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ON $N$-HIGH SUBGROUPS OF ABELIAN GROUPS

BY

JOHN M. IRWIN AND KHALID BENABDALLAH.

1. Introduction.

This paper is based on a curious property of $N$-high subgroups when $N$ is a subgroup of $G^1$ the subgroup of elements of infinite height of a group $G$. Let $G$ be a group, $N$ a subgroup of $G$, we say that a subgroup $H$ of $G$ is $N$-high if $H$ is maximal with respect to the property $H \cap N = \varnothing$. Our first result (Theorem 2.4) is that given a group $G$ and $N$ a subgroup of $G^1$, then $G = \langle H, K \rangle$ whenever $H$ is an $N$-high subgroup of $G$ and $K$ is a pure subgroup of $G$ containing $N$. A close look at the proof of this result shows that the assumption that $K$ is pure can be replaced by the weaker one that $N \subseteq K^1$. An immediate consequence is the classical theorem that divisible subgroups of a group are absolute summands of the group.

$N$-high subgroups where $N \subseteq G^1$ were first introduced and studied by IRWIN and WALKER in [3]. These authors proved that $N$-high subgroups are pure and that the factor groups they induce are divisible. It turns out (Theorem 2.5) that $H$ is an $N$-high subgroup of a group $G$, where $N \subseteq G^1$ if and only if $H$ is pure, $H \cap N = \varnothing$, $G = \langle H, K \rangle$ for all $K$ pure containing $N$ and $G/H$ is divisible. We use this property of $N$-high subgroups where $N \subseteq G^1$, to generalize and simplify many results in [4]. In particular, we obtain a criterion for a pure subgroup of a group $G$ containing $N \subseteq G^1$ to be a summand of $G$ (Theorem 3.1).

In the fourth part, we define the concept of quasi-essential and strongly quasi-essential subsocles of a $p$-group (Definition 4.1) and proceed to characterize those quasi-essential subsocles which are also centers of purity (Theorem 4.4) and those which are strongly quasi-essential (Theorem 4.8).

We use standard notation from [1]. The symbol $\mathbb{Z}^+$ denotes the set of positive integers. If $G$ is a $p$-group, $R$ a subgroup of $G$ and $g \in R$,
the symbol $h_n(g)$ denotes the height of the element $g$ in the subgroup $R$. All groups considered are Abelian.

2. A characterization of $N$-high subgroups of a group $G$, with $N \subseteq G'$.

We need the following lemmas.

**Lemma 2.1.** — Let $G$ be a $p$-group, $N$ subgroup of $G'$ and $K$ a pure subgroup of $G$ containing $N$. Then for any $N$-high subgroup $H$ of $G$, $G = \langle H, K \rangle$.

**Proof.** — Clearly $\langle H, K \rangle \supset H[p] \oplus N[p] = G[p]$. By induction suppose $\langle H, K \rangle \supset G[p^n]$. Let $g \in G$, $o(g) = p^{n+1}$, if $g \notin H$, $\langle g, H \rangle \cap N \neq o$ thus there exists $h \in H$, $g' \in N$, and $m < n + 1$, such that $p^m g + h = g' \neq o$

since $K$ is pure, $g' \in K'$, thus there exists $k \in K$, such that $g' = p^m k$, or $h = p^m (g - k)$. If $h \neq o$, by purity of $H$ (see [3], theorem 5) there exists $h' \in H$, such that $p^m h' = h$, therefore $p^m (g - k - h') = o$. This implies $g - k - h' \in \langle H, K \rangle$, and $g \in \langle H, K \rangle$, thus $\langle H, K \rangle \supset G[p^{n+1}]$. By induction

$G = \langle H, K \rangle$.

**Lemma 2.2.** — Let $G$ be a torsion group, $N$ a subgroup of $G'$ and $K$ a pure subgroup of $G$ containing $N$. Then for any $N$-high subgroup $H$ of $G$, $G = \langle H, K \rangle$.

