H. SUBRAMANIAN

Integer-valued continuous functions


<http://www.numdam.org/item?id=BSMF_1969__97__275_0>
Let $C(X, Z)$ denote the $f$-ring of all integer-valued continuous functions on a topological space $X$. $C^*(X, Z)$ stands for the sub-$f$-ring of all bounded functions in $C(X, Z)$. Pierce [7] and Alling [1] study $C(X, Z)$ from the point of view of the algebra of clopen sets of $X$. We investigate $C(X, Z)$ as an $f$-ring, and, therefore, from the point of view of its maximal $l$-ideals.

We prove the equivalence of the ring, the lattice and the (multiplicative) semigroup structures in $C^*(X, Z)$. We also give characterizations of $C^*(X, Z)$ as a lattice-ordered (l.-o.) ring, as a lattice-ordered (l.-o.) group and as a ring. We obtain these characterizations as soon as we observe that there exist sufficiently many characteristic functions with which every function in $C^*(X, Z)$ is expressed in a natural way. But, it is not clear, at present, how to characterize $C^*(X, Z)$ as a lattice or as a semigroup. Also, all these problems for $C(X, Z)$ remain open.

We consider only commutative rings with unit element. The notion of hull-kernel topology in any given collection of prime ideals of a ring is assumed to be known [1], [4], [5], [6], [7], [9], [10]. An $f$-ring is a l.-o. ring which is a subdirect product of totally-ordered (t.-o.) rings. An $l$-ideal $I$ of a l.-o. ring is a (ring) ideal satisfying the property: $|x| \leq |y|$, $y \in I$ implies that $x \in I$. A maximal $l$-ideal of an $f$-ring is always prime [2]. If the intersection of all maximal $l$-ideals of a l.-o. ring is zero, the l.-o. ring is said to be $l$-semisimple. Thus an $l$-semisimple $f$-ring has no nonzero nilpotent elements.

We fix some notations for the rest of the paper: $X$ and $Y$ for arbitrary topological spaces; $R$ for a ring or an $f$-ring; $Z$ for the t.-o. ring of integers; $\pi$ for the hull-kernel space of all minimal prime ideals of the ring $R$; $\mathcal{M}$ for the hull-kernel space of all maximal $l$-ideals of the $f$-ring $R$; $\mathfrak{X}$ for the Boolean space of the algebra of all clopen sets of $X$ and $\sigma X$ for the space $\mathfrak{X}$ of $C(X, Z)$. 
We lose nothing in the study of the ring-lattice structure of a $C(X, Z)$ if we assume that $X$ is Hausdorff and has a base of clopen sets [7]. Therefore we consider only such spaces in this paper.

**Remark 1.** — For any point $x \in X$, let $M_x$ denote $\{ f \in C(X, Z) \mid f(x) = 0 \}$. It is easy to verify that $M_x$ is a maximal $l$-ideal of $C(X, Z)$. Since $\bigcap_{x \in X} M_x$ is the zero ideal, $\{ M_x \}_{x \in X}$ is a dense subspace of $\sigma X$. Alling [1] has noted that $X$ is homeomorphic to $\{ M_x \}_{x \in X}$ under the correspondence $x \rightarrow M_x$. Obviously, $X$ is compact if, and only if, every maximal $l$-ideal of $C(X, Z)$ is of the form $M_x$ for $x \in X$; that is, $X$ is homeomorphic to $\sigma X$. Thus $\sigma (\sigma X) = \sigma X$.

**Theorem 1.** — A subset $P$ of $C(X, Z)$ is a maximal $l$-ideal if, and only if, it is a minimal prime ideal.

**Proof.** — For any $f \in C(X, Z)$, we denote by $\zeta(f)$ the zero set $\{ x \in X \mid f(x) = 0 \}$ of $f$. If $F$ is a clopen set of $X$, $\zeta_F$ stands for the characteristic function on $F$. Now, if $P$ is a maximal $l$-ideal, it is necessarily prime [2]. To show that it is also minimal prime, consider any $f \in P$. $\zeta(f)$ is a clopen set of $X$, and $f. \zeta(f) = 0$. Since $1 \leq |f + \zeta(f)|$, it follows that $\zeta(f) \in P$. Therefore, $P$ is minimal prime [5], [6]. Conversely, let $P$ be minimal prime. Surely then, $P$ is an $l$-ideal [10]; so, it is contained in a maximal $l$-ideal, but which is also minimal prime. The desired result follows.

