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# ON $N$-HIGH SUBGROUPS OF ABELIAN GROUPS

BY

JOHN M. IRWIN AND KHALID BENABDALLAH.

## 1. Introduction.

This paper is based on a curious property of $N$-high subgroups when $N$ is a subgroup of $G^1$ the subgroup of elements of infinite height of a group $G$. Let $G$ be a group, $N$ a subgroup of $G$, we say that a subgroup $H$ of $G$ is $N$-high if $H$ is maximal with respect to the property $H \cap N = 0$. Our first result (theorem 2.4) is that given a group $G$ and $N$ a subgroup of $G^1$, then $G = \langle H, K \rangle$ whenever $H$ is an $N$-high subgroup of $G$ and $K$ is a pure subgroup of $G$ containing $N$. A close look at the proof of this result shows that the assumption that $K$ is pure can be replaced by the weaker one that $N \subset K^1$. An immediate consequence is the classical theorem that divisible subgroups of a group are absolute summands of the group.

$N$-high subgroups where $N \subset G^1$ were first introduced and studied by IRWIN and WALKER in [3]. These authors proved that $N$-high subgroups are pure and that the factor groups they induce are divisible. It turns out (theorem 2.5) that $H$ is an $N$-high subgroup of a group $G$, where $N \subset G^1$ if and only if $H$ is pure, $H \cap N = 0$, $G = \langle H, K \rangle$ for all $K$ pure containing $N$ and $G/H$ is divisible. We use this property of $N$-high subgroups where $N \subset G^1$, to generalize and simplify many results in [4]. In particular, we obtain a criterion for a pure subgroup of a group $G$ containing $N \subset G^1$ to be a summand of $G$ (theorem 3.1).

In the fourth part, we define the concept of quasi-essential and strongly quasi-essential subsocles of a $p$-group (definition 4.1) and proceed to characterize those quasi-essential subsocles which are also centers of purity (theorem 4.4) and those which are strongly quasi-essential (theorem 4.8).

We use standard notation from [1]. The symbol $Z^+$ denotes the set of positive integers. If $G$ is a $p$-group, $R$ a subgroup of $G$ and $g \in R$, the symbol $h_R(g)$ denotes the height of the element $g$ in the subgroup $R$. All groups considered are Abelian.

## 2. A characterization of $N$-high subgroups of a group $G$, with $N \subset G^1$.

We need the following lemmas.

LEMMA 2.1. — *Let $G$ be a $p$-group, $N$ subgroup of $G^1$ and $K$ a pure subgroup of $G$ containing $N$. Then for any $N$-high subgroup $H$ of $G$, $G = \langle H, K \rangle$.*

*Proof.* — Clearly $\langle H, K \rangle \supset H[p] \oplus N[p] = G[p]$. By induction suppose $\langle H, K \rangle \supset G[p^n]$. Let $g \in G$, $o(g) = p^{n+1}$, if $g \notin H$, $\langle g, H \rangle \cap N \neq 0$ thus there exists $h \in H$, $g' \in N$, and $m < n+1$, such that

$$p^m g + h = g' \neq 0$$

since $K$ is pure, $g' \in K^1$, thus there exists $k \in K$, such that $g' = p^m k$, or $h = p^m(g - k)$. If $h \neq 0$, by purity of $H$ (*see* [3], theorem 5) there exists $h' \in H$, such that $p^m h' = h$, therefore $p^m(g - k - h') = 0$. This implies $g - k - h' \in \langle H, K \rangle$, and $g \in \langle H, K \rangle$, thus $\langle H, K \rangle \supset G[p^{n+1}]$. By induction

$$G = \langle H, K \rangle.$$

LEMMA 2.2. — *Let $G$ be a torsion group, $N$ a subgroup of $G^1$ and $K$ a pure subgroup of $G$ containing $N$. Then for any $N$-high subgroup $H$ of $G$, $G = \langle H, K \rangle$.*

