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l-PRIME IDEALS IN f-RINGS

BY

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Many results ([3], [4]) in the lattice-ordered (l. o.) ring $C(X)$ of real-valued continuous functions on a topological space X are due essentially to the fact that these rings are subdirect products of reals. A direct generalization of these may be attempted for l. o. rings which are sub-direct products of totally ordered (t. o.) rings, viz. f-rings [2]. The point evaluation technique of functions will have to be substituted with evaluation by each t. o. homomorphism. This has been the well-known core of thought in the study of f-rings. We adopt this procedure here in an effort to know how exactly some algebraic properties of $C(X)$ are or are not influenced by the topology on the set X. We confess that the propositions and proofs hereunder broadly follow the same lines as for $C(X)$ to be found in ([3], [4]). For this reason, we also do not dignify any proposition with a formal label such as Theorem, etc., but however merely number them to enable cross-reference purpose.

1. — A convex normal subgroup of a partially ordered group G characterizes the kernel of an order-preserving group homomorphism defined on G. If further G is l. o., the kernel of l. o. preserving group homomorphism is given by a convex normal subgroup which is also a sublattice. We shall call it an *l-ideal.* A normal subgroup S of a l. o. group G such that $|x| \leqq |y|$, $y \in S \Rightarrow x \in S$ will be an l-ideal, and conversely. For a detailed account on these, we may refer to [2]. A natural question in this connection is to ask what are the kernels in a l. o.-group which give t. o. homomorphic images (*). The following simple proposition answers it.

(*) The author thanks the referee for drawing his attention to ([1], [5]) in this connection.

1.1. — If S is any subgroup of a l. o.-group G, the following are equivalent :

(1) $x \wedge y = 0 \Rightarrow x \in S$ or $y \in S$;

(2) $x_1 \wedge x_2 \wedge \ldots \wedge x_n = 0 \Rightarrow$ some $x_i \in S$;

(3) $|x| \wedge |y| = 0 \Rightarrow x \in S$ or $y \in S$;

(4) $|x_1| \wedge |x_2| \wedge \ldots \wedge |x_n| = 0 \Rightarrow$ some $x_i \in S$.

(k') $(k = 1, 2, 3, 4)$: The condition (k) in which the LHS " $\ldots = 0$ " should now read " $\ldots \in S$ ".

It is obvious that $(1') \Leftrightarrow (2') \Rightarrow (2) \Rightarrow (1)$, and $(3') \Leftrightarrow (4') \Rightarrow (4) \Rightarrow (3)$. We will prove $(1) \Rightarrow (1')$. Let $s = x \wedge y \in S$. Since $(x - s) \wedge (y - s) = 0$, either $x - s$ or $y - s$ is in S. That is, either x or y belongs to S. $(3) \Rightarrow (3')$ likewise, once we see that

$$|x| \in S \quad \Rightarrow \quad x \in S,$$

because of the fact

$$x_+ \wedge x_- = 0, \qquad \text{where} \quad x_+ = x \vee 0 \quad \text{and} \quad x_- = (-x) \vee 0.$$

The same remark establishes also that $(1) \Leftrightarrow (3)$. The proof is complete.

1.2 (Šik [5]). — An l-ideal S of a l. o.-group G is such that the canonical order in G/S (i. e. the image of the positive cone being taken as the positive cone in the range) is a total order if and only if S satisfies anyone of the above-mentioned equivalent conditions.

P. Conrad [1] shows that such l-ideals S are also characterized by the *l-prime* property, viz. $A \cap B \subseteq S \Rightarrow A \subseteq S$ or $B \subseteq S$, for any two l-ideals A and B.

Hereafter, let G denote a l. o. Abelian group. G can be taken to be a subdirect product of t. o.-groups [2]. Thus an identity is valid in G if and only if it is true in each t. o.-homomorphic image of G. We list below some interesting and well-known facts. We give no proof because they can be obtained from ([1], [2], [5]) or easily checked.