**Remark 2.** — The space $\mathcal{M}$ of any $\mathfrak{f}$-ring is always compact Hausdorff [4], [9], and the space $\mathcal{P}$ of any ring is always totally-disconnected [5], [6]. Thus $\sigma X$ is compact Hausdorff totally-disconnected (abbreviated in the sequel at CHT). Alling [1] shows that the space $\mathcal{P}$ of $C(X, Z)$ is homeomorphic to $\sigma X$. Therefore, $\sigma X$ is homeomorphic to $\sigma X$. Pierce [7] has shown that $C^*(X, Z)$ and $C(\sigma X, Z)$ are isomorphic as rings. They are indeed isomorphic as $\mathfrak{f}$-rings. It follows that any general theorem concerning $C(X, Z)$'s as $\mathfrak{f}$-rings will also be true of $C^*(X, Z)$'s. Theorem 1 above is a case in instance. We also note that the space $\mathcal{P}$ or $\mathcal{M}$ of $C^*(X, Z)$ ($\approx C(\sigma X, Z) \approx C(\sigma X, Z)$) is $\sigma X$.

The effective content of the following theorem has been announced by Sankaran [8], but no proof seems to have been published so far.

**Theorem 2.** — The following are equivalent:

1. $C^*(X, Z)$ and $C^*(Y, Z)$ are isomorphic as rings;
2. $C^*(X, Z)$ and $C^*(Y, Z)$ are isomorphic as lattices;
3. $C^*(X, Z)$ and $C^*(Y, Z)$ are isomorphic as p.o. groups;
4. $C^*(X, Z)$ and $C^*(Y, Z)$ are isomorphic as semigroups;
5. $\sigma X$ and $\sigma Y$ are homeomorphic.
Proof. — A ring (resp. lattice) isomorphism between two l-semisimple \( f \)-rings induces a homeomorphism between their spaces of maximal \( l \)-ideals \([4]\) (resp. \([9]\)). Hence each of (1) and (2) implies (5). (3) implies (3) because any order-group isomorphism between two \( l \)-o. groups preserves the lattice structures also. A semigroup ideal in a ring is minimal prime if, and only if, it is a minimal prime (ring) ideal \([6]\). Thus, by Remark 2, (4) implies (5). Finally, (5) implies that \( C(\sigma X, Z) \) and \( C(\sigma Y, Z) \) are isomorphic as \( f \)-rings. Remark 2 completes the proof.

Remark 3. — Whether the above theorem is true with \( C'(X, Z) \) and \( C'(Y, Z) \) replaced respectively by \( C(X, Z) \) and \( C(Y, Z) \) is not known. We may refer the analogous case for \( C(X) \) shown to be true by Henriksen \([3]\). Similar to the realcompact spaces (the crucial point in \([3]\)) in the theory of \( C(X) \), Alling \([1]\) considers the space

\[
\delta_0 X = \{ P \in \sigma X \mid C(X, Z)/P \approx Z \}.
\]

It can be proved that \( C(X, Z) \) and \( C(\delta_0 X, Z) \) are isomorphic as \( f \)-rings. Only, we have to observe that, in \( \delta_0 X \), the hull-kernel topology and the weak topology generated by the functions in \( C(X, Z) \) are same. Now \( C(X, Z) \) and \( C(Y, Z) \) are isomorphic as rings if, and only if, \( \delta_0 X \) and \( \delta_0 Y \) are homeomorphic. Such a result with respect to the lattice (resp. semigroup) structure can be brought about if only we can show that any \( P \in \delta_0 X \) can be obtained purely from the lattice (resp. semigroup) structure of \( C(X, Z) \). The following examples counter any hasty predictions on a global generalization of Theorem 2 to all \( f \)-rings.