**Proof.** — Let $G = \sum G[p]$, $H = \sum H[p]$, $K = \sum K[p]$ and $N = \sum N[p]$ then for each prime $p$, $H[p]$ is $N[p]$-high in $G[p]$ (see [2], lemma 11) and since $K[p]$ is pure containing $N[p]$, lemma 2.1 holds and $G[p] = \langle H[p], K[p] \rangle$. Therefore

$G = \sum \langle H[p], K[p] \rangle = \langle H, K \rangle$.

**Lemma 2.3.** — Let $G$ be a group, $N$ a subgroup of $G'$ and $H$ an $N$-high subgroup of $G$. Then $H_i$ is $N_i$-high in $G_i$.

**Proof.** — Clearly $H_i \cap N_i = o$, let $g \in G_i$, $o(g) = b, g \notin H$ then $\langle g, H_i \rangle \cap N_i \neq o$, thus there exists $h \in H$, $n \in N$, and a positive integer $a$ such that $ag + h = n \neq o$. Clearly $a \neq b$. Now $b ag + bh = bn$, thus $bh = bn = o$, and $h \in H_i$, $n \in N_i$, therefore $\langle g, H_i \rangle \cap N_i \neq o$. This implies that $H_i$ is $N_i$-high in $G_i$.

**Theorem 2.4.** — Let $G$ be a group, $N$ a subgroup of $G'$ and $K$ a pure subgroup of $G$ containing $N$. Then for any $N$-high subgroup $H$ of $G$,

$G = \langle H, K \rangle$. 
Proof. — Suppose \( g \in G, g \not\in H \), then \( \langle g, H \rangle \cap N \neq \emptyset \), thus there exists \( h \in H, n \in N \) and a positive integer \( a \), such that
\[
ag + h = n \\
\neq o.
\]
By an argument similar to the one used in lemma 2.1, there exists \( h' \in H \) and \( k \in K \) such that \( a(g + h' - k) = o \). Thus \( g + h' - k \in G \). But, by lemmas 2.3 and 2.2, we know that \( G = \langle H, K \rangle \), therefore \( g \in \langle H, K \rangle \) and \( G = \langle H, K \rangle \).

A classical theorem follows immediately from theorem 2.4.

Corollary. — If \( D \) is a divisible subgroup of a group \( G \), then \( D \) is an absolute summand of \( G \).

Proof. — Let \( D = N \) in theorem 2.4, since \( D \) is divisible it is pure in \( G \). Thus \( G = D \oplus H \), for any \( D \)-high subgroup \( H \) of \( G \).

Theorem 2.5. — Let \( G \) be a group, \( N \) a subgroup of \( G \) and \( H \) a subgroup of \( G \) disjoint from \( N \). Then \( H \) is \( N \)-high in \( G \) if and only if \( H \) is pure, \( G/H \) is divisible and \( G = \langle H, K \rangle \) for any pure subgroup \( K \) of \( G \) containing \( N \).

Proof. — The necessity follows from theorem 2.4. Suppose then that \( H \) satisfies the conditions of the theorem. Since \( H \cap N = \emptyset \) there exists an \( N \)-high subgroup \( H' \) of \( G \) containing \( H \). Since \( H' \) is pure in \( G \), \( H'/H \) is pure in \( G/H \) which is divisible, therefore \( H'/H \) is divisible and \( G/H = (H'/H) \oplus (R/H) \) where \( R \) can be chosen to contain \( N \). Since \( H \) is pure in \( G \) and \( R/H \) is pure in \( G/H \), \( R \) is pure in \( G \), and since \( R \supseteq N \),
\[
R = \langle R, H \rangle = G.
\]
Therefore
\[
H = R \cap H' = G \cap H' = H'
\]
and \( H \) is \( N \)-high in \( G \).

3. Some applications.

We first obtain a criterion for pure subgroups of a group \( G \) to be summands of \( G \).