1.3. — Let S be a subgroup of G. The smallest convex subgroup containing S is

$$\{ x \in G \mid s \leqq x \leqq t;\ s, t \in S \}$$

and the smallest l-ideal containing S is

$$\{ x \in G \,\big|\, |x| \leqq \Sigma |s_i|;\ s_i \in S \}.$$

1.4. — A convex subgroup containing an l-prime ideal is also l-prime.

1.5. — The set union of a chain of convex subgroups (l-ideals) is a convex subgroup (l-ideal). It may be the whole group.

The situation is even more delightful to observe :

1.6. — Any l-ideal is an intersection of l-prime ideals.

A heuristic proof of this will be : G being a subdirect product of t. o.-groups, $\{0\}$ is an intersection of l-prime ideals. But G is equationally definable, hence its homomorphic images (homomorphism with respect to all algebraic operations under considerations) also share this property. We skip the formal proof because it is both simple and can be found in [1].

1.7. — Just as maximal subgroups of a group need not exist, so is about maximal convex subgroups (maximal l-ideals) in l. o.-groups. The existence of these can however be established by Zorn's lemma process under the assumption of a strong order unit 1 in G (i. e. a positive element which does not belong to any proper l-ideal of G). By the preceding results, the maximal l-ideals are precisely the same as maximal l-prime ideals, whether or not a strong order unit exists. But the collection of maximal convex subgroups is different — a larger collection in general than the collection of all maximal l-ideals. For instance ([3], [4]), consider the group $\mathbf{R}\times\mathbf{R}$ with the positive cone consisting of all those (x, y) such that $x \geqq y \geqq 0$. $\mathbf{R}\times\mathbf{R}$ is then a l. o.-Abelian group with a strong order unit $(2, 1)$ in which $(0, \mathbf{R})$ is maximal convex subgroup but not an l-ideal.

1.8. — The group-theoretic sum of any two l-ideals of G is also an l-ideal [2]. Therefore, the same is true of l-prime ideals. We may see that the l-ideals themselves constitute a lattice — in fact, a distributive and complete sublattice of the lattice of all subgroups [2]. The position with respect to convex subgroups, on the otherhand, is not so happy. The group-theoretic sum of even a convex subgroup and an l-ideal need not be convex. For example [4], consider the l. o.-group $C(\mathbf{R})$ in which the subgroup

$$S = \{ fi \mid f \in C(\mathbf{R});\ i(x) = x \}$$

is convex and

$$T = \{ f \in C(\mathbf{R}) \mid f(x) = 0 \text{ for all } x > 0 \}$$

is an l-ideal; but their sum $S + T$ is not convex.

2. — We will now consider l. o.-rings (commutative with unity). Almost all the preceding discussions evidently carry over — with " subgroups " replaced by " ideals ". A typical exception is 1.6, the arguments in which show that this can also be valid provided the l. o.-ring is a subdirect product of t. o.-rings. Such rings are called f-rings [2].

The following result subsumes the same known for $C(X)$ [4].

2.1. — In an f-ring R with no nonzero nilpotent elements, all the three properties of an ideal I stated below are equivalent :

(1) $xy = 0 \Rightarrow x \in I$ or $y \in I$ (I is then called " pseudoprime " [4]);

(2) $x_1 x_2 \ldots x_n = 0 \Rightarrow$ some $x_i \in I$;

(3) I contains a prime ideal.

R can be taken to be a subdirect product of t. o. integral domains [2]. Now, $x_1 x_2 \ldots x_n = 0$ in any t. o. integral domain (and hence, in any f-ring) if and only if

$$|x_1| \wedge |x_2| \wedge \ldots \wedge |x_n| = 0.$$

This remark clinches the proof of (1) $\Leftrightarrow$ (2) in view of 1.1. In any ring, (2) $\Leftrightarrow$ (3) [4].

2.2. — In an f-ring with no nonzero nilpotent elements, an l-prime ideal is given by a convex ideal containing a prime ideal, and conversely.