Example 1 \([9]\). — The \( t \)-o. field \( Q \) of rational numbers and the non-archimedean ordered ring \( Q[x] \) of polynomials over \( Q \) are order-isomorphic, but not ring-isomorphic (not semigroup isomorphic also).

Example 2 \([9]\). — The non-archimedean ordered ring \( Z[x] \) of polynomials over \( Z \) and the subring \( Z[9] \) of the real number field generated by \( Z \) and a transcendental number \( \theta \) (with induced order) are ring-isomorphic, but not order isomorphic.

Example 3. — \( Z \) and \( Z[x] \) are semigroup isomorphic, but not ring isomorphic.

A semigroup isomorphism between \( Z \) and \( Z[x] \) can be constructed as follows: Consider the set of all nonzero irreducible polynomials in \( x \) over \( Z \), the coefficients of whose highest degree are positive. This is a countable set which can be put in one-to-one correspondence with the set of all prime numbers. This one-to-one correspondence is extended to a semigroup isomorphism between \( Z \) and \( Z[x] \) in the natural way, making use of the fact that both \( Z \) and \( Z[x] \) are unique factorization domains with exactly two unit elements, viz. \( 1 \) and \( -1 \). But, in any ring
isomorphism between $\mathbb{Z}$ and $\mathbb{Z}[x]$, $x$ should correspond to some integer, which means $x$ equals the same integer, a contradiction.

However, the following theorem and corollaries are interesting in the face of the above remark and "warning post" examples. Before stating the theorem, we recall from [9] a definition.

**Definition.** — Let $R$ be an $\mathfrak{f}$-ring, $M$ a maximal $l$-ideal of $R$ and $P$ a (proper) lattice-prime ideal of $R$. $P$ is said to be associated with $M$ if $y \in P$, $x(M) < y(M)$ imply that $x \in P$. [$x(M)$ denotes the homomorphic image of $x \in R$ in $R/M$.]

**Theorem 3.** — Let $R$ be an $\mathfrak{f}$-ring, and $M$ a maximal $l$-ideal of $R$. $R/M$ is non-archimedean if, and only if, there exists a lattice-prime ideal $P$ of $R$, associated with $M$, such that $P$ contains all of $\{1, 2, 3, \ldots\}$.

**Proof.** — If $R/M$ is non-archimedean, there is an $f \in R$ such that $f(M) > n$ for every natural number $n$. Consider now

$$P = \{ g \in R \mid g(M) < f(M) \}.$$ 

$P$ is a lattice-prime ideal of $R$, associated with $M$ [9]. Evidently, $P$ contains all of $\{1, 2, 3, \ldots\}$. Conversely, if such a $P$ exists, choose $f \notin P$. Since $P$ is associated with $M$, $f(M) > n$ for every natural number $n$. Surely then, $R/M$ is non-archimedean.

**Corollary 1.** — $C(X, \mathbb{Z})$ and $C(Y, \mathbb{Z})$ are isomorphic as rings if, and only if, they are $\mathbb{Z}$-isomorphic (i.e. mapping constant functions into the same constant functions) as lattices.

**Proof.** — The lattice structure of an $\mathfrak{f}$-ring $R$ determines the space $\mathfrak{m}$ of $R$ [9]. If $M \in \sigma X$, $C(X, \mathbb{Z})/M$ is either $\mathbb{Z}$ or a near $\nu$-set (which is not archimedean) [1]. Therefore, by Theorem 3, a $\mathbb{Z}$-isomorphism between $C(X, \mathbb{Z})$ and $C(Y, \mathbb{Z})$ induces a homeomorphism between $\delta_0X$ and $\delta_0Y$. The result follows by Remark 3.