*Proof.* — Let $G = \sum Gp$, $H = \sum Hp$, $K = \sum Kp$ and $N = \sum Np$ then for each prime $p$, $Hp$ is $Np$-high in $Gp$ (*see* [2], lemma 11) and since $Kp$ is pure containing $Np$, lemma 2.1 holds and $Gp = \langle Hp, Kp \rangle$. Therefore

$$G = \sum \langle Hp, Kp \rangle = \langle H, K \rangle.$$

LEMMA 2.3. — *Let $G$ be a group, $N$ a subgroup of $G^1$ and $H$ an $N$-high subgroup of $G$. Then $H_t$ is $N_t$-high in $G_t$.*

*Proof.* — Clearly $H_t \cap N_t = 0$, let $g \in G_t$, $o(g) = b$, $g \notin H$ then $\langle g, H \rangle \cap N \neq 0$, thus there exists $h \in H$, $n \in N$, and a positive integer $a$ such that $ag + h = n \neq 0$. Clearly $a \neq b$. Now $bag + bh = bn$, thus $bh = bn = 0$, and $h \in H_t$, $n \in N_t$, therefore $\langle g, H_t \rangle \cap N_t \neq 0$. This implies that $H_t$ is $N_t$-high in $G_t$.

THEOREM 2.4. — *Let $G$ be a group, $N$ a subgroup of $G^1$ and $K$ a pure subgroup of $G$ containing $N$. Then for any $N$-high subgroup $H$ of $G$,*

$$G = \langle H, K \rangle.$$

*Proof.* — Suppose $g \in G$, $g \notin H$, then $\langle g, H \rangle \cap N \notin o$, thus there exists $h \in H$, $n \in N$ and a positive integer $a$, such that

$$ag + h = n \neq o.$$

By an argument similar to the one used in lemma 2.1, there exists $h' \in H$ and $k \in K$ such that $a(g + h' - k) = o$. Thus $g + h' - k \in G_t$. But, by lemmas 2.3 and 2.2, we know that $G_t = \langle H_t, K_t \rangle$, therefore $g \in \langle H, K \rangle$ and $G = \langle H, K \rangle$.

A classical theorem follows immediately from theorem 2.4.

COROLLARY. — *If $D$ is a divisible subgroup of a group $G$, then $D$ is an absolute summand of $G$.*

*Proof.* — Let $D = N$ in theorem 2.4, since $D$ is divisible it is pure in $G$. Thus $G = D \oplus H$, for any $D$-high subgroup $H$ of $G$.

THEOREM 2.5. — *Let $G$ be a group, $N$ a subgroup of $G^1$ and $H$ a subgroup of $G$ disjoint from $N$. Then $H$ is $N$-high in $G$ if and only if $H$ is pure, $G/H$ is divisible and $G = \langle H, K \rangle$ for any pure subgroup $K$ of $G$ containing $N$.*

*Proof.* — The necessity follows from theorem 2.4. Suppose then that $H$ satisfies the conditions of the theorem. Since $H \cap N = o$ there exists an $N$-high subgroup $H'$ of $G$ containing $H$. Since $H'$ is pure in $G$, $H'/H$ is pure in $G/H$ which is divisible, therefore $H'/H$ is divisible and $G/H = (H'/H) \oplus (R/H)$ where $R$ can be chosen to contain $N$. Since $H$ is pure in $G$ and $R/H$ is pure in $G/H$, $R$ is pure in $G$, and since $R \supset N$,

$$R = \langle R, H \rangle = G.$$

Therefore

$$H = R \cap H' = G \cap H' = H'$$

and $H$ is $N$-high in $G$.

## 3. Some applications.

We first obtain a criterion for pure subgroups of a group $G$ to be summands of $G$.

THEOREM 3.1. — *Let $G$ be a group, $K$ a pure subgroup of $G$ containing a subgroup $N$ of $G^1$. Then $K$ is a direct summand of $G$ if and only if there exists an $N$-high subgroup $H$ of $G$ such that $H \cap K$ is a direct summand of $H$.*

*Proof.* — Suppose $G = K \oplus L$, let $M$ be any $N$-high subgroup of $K$, then it is easy to see that $H = L \oplus M$ is $N$-high in $G$ and $H \cap K = M$ is a summand of $H$.