In a like manner, we may ask the generalizability of other results in ([3], [4]) to f-rings, at least when there are no nonzero nilpotent elements. For instance, a convex ideal I in $C(X)$ is l-prime if and only if its (nil) radical

$$\sqrt{I} = \{ f \in C(X) \mid f^n \in I \text{ for some } n \in \mathbf{N} \}$$

is prime. We are unable to answer yes or no for such a proposition in the case of f-rings. However, there are rosy aspects in part at least. For the remainder of this section, let R denote an f-ring.

2.3. — Any minimal prime ideal P of R is an l-ideal.

To prove this, we use a known characterization of minimal prime ideals in any commutative ring with unity. A prime ideal P is minimal prime if and only if it satisfies the condition : $x \in P \Rightarrow$ there exists $y \notin P$ such that xy is nilpotent. Let $|z| \leq |x|$, $x \in P$. Then

$$0 \leq |(zy)^n| = |z|^n |y|^n \leq |x|^n |y|^n = |(xy)^n| = 0,$$

for some $n \in \mathbf{N}$ and some $y \notin P$. So, zy is nilpotent, implying that $z \in P$.

2.4. — If I is a convex ideal (l-ideal) of R, so is $\sqrt{I}$.

Suppose that I is a convex ideal and $0 \leq x \leq y$, $y \in \sqrt{I}$. Then $y^n \in I$ for some $n \in \mathbf{N}$; and from $0 \leq x^n \leq y^n$, it follows that $x^n \in I$. Thus $x \in \sqrt{I}$. The other part is similar.

2.5. — If I is a convex pseudoprime ideal of R, then $\sqrt{I}$ is convex prime.

$\sqrt{I}$ is certainly convex as seen above. Indeed, I being l-prime, $\sqrt{I}$ is also so. Suppose now that $xy \in \sqrt{I}$. Then $|x| . |y| \in \sqrt{I}$. But either

$|x| \leq |y|$ modulo I or $|y| \leq |x|$ modulo I. That is, either $|x^2|$ or $|y^2|$ belongs to $\sqrt{I}$. Thus either x or y is in $\sqrt{I}$.

2.6. — Let R be an f-ring with no nonzero nilpotent elements. Then a convex radical ideal I (i. e. $I = \sqrt{I}$) in R is an l-prime if and only if it is prime.

Use 2.2, 2.5 and the fact that $\sqrt{\sqrt{I}} = \sqrt{I}$.

2.7. — The proof of 2.5 is easier in the case of $C(X)$, where every prime ideal is an l-ideal. Taking P to be a minimal prime ideal contained in the convex pseudoprime ideal I, $C(X)/P$ is t. o. in which the convex ideals being intervals form a chain. Thus the prime divisors of I form a chain; so $\sqrt{I}$, being the intersection of them, is prime. A similar argument will work in R, using 2.3, provided we are able to show that every convex radical ideal is an intersection of convex prime ideals. We first settle that this is the case for radical ideals which are l-ideals, and then extend the result to convex radical ideals also.

2.8. — For an l-ideal I of R, $\sqrt{I}$ is equal to an intersection of prime l-ideals.

$\sqrt{I}$ is equal to the intersection of all minimal prime divisors of I, and the minimal prime ideals of R/I are l-ideals by 2.3. The correspondence between convex ideals (l-ideals) of R and R/I does the rest.

2.9. — A convex radical ideal I is also an l-ideal.

If $x = x_+ - x_- \in I$, then $x_+^2 = x x_+ \in I$. Thus $x_+ \in I$, proving that I is an l-ideal.

2.10. — Every l-prime ideal in R contains a minimal one.

If we show that the intersection of a decreasing chain of l-prime ideals is an l-prime ideal, Zorn's lemma clinches the proof. Let $\{K_\alpha\}$ be a decreasing chain of l-prime ideals. If $x \notin \cap K_\alpha$, $y \notin \cap K_\alpha$, then both x and y are outside some particular K_α. So, $x \wedge y \neq 0$. By 1.2, $\cap K_\alpha$ is an l-prime ideal.