**Corollary 2.** — $C(X)$ and $C(Y)$ are isomorphic as rings if, and only if, they are $R$-isomorphic as lattices [$C(X)$ denotes the $\mathfrak{f}$-ring of all real-(R)-valued continuous functions on $X$].

**Proof.** — The same as in Corollary 1, with $\delta_0X$ and $\delta_0Y$ replaced respectively by $\nu X$ and $\nu Y$ ($\nu X$ stands for the realcompactification of $X$).

For use in the following two lemmas, we fix some terminology. If $R$ is an $\mathfrak{f}$-ring, let us suppose that $R$ is the subdirect product of the $t$.-o. rings $R_\alpha$; that is, if we denote the projection from $R$ onto $R_\alpha$ by $\varphi_\alpha$ with the $l$-ideal $P_\alpha$ as the kernel $\bigcap P_\alpha$ is $(\alpha)$.

**Lemma 1.** — Any idempotent $e$ in a $t$.-o. ring $R$ is either $0$ or $1$. 
Proof. — Since \( e = e^2 \), \( e \geq 0 \). Either \( e \leq (1 - e) \) or \( (1 - e) \leq e \).
In the first case, \( e = e^2 \leq e(1 - e) = 0 \). Thus \( e = 0 \). Since also \( (1 - e)^2 = (1 - e) \), the second case shows that \( e = 1 \).

Corollary. — If \( e \) is any idempotent in an \( \ell \)-ring, \( R, e \wedge (1 - e) = 0 \).

Proof. — Follows by observing that \( \varphi_e(e) = 0 \) or \( 1 \) in \( R_e \).

Lemma 2. — An \( \ell \)-ring \( R \) in which every element is a finite integral combination of idempotents is \( \ell \)-semisimple.

Proof. — If \( \{ M_\beta \} \) is the collection of all maximal \( \ell \)-ideals of \( R \), consider any \( x \in \bigcap M_\beta \). Let \( y = |x| \wedge 1 \). Since \( y \) is an integral combination of idempotents, by Lemma 1, \( \varphi_e(y) \) is an integer in \( R_e \). Since also \( 0 \leq y \leq 1 \), \( \varphi_e(y) \) is \( 0 \) or \( 1 \); and, every \( P_\alpha \) contains either \( y \) or \( (1 - y) \).
If some \( P_\alpha \) contains \( (1 - y) \), we get \( (1 - y) \in M_\beta \), where \( M_\beta \) is a maximal \( \ell \)-ideal containing \( P_\alpha \). Since also \( y \in M_\beta \), this is a contradiction. Therefore \( y \in \bigcap P_\alpha = (0) \). Now \( |x| \wedge 1 = 0 \) implies that \( x = 0 \), because \( |\varphi_e(x)| \wedge 1 = 0 \).

Theorem 4. — An \( \ell \)-ring \( R \) as described in Lemma 2 is ring-lattice isomorphic to a \( C(X, Z) \) for some (unique up to homeomorphism) CHT space \( X \).

Proof. — The obvious choice for the space \( X \) should be the space \( \mathfrak{M} \) of \( R \). Given \( x \in R \) and \( M \in \mathfrak{M} \), we have

\[
x = n_0 \cdot 1 + n_1 e_1 + \ldots + n_r e_r
\]

where \( n_k \in Z \) and \( e_k \in M \). For,

\[
x = \sum m_k i_k, \quad m_k \in Z, \quad i_k = i_k
\]

can be rewritten by changing an \( i_k \) into \( 1 - (1 - i_k) \) whenever \( i_k \notin M \).
If possible, let also

\[
x = n_0' \cdot 1 + n_1' e_1' + \ldots + n_r' e_r', \quad n_k' \in Z, \quad e_k' = e_k \in M.
\]

If \( n_0 > n_0' \), then \( 1 \leq (n_0 - n_0') \cdot 1 \in M \); thus \( 1 \in M \), which is a contradiction.
If \( n_0 < n_0' \), then \( 1 \leq (n_0 - n_0') \cdot 1 \in M \); thus \( 1 \in M \), which is a contradiction.
Every \( M \in \mathfrak{M} \) now induces a map \( \psi_M : R \to Z \), \( \psi_M(x) \) being the integer \( n_0 \) as we have just obtained.