Suppose now that there exists an $N$-high subgroup $H$ of $G$ such that $H = (H \cap K) \oplus R$, by theorem 2.4 :

$$G = \langle H, K \rangle = \langle (H \cap K) \oplus R, K \rangle = \langle R, K \rangle$$

and since $R \cap K = 0$, $G = R \oplus K$.

The following corollary contains theorem 2 in [4].

COROLLARY. — *A reduced group $G$ splits over its maximal torsion subgroup $G_t$ if and only if some $N$-high subgroup of $G$ splits, where $N \subset G^1 \cap G_t$.*

*Proof.* — If $G$ is reduced and $G = G_t \oplus L$ then $G^1 \subset G_t$ and since $G_t$ is pure theorem 3.1 implies there exists an $N$-high subgroup such that $H \cap G_t = H_t$ is a summand of $H$. Now if $H$ is $N$-high and $H = H_t \oplus L$ since $N \subset G_t \cap G^1$ by theorem 3.1, $G_t$ is a summand of $G$.

For what follows we need the following lemmas.

LEMMA 3.2. — *Let $G$ be a group, $H$ a subgroup of $G$ then if $K/H_t$ is an $(H/H_t)$-high subgroup of $G/H_t$, then $K$ is pure in $G$ and $K \supset G_t$.*

*Proof.* — Suppose $ng \in K$ where $g \in G$. Let $0 \neq h = ag + k \in \langle K, g \rangle \cap H$ then $nag + nk = nh \in K \cap H = H_t$, therefore $h \in H_t$, thus $\langle K, g \rangle \cap H = H_t$ which implies $\langle K, g \rangle = K$, therefore $g \in K$ and thus $K$ is pure in $G$. Now if $g \in G_t$ then letting $n = o(g)$ in the above argument we see that $K \supset G_t$.

LEMMA 3.3. — *Let $G$ be a group, $N$ a subgroup of $G^1$, $H$ an $N$-high subgroup of $G$ and $K$ a pure subgroup of $G$ containing $\langle N, G_t \rangle$ and such that $K \cap H = H_t$. Then for any $N$-high subgroup $H'$ we have $K \cap H' = H'_t$.*

*Proof.* — Such $K$ do exist (lemma 3.2). Clearly $K \cap H' \supset H'_t$. Let $h' \in K \cap H'$ and suppose $h' \notin H$ then there exists $h \in H$, $g \in N$ and a positive integer $a$ such that $ah' + h = g \neq 0$, thus $h \in K \cap H = H_t$ let $b = o(h)$, then

$$bah' = bah' + bh = bg \in H' \cap N = 0$$

thus $bah' = 0$ and consequently $h' \in H'_t$. Therefore $K \cap H' = H'_t$.

COROLLARY 1 ([4], lemma). — *If $G$ is a group, $N$ a subgroup of $G^1$ and $H$ is an $N$-high subgroup of $G$, then $H/H_t$ is a summand of $G/H_t$.*

*Proof.* — Let $K/H_t$ be $H/H_t$-high in $G/H_t$. Choose $K \supset N$. Then, since $K$ is pure in $G$ (lemma 3.2), it follows from theorem 2.4 that $G = \langle K, H \rangle$. Therefore $G/H_t = (H/H_t) \oplus (K/H_t)$.

COROLLARY 2 ([4], theorem 4). — *Let $H$ and $H'$ be two $N$-high subgroups of a reduced group $G$ where $N$ is a subgroup of $G^1$. Then $H/H_t \simeq H'/H'_t$ and $G/H_t \simeq G/H'_t$.*

*Proof.* — From corollary 1, $G/H_t = (H/H_t) \oplus (K/H_t)$. From lemma 3.3, $K \cap H' = H'_t$, therefore $G/H'_t = (H'/H'_t) \oplus (K/H'_t)$. The result follows from this and the fact that $G/H \simeq G/H'$ (*see* [3]).