2.11. — Let R be an f-ring with no nonzero nilpotent elements. Then, the minimal l-prime ideals and the minimal prime ideals are same in R.

Follows from 2.2, 2.3 and 2.10.

2.12. — The question which tries our wit now is a converse of 2.5 : If I is an l-ideal such that $\sqrt{I}$ is prime, is I l-prime? This is tantamount to asking : If I is an l-ideal such that $\sqrt{I}$ is minimal prime, does I coincide with $\sqrt{I}$? This is proved to be so in $C(X)$ [4], where of course notions of uniform convergence are employed. We do not know whether they are indispensable to prove this result.

2.13. — Every prime ideal in R is convex if and only if $|y| \leq |x|$ (or, $0 \leq y \leq x$) implies that some integral power of y is a multiple of x in R.

When every prime ideal is convex, by 2.9, every radical ideal will be an l-ideal. So, $|y| \leq |x| \Rightarrow y \in \sqrt{(x)}$. Conversely, let P be a prime ideal with $x \in P$ and $|y| \leq |x|$. We have $y^n = rx$ for some $n \in \mathbf{N}$, and some $r \in R$. Thus $y \in P$.

2.14. — Every maximal ideal in R is convex if and only if R is of bounded inversion, viz. $x \geq 1$ is a unit in R.

2.15. — In Canadian Bulletin (*Thesis Abstracts*, 1965, vol. 8, No. 5), Armstrong has announced that any f-ring can be embedded as a sub-f-ring in an f-ring with bounded inversion. We shall indicate the outline of a proof of this, since it does not seem to have been published so far. All the elements $x \geq 1$ in an f-ring will form a multiplicatively closed set. Consider then the formal quotients r/x, $x \geq 1$. Naturally this new set can be endowed with ring-lattice operations in a similar way of construction of rational numbers from integers. As Armstrong has observed, the embedding is also minimal.

3. — In this section, we will see how some of the results obtained for $C(X)$ [3] can be viewed in the general set up of f-rings. We recall that in the set $\mathfrak{M}$ of all maximal l-ideals of an f-ring a hull-kernel topology can be introduced : If $\{M_\alpha\}$ is a subset of $\mathfrak{M}$, cl $\{M_\alpha\}$ is defined as $\{M \in \mathfrak{M} \mid M \supseteq \cap M_\alpha\}$. This hull-kernel space $\mathfrak{M}$ is known to be compact Hausdorff. We have shown in [6] the following proposition.

3.1. — If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two disjoint closed sets in the hull-kernel space $\mathfrak{M}$ of an f-ring R, then for every a, $b \in R$ there exists $f \in R$ such that

$$f(M) \leq (a - 1)(M) \quad \text{whenever} \quad M \in \mathcal{F}_1$$

and

$$f(M) \geq (b + 1)(M) \quad \text{whenever} \quad M \in \mathcal{F}_2,$$

where $r(M)$, $r \in R$ denotes the homomorphic image of r in R/M.

Note : The general stipulation in [6] of l-semisimplicity (i. e. intersection of all maximal l-ideals is zero) of the f-ring is not of course needed to prove 3.1.

3.2. — With $\mathcal{F}_1$ and $\mathcal{F}_2$ as in 3.1, there exist g, $h \in R$ such that

$$|g| \leq 1, \qquad 0 \leq h \leq 1,$$
$$g(M) = -1 \quad \text{whenever } M \in \mathcal{F}_1,$$
$$g(M) = 1 \quad \text{whenever } M \in \mathcal{F}_2,$$

and

$$h(M) = 0 \quad \text{whenever } M \in \mathcal{F}_1 \qquad \text{and} \qquad h(M) = 1 \quad \text{whenever } M \in \mathcal{F}_2.$$

Choose $a = b = 0$ in 3.1. Put $g = (-1) \vee (f \wedge 1)$.
Choose $a = 1$ and $b = 0$ in 3.1. Put $h = 0 \vee (f \wedge 1)$.