Theorem 3.1. — Let \( G \) be a group, \( K \) a pure subgroup of \( G \) containing a subgroup \( N \) of \( G \). Then \( K \) is a direct summand of \( G \) if and only if there exists an \( N \)-high subgroup \( H \) of \( G \) such that \( H \cap K \) is a direct summand of \( H \).

Proof. — Suppose \( G = K \oplus L \), let \( M \) be any \( N \)-high subgroup of \( K \), then it is easy to see that \( H = L \oplus M \) is \( N \)-high in \( G \) and \( H \cap K = M \) is a summand of \( H \).
Suppose now that there exists an $N$-high subgroup $H$ of $G$ such that $H = (H \cap K) \oplus R$, by theorem 2.4:

$$G = \langle H, K \rangle = \langle (H \cap K) \oplus R, K \rangle = \langle R, K \rangle$$

and since $R \cap K = o$, $G = R \oplus K$.

The following corollary contains theorem 2 in [4].

**Corollary.** — A reduced group $G$ splits over its maximal torsion subgroup $G_t$, if and only if some $N$-high subgroup of $G$ splits, where $N \subseteq G' \cap G_t$.

**Proof.** — If $G$ is reduced and $G = G_t \oplus L$ then $G_t \subset G_t$ and since $G_t$ is pure theorem 3.1 implies there exists an $N$-high subgroup such that $H \cap G_t = H_t$ is a summand of $H$. Now if $H$ is $N$-high and $H = H_t \oplus L$ since $N \subseteq G_t \cap G'$ by theorem 3.1, $G_t$ is a summand of $G$.

For what follows we need the following lemmas.

**Lemma 3.2.** — Let $G$ be a group, $H$ a subgroup of $G$ then if $K/H_t$ is an $(H/H_t)$-high subgroup of $G/H_t$, then $K$ is pure in $G$ and $K \subseteq G_t$.

**Proof.** — Suppose $ng \in K$ where $g \in G$. Let $n \neq h = ag + k \in \langle K, g \rangle \cap H$ then $nag + nk = nh \in K \cap H = H_t$, therefore $h \in H_t$, thus $\langle K, g \rangle \cap H = H_t$ which implies $\langle K, g \rangle = K$, therefore $g \in K$ and thus $K$ is pure in $G$. Now if $g \in G$, then letting $n = o(g)$ in the above argument we see that $K \supseteq G_t$.

**Lemma 3.3.** — Let $G$ be a group, $N$ a subgroup of $G$, $H$ an $N$-high subgroup of $G$ and $K$ a pure subgroup of $G$ containing $\langle N, G_t \rangle$ and such that $K \cap H = H_t$. Then for any $N$-high subgroup $H'$ we have $K \cap H' = H'_t$.

**Proof.** — Such $K$ do exist (lemma 3.2). Clearly $K \cap H' \supseteq H_t$. Let $h' \in K \cap H'$ and suppose $h' \notin H$ then there exists $h \in H, g \in N$ and a positive integer $a$ such that $ah' + h = g \neq o$, thus $h \in K \cap H = H_t$, let $b = o(h)$, then

$$bah' = bah' + bh = bg \in H' \cap N = o$$

thus $bah' = o$ and consequently $h' \in H_t$. Therefore $K \cap H' = H'_t$.

**Corollary 1 ([4], lemma).** — If $G$ is a group, $N$ a subgroup of $G$ and $H$ is an $N$-high subgroup of $G$, then $H/H_t$ is a summand of $G/H_t$.

**Proof.** — Let $K/H_t$ be $H/H_t$-high in $G/H_t$. Choose $K \supset N$. Then, since $K$ is pure in $G$ (lemma 3.2), it follows from theorem 2.4 that $G = \langle K, H \rangle$. Therefore $G/H_t = (H/H_t) \oplus (K/H_t)$.