COROLLARY 3 ([4], theorem 1). — *Let $G$ be a reduced group, $N$ a subgroup of $G^1$ and $H$ an $N$-high subgroup of $G$ then if $H = H_t \oplus L$, we have $G = K \oplus L$ where $K/G_t$ is the divisible part of $G/G_t$.*

*Proof.* — $G/H_t = H/H_t \oplus K/H_t$ from corollary 1.

Now $K/H_t$ is divisible since $K/H_t \simeq G/H$, and $H/H_t \simeq L$ is reduced. Thus $K/G_t$ is the divisible part of $G/H_t$. Now $K \cap H = H_t$ implies $K \cap L = 0$ and $\langle K, H \rangle = G$ implies $\langle K, L \rangle = G$, therefore

$$G = K \oplus L.$$

## 4. Generalizations. Quasi-Essential subsocles of $p$-groups.

It is natural to ask, what kind of subgroups of a group $G$ have properties similar to subgroups of $G^1$. We consider first $p$-groups. It is trivial to verify that two subgroups of a $p$-group are disjoint if and only if their socles are. Thus it suffices to consider subgroups of the socle of a $p$-group which we will call subsocles.

DEFINITION 4.1. — Let $G$ be a $p$-group, a subsocle $S$ of $G$ is said to be quasi-essential (q. e.) if $G = \langle H, K \rangle$ whenever $H$ is an $S$-high subgroup of $G$ and $K$ a pure subgroup of $G$ containing $S$. $S$ is said to be strongly quasi-essential (s. q. e.) if every subgroup of $S$ is q. e.

We now proceed to characterize those quasi-essential subsocles of $G$ a $p$-group $G$ which are also centers of purity (*see* [7] and [6]).

THEOREM 4.2. — *Let $G$ be a $p$-group, $S$ a center of purity, $S \subset G[p]$. If $S$ is not quasi-essential in $G$ then there exists $n \in Z$, $g \in G[p]$, $g \notin S$ and $s \in S$ such that*

$$h(s) = h(g) = n \qquad \text{and} \qquad h(s+g) = n+1.$$

*Proof.* — Set $P_n = (p^n G)[p]$, $P_\infty = G^1[p]$ and $P_{\infty+1} = 0$ then it is known (*see* [6]) that $S$ is a center of purity if and only if

$$P_n \supset S \supset P_{n+2} \qquad \text{for some} \quad n \in \{1, 2, \ldots, \infty, \infty+1\}.$$

From lemma 2.1, we see that if $n = \infty$, i. e. $S \subset G^1$, $S$ is q. e. Thus if $S$ is not q. e. there exists $n \in Z^+$, such that

$$P_n \supset S \supset P_{n+2}.$$

Also $S$ is not q. e. implies that there exist a pure subgroup $K$ of $G$ containing $S$ and an $S$-high subgroup $H$, of $G$ such that $\langle H, K \rangle \neq G$. Let

$\langle H, K \rangle = R$. Since $R \supset G[p]$ and $R \neq G$, $R$ is not pure in $G$ (*see* [5], lemma 12). Therefore there exists an element $x \in R[p]$ such that $h(x) > h_R(x)$. $H$ and $K$ being both pure in $G$ implies that $x \notin H$ and $x \notin K$ Therefore there exists $g \in H[p]$ and $s \in S$ such that $x = g + s$, $g \neq 0 \neq s$. It is easy to verify that

$$h_R(g) = h_H(g) = h(g) \qquad \text{and} \qquad h_R(s) = h_K(s) = h(s),$$

therefore

$$h(g) = h(s) \leqq h_R(g+s) < h(g+s).$$

Now $s \in S$ implies $h(s) \geqq p^n$, $g \notin S$ implies $h(g) \leqq n+1$ and since $S \supset P_{n+2}$ we conclude that $h(s) = h(g) = n$ and $h(g+s) = n+1$ as stated.