Notation : If $f \in R$,

$$h(f) = \{ M \in \mathfrak{M} \mid f \in M \};$$
$$h'(f) = \{ M \in \mathfrak{M} \mid f \notin M \};$$
$$\operatorname{pos} f = \{ M \in \mathfrak{M} \mid f(M) > 0 \} = h'(f_+);$$
$$\operatorname{neg} f = \{ M \in \mathfrak{M} \mid f(M) < 0 \} = h'(f_-).$$

3.3. — In the hull-kernel space of an f-ring R, define two subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ to be *completely separated* if there exist $x, y \in R$ such that

$$\mathcal{F}_1 \subseteq h(x), \quad \mathcal{F}_2 \subseteq h(y) \qquad \text{and} \qquad h(x) \cap h(y) = \emptyset.$$

Clearly two subsets are completely separated if and only if their closures are.

3.4. — Let R be an f-ring. Two subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ are completely separated if and only if there exists $h \in R$, $0 \leqq h \leqq 1$ such that

$$h(M) = 0 \quad \text{whenever } M \in \mathcal{F}_1 \qquad \text{and} \qquad h(M) = 1 \quad \text{whenever } M \in \mathcal{F}_2.$$

By definition, there exist $x, y \in R$ such that

$$\mathcal{F}_1 \subseteq h(x), \quad \mathcal{F}_2 \subseteq h(x) \qquad \text{and} \qquad h(x) \cap h(y) = \emptyset.$$

From 3.2, we obtain $h \in R$ as required. Conversely, put

$$x = (3h - 1) \vee 0 \qquad \text{and} \qquad y = (3h - 2) \wedge 0.$$

Then

$$h(x) = \{ M \in \mathfrak{M} \mid 3h(M) \leqq 1 \} \qquad \text{and} \qquad h(y) = \{ M \in \mathfrak{M} \mid 3h(M) \geqq 2 \}.$$

Obviously, $\mathcal{F}_1 \subseteq h(x)$, $\mathcal{F}_2 \subseteq h(y)$ and $h(x) \cap h(y) = \emptyset$.

3.5. — Two subsets of the hull-kernel space of an f-ring are completely separated if and only if their closures are disjoint.

The following is a direct generalization of a result [3] of $C(X)$ to f-rings, the sketch of the proof of which in comparison with [3] reveals the wholesome trend of this paper.

3.6. — Let I be an ideal of an f-ring R, and let $f \in R$. If $(f(I), |f|(I))$ is a principal ideal (perhaps improper) in R/I, then there exists $x \in l(I)$, where $l(I)$ is the l-ideal generated by I, such that $h(x) \cap \operatorname{pos} f$ and and $h(x) \cap \operatorname{neg} f$ are completely separated int the hull-kernel space of R.

We will have $d, g, h, s, t \in R$ such that

$$f = gd \pmod{I}, \qquad f = hd \pmod{I} \qquad \text{and} \qquad sf + t|f| = d \pmod{I}.$$

Put $x = |f - gd| + \big||f| - hd| + |sf + t|f| - d\big|$. Then $x \in l(I)$. Surely the above congruences hold modulo any $M \in l(x)$ and we can see that $(sg + th)\,d(M) = d(M)$ whenever $M \in l(x)$. Now, if $M \in h(x) \cap h'(f)$, then $d \notin M$. This yields, since R/M is an integral domain, $(sg + th)(M) = 1$ for any $M \in h(x) \cap h'(f)$. Note that $h'(f) = h'(f_+) \cup h'(f_-)$. We next obtain $g(M) = h(M)$ for any $M \in h(x) \cap h'(f_+)$ and $g(M) = -h(M)$ for any $M \in h(x) \cap h'(f_-)$. Consequently,

$$h(x) \cap \operatorname{pos} f \subseteq h(g - h) \cap h(sg + th - 1) = h(|g - h| + |sg + th - 1|)$$

and

$$h(x) \cap \operatorname{neg} f \subseteq h(g + h) \cap h(sg + th - 1) = h(|g + h| + |sg + th - 1|).$$

If a maximal l-ideal contains $g - h$, $g + h$ and $sg + th - 1$, it will contain 2 and hence 1. Therefore, the closed sets on the right side of the above two inequalities are disjoint. The proof is complete.