**Corollary 2 ([4], theorem 4).** — Let $H$ and $H'$ be two $N$-high subgroups of a reduced group $G$ where $N$ is a subgroup of $G'$. Then $H/H_t \sim H'/H_t'$ and $G/H_t \sim G/H_t'$. 
Proof. — From corollary 1, $G/H_t = (H/H_t) \oplus (K/H_t)$. From lemma 3.3, $K \cap H' = H_t$, therefore $G/H_t = (H'/H_t) \oplus (K/H_t)$. The result follows from this and the fact that $G/H \simeq G/H'$ (see [3]).

Corollary 3 ([4], theorem 1). — Let $G$ be a reduced group, $N$ a subgroup of $G$ and $H$ an $N$-high subgroup of $G$ then if $H = H_t \oplus L$, we have $G = K \oplus L$ where $K/G_t$ is the divisible part of $G/G_t$.

Proof. — $G/H_t = H/H_t \oplus K/H_t$ from corollary 1.

Now $K/H_t$ is divisible since $K/H_t \simeq G/H_t$ and $H/H_t \simeq L$ is reduced. Thus $K/G_t$ is the divisible part of $G/H_t$. Now $K \cap H = H_t$ implies $K \cap L = 0$ and $\langle K, H \rangle = G$ implies $\langle K, L \rangle = G$, therefore $G = K \oplus L$.


It is natural to ask, what kind of subgroups of a group $G$ have properties similar to subgroups of $G^1$. We consider first $p$-groups. It is trivial to verify that two subgroups of a $p$-group are disjoint if and only if their socles are. Thus it suffices to consider subgroups of the socle of a $p$-group which we will call subsocles.

Definition 4.1. — Let $G$ be a $p$-group, a subsocle $S$ of $G$ is said to be quasi-essential (q. e.) if $G = \langle H, K \rangle$ whenever $H$ is an $S$-high subgroup of $G$ and $K$ a pure subgroup of $G$ containing $S$. $S$ is said to be strongly quasi-essential (s. q. e.) if every subgroup of $S$ is q. e.

We now proceed to characterize those quasi-essential subsocles of $G$ a $p$-group $G$ which are also centers of purity (see [7] and [6]).

Theorem 4.2. — Let $G$ be a $p$-group, $S$ a center of purity, $S \subseteq G[p]$. If $S$ is not quasi-essential in $G$ then there exists $n \in Z$, $g \in G[p]$, $g \notin S$ and $s \in S$ such that

$$h(s) = h(g) = n \quad \text{and} \quad h(s + g) = n + 1.$$  

Proof. — Set $P_n = (p^n G)[p]$, $P_x = G'[p]$ and $P_{x+1} = p$ then it is known (see [6]) that $S$ is a center of purity if and only if

$$P_n \supset S \supset P_{n+2} \quad \text{for some} \quad n \in \{1, 2, \ldots, \infty, \infty + 1\}.$$  

From lemma 2.1, we see that if $n = \infty$, i.e. $S \subset G^1$, $S$ is q. e. Thus if $S$ is not q. e. there exists $n \in Z^+$, such that

$$P_n \supset S \supset P_{n+2}.$$  

Also $S$ is not q. e. implies that there exist a pure subgroup $K$ of $G$ containing $S$ and an $S$-high subgroup $H$, of $G$ such that $\langle H, K \rangle \not= G$. Let
\[ \langle H, K \rangle = R. \] Since \( R \supset G[p] \) and \( R \not\subset G \), \( R \) is not pure in \( G \) (see [5], lemma 12). Therefore there exists an element \( x \in R[p] \) such that \( h(x) \gg h_R(x) \). \( H \) and \( K \) being both pure in \( G \) implies that \( x \in H \) and \( x \notin K \). Therefore there exists \( g \in H[p] \) and \( s \in S \) such that \( x = g + s \), \( g \not= 0 \not= s \). It is easy to verify that
\[
h_R(g) = h_H(g) = h(g) \quad \text{and} \quad h_R(s) = h_K(s) = h(s),
\]
therefore
\[
h(g) = h(s) \leq h_R(g + s) < h(g + s).
\]
Now \( s \in S \) implies \( h(s) \geq p^n \), \( g \notin S \) implies \( h(g) \leq n + 1 \) and since \( S \supset P_{n+2} \) we conclude that \( h(s) = h(g) = n \) and \( h(g + s) = n + 1 \) as stated.