Corollary 1. — *Let $G$ be a p-group, $S$ a subsocle of $G$ such that*

$$P_n \supset S \supset P_{n+1} \quad \textit{then } S \textit{ is quasi-essential.}$$

*Proof.* — $S$ is a center of purity, thus theorem 4.2 applies and clearly there exists no pair $g \in G[p]$, $g \notin S$ and $s \in S$ that satisfy the conditions of the theorem. Thus $S$ is q. e.

Corollary 2. — *Let $G$ be a p-group, $S$ subsocle of $G$ such that $S$ supports an absolute summand $A$ of $G$ then $S$ is quasi-essential.*

*Proof.* — $S$ is a center of purity, thus theorem 4.2 holds and again if $g \notin S$ and $s \in S$ and $h(g) = h(s)$ then, since $g$ can be embedded in a complementary summand of $A$ in $G$, $h(g+s) = h(g) = h(s)$. Therefore the condition of the theorem cannot be satisfied and $S$ must be q. e.

Corollary 3. — *Let $G$ be a p-group, $K$ a pure subgroup of $G$ containing $P_n$ for some $n \in Z^+$, then $K$ is a direct summand containing $p^n G$.*

*Proof.* — Since $P_n$ is q. e., $G = \langle K, H \rangle$, where $H$ is a $P_n$-high subgroup of $G$. Now $H$ is bounded, in fact $p^n H = 0$, and $G/K \simeq H/H \cap K$, therefore $K$ is a direct summand of $G$ and $p^n G \subset K$.

In fact, it turns out that the conditions on $S$ in corollary 1 and 2 as well as the condition that $S$ be quasi-essential and a center of purity, are equivalent provided $S \not\subset G^1$. To prove this, we need the following lemma.

Lemma 4.3. — Let $G$ be a $p$-group, $H$ a pure subgroup of $G$ such that $G/H$ is pure-complete. Let $S$ be a subsocle of $G$ such that $H[p] \subset S$. Then $S$ supports a pure subgroup $K$ of $G$ containing $H$.

*Proof.* — Since $G/H$ is a pure-complete group, by definition, every subsocle of $G/H$ supports a pure subgroup of $G/H$. Now $\langle S, H \rangle / H$ is clearly a subsocle of $G/H$, therefore there exists $K/H$ a pure subgroup of $G/H$ such that

$$(K/H)[p] = \langle S, H \rangle / H.$$

Since $H$ is pure in $G$, $K$ is pure in $G$ (*see* [5], lemma 2). Clearly $K[p] \supset S$, let $k \in K[p]$, then $k + H \in (K/H)[p] = \langle S, H \rangle / H$, thus there exists $s \in S$ and $h \in H$ such that $k - s = h$, but $ph = p(k - s) = 0$, and since $S \supset H[p]$, we conclude that $k \in S$. Therefore $K[p] = S$.

Corollary. — *Let $G$ be a $p$-group, $S$ a subsocle containing $P^u$* (*see* theorem 4.2) *for some $n \in Z^+$, then $S$ supports a pure subgroup of $G$ containing $p^n G$.*

*Proof.* — Let $G_n$ be as in [1], p. 98. Then $G_n$ is pure in $G$, $G_n[p] = P_n$ and $G/G_n$ is bounded and therefore pure complete. Thus lemma 4.3 holds, and $S$ supports a pure subgroup of $G$ containing $G_n$.

Theorem 4.4. — *Let $G$ be a $p$-group, $S$ subsocle of $G$ not contained in $G^1$ then the following are equivalent:*

(i) *$S$ is both a center of purity and a quais-essential subsocle of $G$;*
(ii) *$S$ supports an absolute direct summand of $G$;*
(iii) *There exists $n \in Z^+$ such that $P_n \supset S \supset P_{n+1}$.*

*Proof.* — (i) implies (ii). Suppose $S$ satisfies (i), then since $S$ is a center of purity $S \supset P_m$ for some $m \in Z^+$ and by the corollary to lemma 4.3, $S$ supports a pure subgroup $K$ of $G$. Since $S$ is also quasi-essential $K$ is an absolute summand of $G$.