3.7. — In an l-semisimple f-ring R, the following are equivalent for any $f \in R$:

(1) pos f and neg f are completely separated;

(2) There exists $k \in R$ such that $f = k|f|$ (obviously, if and only if $|f| = kf$ for the same $k \in R$; k may be so chosen that $|k| \leq 1$);

(3) $(f, |f|)$ is a principal ideal (perhaps improper) in R.

(1) $\Rightarrow$ (2). There exists $h \in R$ such that

$$h(M) = 0 \quad \text{whenever } M \in \operatorname{pos} f$$

and

$$h(M) = 1 \quad \text{whenever } M \in \operatorname{neg} f.$$

Put $k = 1 - 2h$. Then

$$k(M) = 1 \qquad \text{whenever } M \in \operatorname{pos} f$$

and

$$k(M) = -1 \quad \text{whenever } M \in \operatorname{neg} f.$$

$f(M) = k|f|(M)$ can be checked for any $M \in \mathfrak{M}$. (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Particular case of 3.6, choosing $I = \{0\}$.

4. — We provide here a few more concepts and results for f-rings, the origin of which can again be traced back to $C(X)$ [3]. From now on, let R denote an l-semisimple f-ring.

4.1. — Let M be any maximal l-ideal of R. If $x \in R$, the following are equivalent :

(1) There exists $e \in M$ such that $xe = x$;

(2) $h(x)$ is a neighbourhood of M in the hull-kernel space of R;

(3) There exists $y \notin M$ such that $xy = 0$.

Let $xe = x$, $e \in M$. Since M is prime, $M \in h'(e - 1) \subseteq h(x)$, which is tantamount to saying that $h(x)$ is a neighbourhood of M in the hull-kernel space of R. If $M \in h'(y) \subseteq h(x)$, then xy belongs to every maximal l-ideal, and hence $xy = 0$. Suppose now that $xy = 0$ for some $y \notin M$. Then $z + e = 1$, where $z \in l(y)$ and $e \in M$. And, $xy = 0$ implies that $xz = 0$. Thus $xe = x$.

4.2. — For any maximal l-ideal M of R, define $\pi(M)$ to be the subset of R consisting of all those $x \in R$ satisfying the equivalent conditions mentioned in 4.1.

4.3. — Define an ideal I of R to be h-ideal if $x \in I$ and $h(x) = h(y)$ implies that $y \in I$.

4.4. — An ideal I of R is a h-ideal if and only if from $x \in I$ it follows that the intersection of all maximal l-ideals containing x is included in I.

Follows because $h(x) \subseteq h(y) \Leftrightarrow h(y) = h(xy)$.

4.5. — Any h-ideal is an l-ideal and also a radical ideal.

4.6. — $\pi(M)$ is a h-ideal contained in M.

4.7. — Every prime l-ideal lies between $\pi(M)$ and M for some unique maximal l-ideal M of R.

The l-semisimplicity guarantees the nonexistence of nonzero nilpotent elements. Thus a prime l-ideal is also l-prime; so, prime l-ideals containing a prime l-ideal form a chain. The rest is a routine check.

4.8. — The only maximal l-ideal containing $\pi(M)$ is M.

Since $\mathcal{M}$ is Hausforff [6], we can find for every two distinct maximal l-ideals M_1 and M_2, $x \notin M_1$ and $y \notin M_2$ such that $xy = 0$. Obviously, $y \in \pi(M_1) \sim M_2$.

4.9. — If I is any ideal of R, $I = \cap I + \pi(M_\alpha)$, where α runs through the index set $\mathcal{M}$ of all maximal l-ideals.

