$$

of k-algebras, which sends X_i into x_i for each i, is injective, and A is flat as a T-module.

Proof. — We show φ is injective by induction on r, the case $r = 0$ being trivial. Let $r > 0$ be given. Then $x_2, \ldots, x_r$ is an $(A/x_1 A)$-sequence, so by the induction hypothesis, we may assume that

$$\bar{\varphi} : \quad k[X_2, \ldots, X_r] \to A/x_1 A$$

is injective. Now let $t \in T$ be given and write

$$t = \sum_{n=0}^{\infty} X_1^n f_n(X_2, \ldots, X_r),$$

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where each $f_n(X_2, \ldots, X_r) \in k[X_2, \ldots, X_r]$. Suppose that $\varphi(t) = 0$. If $t \neq 0$, let f_s be the first of the f_n which is non-zero. Then

$$\varphi(t) = x_1^s\left(\sum_{n=s}^{\infty} x_1^{n-s} f_n(x_2, \ldots, x_r)\right).$$

Since x_1 is a non-zero-divisor in A, we have

$$\sum_{n=s}^{\infty} x_1^{n-s} f_n(x_2, \ldots, x_r) = 0.$$

Reducing modulo x_1, we find $f_s(x_2, \ldots, x_r) = 0$ in A/x_1A. Now since $\overline{\varphi}$ is injective by the induction hypothesis, $f_s(X_2, \ldots, X_r) = 0$, which is a contradiction. Hence $t = 0$ and φ is injective.

Now to show A is flat over T, we use the local criterion of flatness ([3], chap. III, § 5, n° 2, theorem 1, (iii)) applied to the ring T, the ideal $J = (x_1, \ldots, x_r)$, and the T-module A. We must verify the four following statements :

(*a*) T is noetherian (well-known).

(*b*) A is separated for the J-adic topology, i. e. $\bigcap J^n A = 0$. This is true since JA is contained in the radical $\mathfrak{m}$ of A, and $\bigcap \mathfrak{m}^n = 0$ by Krull's theorem ([3], chap. III, § 3, n° 2).

(*c*) A/JA is flat over $k = T/J$. This is true since anything is flat over a field.

(*d*) $\mathrm{Tor}_1^T(T/J, A) = 0$. To calculate this Tor, we use the Koszul complex $K.(X_1, \ldots, X_r; T)$ ([4], EGA, III, 1.1) which is a resolution of T/J since $X_1, \ldots, X_r$ is a T-sequence. $\mathrm{Tor}_i(T/J, A)$ is the i^{th} homology group of the complex

$$K.(X_1, \ldots, X_r; T) \otimes_T A = K.(x_1, \ldots, x_r; A).$$

But since $x_1, \ldots, x_r$ is an A-sequence, this homology is zero in degrees $i > 0$ ([4], EGA, III, 1.1.4). In particular $\mathrm{Tor}_1^T(T/J, A) = 0$, which completes the proof of the proposition.

COROLLARY 1. — *With the notations of the proposition, let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be any two ideals in* T. *For any ideal* $\mathfrak{c}$ *in* T, *denote by* $\mathfrak{c}A$ *its extension to* A. *Then*

(i) $(\mathfrak{a} + \mathfrak{b})A = \mathfrak{a}A + \mathfrak{b}A$;
(ii) $(\mathfrak{a}.\mathfrak{b})A = (\mathfrak{a}A).(\mathfrak{b}A)$;
(iii) $(\mathfrak{a} \cap \mathfrak{b})A = (\mathfrak{a}A) \cap (\mathfrak{b}A)$;
(iv) $(\mathfrak{a} : \mathfrak{b})A = (\mathfrak{a}A) : (\mathfrak{b}A)$.

(Recall that for any two ideals $\mathfrak{a}$, $\mathfrak{b}$ in a ring R, $\mathfrak{a} : \mathfrak{b} = \{x \in R \mid x.\mathfrak{b} \subseteq \mathfrak{a}\}$.

Proof. — (i) and (ii) are trivially true for any ring extension and are repeated here for convenience. (iii) and (iv) are true for any flat ring extension. (iii) is proved in ([3], chap. I, § 2, n° 6, Prop. 6).

To prove (iv), let $y_1, \ldots, y_s$ be a set of generators for $\mathfrak{b}$. Then $\mathfrak{a} : \mathfrak{b} = \bigcap (\mathfrak{a} : (y_i))$, and so using (iii) we are reduced to the case where $\mathfrak{b}$ is generated by a single element y. Now $\mathfrak{a} : (y)$ is characterized by the exact sequence of T-modules

$$0 \to \mathfrak{a} : (y) \to T \xrightarrow{y} T/\mathfrak{a},$$

where the last map is multiplication by y. Tensoring with A we have an exact sequence of A-modules

$$0 \to (\mathfrak{a} : (y))\, A \to A \xrightarrow{y} A/\mathfrak{a}A$$

from which we deduce that $(\mathfrak{a} : (y))\, A = \mathfrak{a}A : yA$ (Note that for any ideal $\mathfrak{b}$ in T, the natural map $\mathfrak{b} \otimes_T A \to \mathfrak{b}A$ is an isomorphism, since A is flat over T, so we identify the two).