(ii) implies (i). Suppose $S$ supports an absolute summand $K$. Then $S$ is clearly a center of purity and by corollary 2 to theorem 4.2, $S$ is q. e.

(i) implies (iii). Suppose $S$ satisfies (i), then $S$ supports an absolute summand $K$ of $G$. Since $S$ is a center of purity, we know there exists $m \in Z^+ \ni P_m \supset S \supset P_{m+2}$. Suppose $P_{m+1} \not\supset S$, we will show, by contradiction, that $S \supset P_{m+1}$. Indeed, suppose not, i. e. there is $x \in G[p]$ such that $x \notin S$ and $h(x) = m + 1$. Now $P_{m+1} \not\subset S$ implies there exists $s \in S$, $h(s) = m$, otherwise $P_m \subset S \subset P_{m+1}$, and we would be done. Let

$$y = x - s \quad \text{then } h(y) = m, \qquad y \notin S \qquad \text{and} \qquad h(y + s) = m + 1.$$

Since $y \notin S$ there is an $S$-high subgroup $H$ of $G$ such that $y \in H$. But, $G = K \oplus H$ and $h(y) = h(s)$ imply that $h(y + s) = h(y) = h(s)$ which is a contradiction. Therefore $S \supset P_{m+1}$.

(iii) implies (i). If $S$ satisfies (iii) it is a center of purity (*see* theorem 4.2, proof) and by corollary 1 to theorem 4.2 it is also q. e.

At this point we have completely characterized those quasi-essential subsocles of a $p$-group which are also centers of purity. An immediate consequence is the following.

COROLLARY. — *Let $G$ be a p-group, $A$ a pure subgroup of $G$, then $A$ is an absolute direct summand of $G$ if and only if $A$ is divisible or $P_n \supset A[p] \supset P_{n+1}$ for some $n \in Z^+$.*

The strongly quasi-essential subsocles have also a simple characterization which can be obtained from the previous result. We need the following lemmas.

LEMMA 4.5. — *Let $G$ be a group; $A$, $B$, $C$ three subgroups of $G$ then*

$$\langle A \cap B, C \cap B \rangle = \langle A \cap B, C \rangle \cap B = \langle A, C \cap B \rangle \subset B.$$

LEMMA 4.6. — *Let $G$ be a group, $N$ a subgroup of $M$ a subgroup of $G$, if a subgroup $H$ is N-high in $G$ then $H \cap M$ is N-high in $M$. Conversely if $H'$ is an N-high subgroup of $M$ then $H' = H \cap M$ for any N-high subgroup $H$ of $G$ containing $H'$.*

*Proof.* — Let $H$ be $N$-high in $G$ then for all $x \notin H$, we have

$$\langle H, x \rangle \cap N \neq 0.$$

Suppose $m \in M$, $m \notin H$, then

$$\langle H \cap M, m \rangle \cap N = \langle H, m \rangle \cap M \cap N = \langle H, m \rangle \cap N \neq 0.$$

and since $(H \cap M) \cap N = 0$, $H \cap M$ is $N$-high in $M$.

Let $H'$ be an $N$-high subgroup of $M$, and let $H$ be any $N$-high subgroup of $G$ then $H \cap M \supset H'$ and $(H \cap M) \cap N = 0$. The maximality of $H'$ implies $H \cap M = H'$.

LEMMA 4.7. — *Let $G$ be a p-group, $S$ a quasi-essential subsocle of $G$. Let $K$ be a pure subgroup of $G$ containing $S$. Then $S$ is quasi-essential in $K$.*

*Proof.* — Let $M$ be a pure subgroup of $K$ containing $S$ and let $H$ be an $S$-high subgroup of $K$. Let $H'$ be an $S$-high subgroup of $G$ containing $H$ then, since $S$ is q. e. and $M$ is also pure in $G$ we have $\langle M, H' \rangle = G$, thus by lemma 4.5 and 4.6,

$$K = \langle M, H' \rangle \cap K = \langle M \cap K, H' \rangle \cap K = \langle M \cap K, H' \cap K \rangle = \langle M, H \rangle,$$

and $S$ is q. e. in $K$ as stated.