**Corollary 1.** — Let \( G \) be a \( p \)-group, \( S \) a subsocle of \( G \) such that \( P_n \supset S \supset P_{n+1} \) then \( S \) is quasi-essential.

**Proof.** — \( S \) is a center of purity, thus theorem 4.2 applies and clearly there exists no pair \( g \in G[p] \), \( g \notin S \) and \( s \in S \) that satisfy the conditions of the theorem. Thus \( S \) is q.e.

**Corollary 2.** — Let \( G \) be a \( p \)-group, \( S \) subsocle of \( G \) such that \( S \) supports an absolute summand \( A \) of \( G \) then \( S \) is quasi-essential.

**Proof.** — \( S \) is a center of purity, thus theorem 4.2 holds and again if \( g \notin S \) and \( s \in S \) and \( h(g) = h(s) \) then, since \( g \) can be embedded in a complementary summand of \( A \) in \( G \), \( h(g + s) = h(g) = h(s) \). Therefore the condition of the theorem cannot be satisfied and \( S \) must be q.e.

**Corollary 3.** — Let \( G \) be a \( p \)-group, \( K \) a pure subgroup of \( G \) containing \( P_n \) for some \( n \in \mathbb{Z}^+ \), then \( K \) is a direct summand containing \( p^n G \).

**Proof.** — Since \( P_n \) is q.e., \( G = \langle K, H \rangle \), where \( H \) is a \( P_n \)-high subgroup of \( G \). Now \( H \) is bounded, in fact \( p^n H = 0 \), and \( G/H \cong H/H \cap K \), therefore \( K \) is a direct summand of \( G \) and \( p^n G \subset K \).

In fact, it turns out that the conditions on \( S \) in corollary 1 and 2 as well as the condition that \( S \) be quasi-essential and a center of purity, are equivalent provided \( S \notin G' \). To prove this, we need the following lemma.

**Lemma 4.3.** — Let \( G \) be a \( p \)-group, \( H \) a pure subgroup of \( G \) such that \( G/H \) is pure-complete. Let \( S \) be a subsocle of \( G \) such that \( H[p] \subset S \). Then \( S \) supports a pure subgroup \( K \) of \( G \) containing \( H \).

**Proof.** — Since \( G/H \) is a pure-complete group, by definition, every subsocle of \( G/H \) supports a pure subgroup of \( G/H \). Now \( \langle S, H \rangle/H \) is
clearly a subsocle of $G/H$, therefore there exists $K/H$ a pure subgroup of $G/H$ such that

$$(K/H)[p] = \langle S, H \rangle/H.$$ 

Since $H$ is pure in $G$, $K$ is pure in $G$ (see [5], lemma 2). Clearly $K[p] > S$, let $k \in K[p]$, then $k + H \in (K/H)[p] = \langle S, H \rangle/H$, thus there exists $s \in S$ and $h \in H$ such that $k - s = h$, but $ph = p(k - s) = 0$, and since $S \supset H[p]$, we conclude that $k \in S$. Therefore $K[p] = S$.

**Corollary.** Let $G$ be a $p$-group, $S$ a subsocle containing $P^n$ (see theorem 4.2) for some $n \in \mathbb{Z}^+$, then $S$ supports a pure subgroup of $G$ containing $p^n G$.

**Proof.** Let $G_n$ be as in [1], p. 98. Then $G_n$ is pure in $G$, $G_n[p] = P_n$ and $G/G_n$ is bounded and therefore pure complete. Thus lemma 4.3 holds, and $S$ supports a pure subgroup of $G$ containing $G_n$.

**Theorem 4.4.** Let $G$ be a $p$-group, $S$ subsocle of $G$ not contained in $G'$ then the following are equivalent:

(i) $S$ is both a center of purity and a quasi-essential subsocle of $G$;

(ii) $S$ supports an absolute direct summand of $G$;

(iii) There exists $n \in \mathbb{Z}^+$ such that $P_n \supset S \supset P_{n+1}$.