COROLLARY 2 (Theorem of Rees). — *Let A be a noetherian local ring containing a field, and let J be an ideal generated by an A-sequence $x_1, \ldots, x_r$. Then the images $\bar{x}_1, \ldots, \bar{x}_r$ of the x_i in the graded ring*

$$\mathrm{gr}_J(A) = \sum_{n=0}^{\infty} J^n/J^{n+1}$$

are algebraically independent, so that $\mathrm{gr}_J(A)$ *is isomorphic to the polynomial ring* $A/J[X_1, \ldots, X_r]$.

Proof (*see* also [7], Appendix 6, theorem 3). — It is sufficient to show that for each n, J^n/J^{n+1} is a free A/J-module, with the images of the monomials in $x_1, \ldots, x_r$ of degree n for basis. It is clear that these monomials generate J^n/J^{n+1}. To show they are linearly independent, let z be a monomial of degree n in $x_1, \ldots, x_r$, and let J' be the ideal generated by all the other monomials of degree n and by J^{n+1}. Then we must show that $J' : z = J$, which follows from Corollary 1.

COROLLARY 3. — *Let A be a noetherian local ring containing a field k, and let $x_1, \ldots, x_r$ be an A-sequence. Then any ideal of A generated by polynomials in the x_i, with coefficients in k, is of finite homological dimension over A.*

Proof. — Using the notations of the proposition, any such ideal can be written as $\mathfrak{a}A$, where $\mathfrak{a}$ is an ideal in the polynomial ring $T = k[X_1, \ldots, X_r]$. Over T, a has $\mathfrak{a}$ finite projective resolution ([7], chap. VII, § 13, theorem 43)

$$0 \to L_n \to \ldots \to L_1 \to L_0 \to \mathfrak{a} \to 0.$$

Tensoring with A gives an exact sequence

$$0 \to L_n \otimes A \to \ldots \to L_1 \otimes A \to L_0 \otimes A \to \mathfrak{a}A \to 0$$

which is a finite projective resolution of $\mathfrak{a}A$.

Remark. — A refinement of the proof of proposition 1 due to D. QUILLEN allows one to dispense with the hypothesis that A contains a field, provided that one is interested only in ideals of A generated by monic monomials in the x_i. In particular this is sufficient for the result of Corollary 2, and of Proposition 2 below.

As an application we give the following:

PROPOSITION 2. — *Let A be a noetherian local ring containing a field. Let I be a radical ideal in A* (i. e. an ideal which is a finite intersection of prime ideals), *and let J be any ideal generated by an A-sequence whose radical is I. Then, to within isomorphism, the A/I-module*

$$M = \mathrm{Hom}_A(A/I, A/J)$$

is independent of J.

Example. — An interesting case (already known [2]) is that of a local Cohen-Macaulay ring A, with $I = \mathfrak{m}$ the maximal ideal. Then there are ideals J generated by an A-sequence with radical $\mathfrak{m}$, so that M is defined. Its dimension as an $A/\mathfrak{m}$-vector space is an invariant of A, which is equal to 1 if and only if A is a Gorenstein ring. (*See* [2], where if n is the dimension of M, then A is called a MCn-ring. This number is also the " vordere Loewysche Invariante " of A/J in [6], p. 28, and is the number e of the exercises in [5], § 4, p. 67.)

Proof of Proposition. — Let J be generated by the A-sequence $x_1, \ldots, x_r$. Then r is the height of I, and so is independent of J. We consider the r^{th} local cohomology group (*see* [5] for definition and methods of calculation)

$$H = H^r_J(A) = \varinjlim_n \mathrm{Ext}^r(A/J^{(n)}, A),$$

where $J^{(n)} = (x_1^n, \ldots, x_r^n)$. Using the Koszul complex $K.(x_1^n, \ldots, x_r^n; A)$ to calculate the Ext, we find an isomorphism

$$\varphi_n : \quad \mathrm{Ext}^r(A/J^{(n)}, A) \xrightarrow{\sim} A/J^{(n)}$$

which transforms the maps of the direct system into the maps

$$f_n : \quad A/J^{(n)} \to A/J^{(n+1)}$$

which are defined by multiplication by $x_1 \cdots x_r$.

I claim that the maps f_n are all injective. Indeed, it is sufficient to see that

$$J^{(n+1)} : \quad (x_1 \cdots x_r) = J^{(n)}.$$

This follows from Corollary 1 and the fact that the analogous relation holds in a polynomial ring. Therefore we can write H as an increasing union

$$H = \bigcup_{n=1}^{\infty} E_n,$$

where E_n is the isomorphic image of $A/J^{(n)}$ in H. Furthermore, I claim that for each n, E_n is the set of elements of H annihilated by $J^{(n)}$. Indeed, we have only to observe that for each n, $k > 0$,

$$J^{(n+k)} : J^{(n)} = (x_1 \cdots x_r)^k$$

which follows from Corollary 1 and the analogous formula in a polynomial ring. Now since $J \subseteq I$, anything in H annihilated by I is annihilated by J. Hence

$$M = \mathrm{Hom}_A(A/I, A/J) = \mathrm{Hom}_A(A/I, E_1) = \mathrm{Hom}_A(A/I, H).$$

But by definition, H depends only on the radical of J [5], so we are done.

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