THEOREM 4.8. — *Let $G$ be a p-group, $S$ a subsocle of $G$ then $S$ is strongly quasi-essential if and only if either $S \subset G^1$ or there exists $n \in Z^+$, such that $p^n G = 0$ and $(p^{n-1} G)[p] \supset S$.*

*Proof.* — If $S \subset G^1$, $S$ is s. q. e. follows from lemma 2.1. If there exists $n \in Z^+$ such that $p^{n-1} G[p] \supset S \supset p^n G = 0$ then $S$ is s. q. e. as a consequence of corollary 1 to theorem 4.2. Suppose now that $S$ is s. q. e. and $S \not\subset G^1$ then there exists $s \in S$ such that $h(s) < \infty$. By corollary 24.2 in [1], $s$ can be embedded in a finite pure subgroup $K$ such that $K[p] = \langle s \rangle$. Since $S$ is s. q. e., $K$ is an absolute summand of $G$. Thus by theorem 4.4, there exists $m \in Z^+$, such that $(p^m G)[p] \subset \langle s \rangle \subset (p^{m-1} G)[p]$ but $\langle s \rangle$ is a cyclic group of order $p$, therefore $G$ is a bounded group. This implies that $S$ supports a pure subgroup $M$ of $G$, and since $S$ is q. e., $M$ is an absolute summand of $G$. From lemma 4.7, we see that every subsocle of $M$ is q. e. in $M$, and thus every summand of $M$ is an absolute summand. By problem 11 (*b*), p. 93 in [1], $M = \sum C(p^n)$ for some $n \in Z^+$ and

$$S = M[p] \subset (p^{n-1} G)[p].$$

Clearly $M[p] \not\subset (p^n G)[p]$, therefore $(p^n G)[p] \subset S \subset (p^{n-1} G)[p]$, and since $M$ is pure $P^n G \subset M$. Thus

$$p^n G = (p^n G) \cap M = p^n M = 0,$$

and the proof is complete.

The following characterization follows immediately from theorem 4.8.

THEOREM 4.9. — *Let $G$ be a $p$-group, every subsocle of $G$ is quasi-essential if and only if $G$ is divisible or $G$ is a direct sum of cyclic groups of same order.*

We have not been able to decide whether a quasi-essential subsocle is necessarily a center of purity or not. But in the next theorem, we have a case where quasi-essential subsocles are centers of purity.

THEOREM 4.10. — *Let $G$ be a $p$-group, if $G$ is pure-complete then every quasi-essential subsocle of $G$ is a center of purity.*

*Proof.* — Let $S$ be q. e. Since $G$ is pure-complete, $S$ supports a pure subgroup $K$ of $G$. This $K$ is an absolute summand and therefore the result follows from corollary 2 to theorem 4.2

## BIBLIOGRAPHY.

[1] FUCHS (Laszlo). — *Abelian groups.* — Budapest, Hungarian Academy of Science, 1958.

[2] IRWIN (John M.). — High subgroups of Abelian torsion groups, *Pacific J. Math.*, vol. 11, 1961, p. 1375-1384.

3] IRWIN (John M.) and WALKER (Elbert A.). — On $N$-high subgroups of Abelian groups, *Pacific J. Math.*, vol. 11, 1961, p. 1363-1374.

[4] IRWIN (John M.), PEERCY (Carol) and WALKER (Elbert A.). — Splitting properties of high subgroups, *Bull. Soc. math. France*, t. 90, 1962, p. 185-192.

[5] Kaplansky (Irving). — *Infinite abelian groups.* — Ann Arbor, University of Michigan Press, 1954 (*University of Michigan Publications in Mathematics*, 2).

[6] Pierce (R. S.). — Centers of purity in Abelian groups, *Pacific J. Math.*, vol. 13, 1963, p. 215-219.

[7] Reid (J. D.). — On subgroups of an abelian group maximal disjoint from a given subgroup, *Pacific J. Math.*, vol. 13, 1963, p. 657-664.

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