**Proof.** (i) implies (ii). Suppose $S$ satisfies (i), then since $S$ is a center of purity $S \supset P_m$ for some $m \in \mathbb{Z}^+$ and by the corollary to lemma 4.3, $S$ supports a pure subgroup $K$ of $G$. Since $S$ is also quasi-essential $K$ is an absolute summand of $G$.

(ii) implies (i). Suppose $S$ supports an absolute summand $K$. Then $S$ is clearly a center of purity and by corollary 2 to theorem 4.2, $S$ is q. e.

(i) implies (iii). Suppose $S$ satisfies (i), then $S$ supports an absolute summand $K$ of $G$. Since $S$ is a center of purity, we know there exists $m \in \mathbb{Z}^+ \ni P_m \supset S \supset P_{m+2}$. Suppose $P_{m+1} \notin S$, we will show, by contradiction, that $S \supset P_{m+1}$. Indeed, suppose not, i.e. there is $x \in G[p]$ such that $x \notin S$ and $h(x) = m + 1$. Now $P_{m+1} \notin S$ implies there exists $s \in S$, $h(s) = m$, otherwise $P_m \subset S \subset P_{m+1}$, and we would be done. Let $y = x - s$ then $h(y) = m$, $y \notin S$ and $h(y + s) = m + 1$.

Since $y \notin S$ there is an $S$-high subgroup $H$ of $G$ such that $y \in H$. But, $G = K \oplus H$ and $h(y) = h(s)$ imply that $h(y + s) = h(y) = h(s)$ which is a contradiction. Therefore $S \supset P_{m+1}$.

(iii) implies (i). If $S$ satisfies (iii) it is a center of purity (see theorem 4.2, proof) and by corollary 1 to theorem 4.2 it is also q. e.

At this point we have completely characterized those quasi-essential subsocles of a $p$-group which are also centers of purity. An immediate consequence is the following.
Let \( G \) be a \( p \)-group, \( A \) a pure subgroup of \( G \), then \( A \) is an absolute direct summand of \( G \) if and only if \( A \) is divisible or \( P_n \triangleright A[p] \triangleright P_{n+1} \) for some \( n \in \mathbb{Z}^+ \).

The strongly quasi-essential subsocles have also a simple characterization which can be obtained from the previous result. We need the following lemmas.

**Lemma 4.5.** Let \( G \) be a group; \( A, B, C \) three subgroups of \( G \) then
\[
\langle A \cap B, C \cap B \rangle = \langle A \cap B, C \cap B \rangle = \langle A, C \cap B \rangle \leq B.
\]

**Lemma 4.6.** Let \( G \) be a group, \( N \) a subgroup of \( M \) a subgroup of \( G \), if a subgroup \( H \) is \( N \)-high in \( G \) then \( H \cap M \) is \( N \)-high in \( M \). Conversely if \( H' \) is an \( N \)-high subgroup of \( M \) then \( H' = H \cap M \) for any \( N \)-high subgroup \( H \) of \( G \) containing \( H' \).

**Proof.** Let \( H \) be \( N \)-high in \( G \) then for all \( x \notin H \), we have
\[
\langle H, x \rangle \cap N = o.
\]
Suppose \( m \in M, m \notin H \), then
\[
\langle H \cap M, m \rangle \cap N = \langle H, m \rangle \cap M \cap N = \langle H, m \rangle \cap N = o.
\]
and since \( (H \cap M) \cap N = o \), \( H \cap M \) is \( N \)-high in \( M \).

Let \( H' \) be an \( N \)-high subgroup of \( M \), and let \( H \) be any \( N \)-high subgroup of \( G \) then \( H \cap M \triangleright H' \) and \( (H \cap M) \cap N = o \). The maximality of \( H' \) implies \( H \cap M = H' \).