Let $x = i_\alpha + x_\alpha$, where $i_\alpha \in I$ and $x_\alpha \in \pi(M_\alpha)$. Then there exist $y_\alpha \notin M_\alpha$ such that $x_\alpha y_\alpha = 0$. Since there is no l-ideal containing all y_α's, we have

$$z_1 + z_2 + \ldots + z_n = 1, \qquad \text{where } z_i \in l(y_i) \qquad (i = 1, 2, \ldots, n).$$

But $x_i y_i = 0$ implies that $x_i z_i = 0$. Thus $xz_i \in I$, which gives that $x \in I$. The other way round is palpably clear.

4.10. — Let I be a h-ideal containing $\pi(M)$ for some maximal l-ideal M of R. If $(f(I)), |f|(I)$ is a principal ideal in R/I for every $f \in R$, then I is prime.

There exist $x \in I$, h_1, $h_2 \in R$ such that

$$h(x) \cap h'(f_+) \subseteq h(h_1), \quad h(x) \cap h'(f_-) \subseteq h(h_2) \quad \text{and} \quad h(h_1) \cap h(h_2) = \emptyset.$$

Let $M \notin h(h_1)$. So, $\pi(M)$ and h_1 are not together contained in any maximal l-ideal. We then have $1 \leq |h| + |r|.|h_1|$ for some $h \in \pi(M)$ and $r \in R$. This implies that $h(h) \cap h(x) \cap h'(f_+) = \emptyset$. That is,

$$h(|h| + |x|) = h(h) \cap h(x) \subseteq h(f_+).$$

Thus $f_+ \in I$. It follows that I is prime, since R/I becomes totally ordered.

4.11. — If I is any h-ideal containing $\pi(M)$ for some maximal ideal M of an f-ring with bounded inversion, the following are equivalent in R/I :

(1) Every ideal is convex;

(2) The ideals form a chain;

(3) The principal ideals form a chain, i. e., R/I is a valuation ring;

(4) Every finitely generated ideal is principal.

We leave the proof as it is a repetition of the same for $C(X)$ [3] under the hypothesis specified.

4.12. — In any l. o.-ring, the following are equivalent :

(1) Every ideal is convex;

(2) Every principal ideal is convex;

(3) Every ideal is an l-ideal;

(4) Every principal ideal is an l-ideal.

The proof is obvious and hence omitted.

4.13. — The following are equivalent in R :

(1) For every $f \in R$, there exists $k \in R$ such that $f = k|f|$;

(2) $\pi(M)$ is prime for every maximal l-ideal M of R;

(3) The prime l-ideals contained in any maximal l-ideal M of R form a chain;

(4) Every ideal is an intersection of pseudoprime ideals;

(5) Every principal ideal is an intersection of pseudoprime ideals.

(1) $\Rightarrow$ (2) : Since $(f(\pi(M)), |f|(\pi(M))$ is a principal ideal in $R/\pi(M)$, the implication follows from 4.10. (2) $\Rightarrow$ (3) : By 4.7 and 4.8, the prime l-ideals contained in M are exactly those containing $\pi(M)$. Use

the fact $\pi(M)$ is l-prime. (3) $\Rightarrow$ (2) : Follows from 2.8 and the above remark. (2) $\Rightarrow$ (4) : Follows from 4.9. (4) $\Rightarrow$ (5) : Trivial. (5) $\Rightarrow$ (1) : Follows because

$$f = f_+ - f_-, \qquad f = f_+ + f_- \qquad \text{and} \qquad f_+ f_- = 0.$$

4.14. — In an f-ring with bounded inversion, the following are equivalent :

(1) Every ideal is convex;
(2) For every f, g, $\in R$, $(f.g) = (|f| + |g|)$;
(3) Every finitely generated ideal is principal.

We omit the proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) as it is identical to that obtained for $C(X)$ [3]. (3) $\Rightarrow$ (1) : The property (3) of R is shared by its homomorphic images. Use 4.9 and 4.11.

4.15. — In 4.14, the property of bounded inversion in R cannot be omitted. The counter example needed is the t. o.-ring of integers.

4.16. — If every pseudoprime ideal is convex, the conditions in 4.13 and 4.14 are all equivalent with each other. This is not decided however in general cases.

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