**Lemma 4.7.** Let \( G \) be a \( p \)-group, \( S \) a quasi-essential subsocle of \( G \). Let \( K \) be a pure subgroup of \( G \) containing \( S \). Then \( S \) is quasi-essential in \( K \).

**Proof.** Let \( M \) be a pure subgroup of \( K \) containing \( S \) and let \( H \) be an \( S \)-high subgroup of \( K \). Let \( H' \) be an \( S \)-high subgroup of \( G \) containing \( H \) then, since \( S \) is q. e. and \( M \) is also pure in \( G \) we have \( \langle M, H' \rangle = G \), thus by lemma 4.5 and 4.6,\n\[
K = \langle M, H' \rangle \cap K = \langle M \cap K, H' \rangle \cap K = \langle M \cap K, H' \cap K \rangle = \langle M, H \rangle,
\]
and \( S \) is q. e. in \( K \) as stated.

**Theorem 4.8.** Let \( G \) be a \( p \)-group, \( S \) a subsocle of \( G \) then \( S \) is strongly quasi-essential if and only if either \( S \subset G \) or there exists \( n \in \mathbb{Z}^+ \), such that \( p^n G = o \) and \( (p^{n-1} G)[p] \triangleright S \).

**Proof.** If \( S \subset G \), \( S \) is s. q. e. follows from lemma 2.1. If there exists \( n \in \mathbb{Z}^+ \) such that \( p^{n-1} G[p] \triangleright S \triangleright p^n G = o \) then \( S \) is s. q. e. as a
consequence of corollary 1 to theorem 4.2. Suppose now that \( S \) is s. q. e. and \( S \not\subseteq G^i \) then there exists \( s \in S \) such that \( h(s) < \infty \). By corollary 24.2 in [1], \( s \) can be embedded in a finite pure subgroup \( K \) such that \( K[p] = \langle s \rangle \). Since \( S \) is s. q. e., \( K \) is an absolute summand of \( G \). Thus by theorem 4.4, there exists \( m \in \mathbb{Z}^+ \), such that \( (p^m G)[p] \subseteq \langle s \rangle \subseteq (p^{m-1} G)[p] \) but \( \langle s \rangle \) is a cyclic group of order \( p \), therefore \( G \) is a bounded group. This implies that \( S \) supports a pure subgroup \( M \) of \( G \), and since \( S \) is q. e., \( M \) is an absolute summand of \( G \). From lemma 4.7, we see that every subsocle of \( M \) is q. e. in \( M \), and thus every summand of \( M \) is an absolute summand.

By problem 11 (b), p. 93 in [1], \( M = \sum C(p^n) \) for some \( n \in \mathbb{Z}^+ \) and
\[
S = M[p] \subseteq (p^{n-1} G)[p].
\]
Clearly \( M[p] \not\subseteq (p^n G)[p] \), therefore \( (p^n G)[p] \subseteq S \subseteq (p^{n-1} G)[p] \), and since \( M \) is pure \( P^n G \subseteq M \). Thus
\[
p^n G = (p^n G) \cap M = p^n M = 0,
\]
and the proof is complete.

The following characterization follows immediately from theorem 4.8.

**Theorem 4.9.** — Let \( G \) be a \( p \)-group, every subsocle of \( G \) is quasi-essential if and only if \( G \) is divisible or \( G \) is a direct sum of cyclic groups of same order.

We have not been able to decide whether a quasi-essential subsocle is necessarily a center of purity or not. But in the next theorem, we have a case where quasi-essential subsocles are centers of purity.

**Theorem 4.10.** — Let \( G \) be a \( p \)-group, if \( G \) is pure-complete then every quasi-essential subsocle of \( G \) is a center of purity.

**Proof.** — Let \( S \) be q. e. Since \( G \) is pure-complete, \( S \) supports a pure subgroup \( K \) of \( G \). This \( K \) is an absolute summand and therefore the result follows from corollary 2 to theorem 4.2.

**Bibliography.**


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