It is easily seen that \( \psi_M \) is a ring homomorphism from \( R \) onto \( Z \) with \( M \) as the kernel. Since \( M \) is a maximal \( \ell \)-ideal, there is a canonical total order in the ring \( R/M \), i.e., \( Z \). The uniqueness of a compatible total order in the ring \( Z \) shows that \( \psi_M \) preserves lattice structure also.
Because of \( \ell \)-semisimplicity, by Lemma 2, \( R \) is thus lattice-ring isomorphic [by the transform \( x \to \bar{x}; \bar{x}(M) = \psi_M(x) \)] to a sublattice-subring of the \( \ell \)-ring of all integer-valued functions on \( \mathfrak{M} \).
If \( x = \sum n_k e_i, n_k \in \mathbb{Z} \) and \( e_i^2 = e_i \), then \( |\bar{x}| \leq \sum |n_k| \). So \( x \) has finite range which effects a finite partition of the space \( \mathfrak{M} \). Each coset, \( x \) being a constant in it, is exactly the collection of all maximal \( l \)-ideals containing \( x - k \cdot 1 \) for some \( k \in \mathbb{Z} \); and so it is closed. It follows that each coset is open. Hence \( \bar{x} \) is continuous because the preimage of every single point open set in the discrete space \( Z \) is open.

The map \( x \rightarrow \bar{x} \) from \( R \) to \( C(\mathfrak{M}, Z) \) is also onto. Since \( \mathfrak{M} \) is always compact Hausdorff \([4],[9]\), any \( f \in C(\mathfrak{M}, Z) \) is a finite sum \( \sum n_k \bar{E}_i \), where each \( E_i \) is a clopen set of \( \mathfrak{M} \). But \( E_i \) is the hull of a direct summand of \( R \), and therefore he hull of an idempotent \( i_k \in R \). Obviously, \( \bar{f} = \sum \bar{E}_i \), where \( j_k = \bar{1} - \bar{i}_k \). Thus if \( x = \sum n_k j_k \), then \( \bar{x} = f \).

The proof is complete on showing that \( \mathfrak{M} \) is a totally disconnected space. Every \( x \in M \in \mathfrak{M} \) is a finite integral combination of idempotents within \( M \); so distinct elements of \( \mathfrak{M} \) contain in them different collections of idempotents. Therefore any two points of \( \mathfrak{M} \) are separated by a clopen set, and \( \mathfrak{M} \) is totally disconnected. The uniqueness follows from Theorem 2 and Remark 1.

Let \( G \) denote a l.-o. Abelian group with strong order unit \( 1 \) (i.e. an element contained in no maximal \( l \)-subgroup). We define an idempotent in \( G \) to be a relatively complemented element in the interval \((0,1)\). The property of being a strong order unit is preserved under any group-lattice homomorphism. Since \( G \) is a subdirect product of t.o. groups, \( e \in (G, 1) \) is an idempotent if, and only if, \( e \wedge (1 - e) = 0 \). Any maximal \( l \)-subgroup of \( G \) is the kernel of a group-lattice homomorphism of \( G \) onto a t.o. group, and contains one, and only one, of the conjugate idempotents \( e \) and \((1-e)\). Any clopen set in the hull-kernel space (certainly compact \( T_1 \)) of maximal \( l \)-ideals of \((G, 1)\) is given by the hull of an idempotent. In Summary, we obtain the following theorem.

**Theorem 5.** — Lemma 2 and Theorem 4 are true in the set up of a l.-o. Abelian group with strong order unit in the place of an \( \mathfrak{J} \)-ring.

**Lemma 3.** — A ring \( R \), whose additive group is torsion-free, and whose elements are finite integral combinations of idempotents, does not have any nonzero nilpotent elements.

**Proof.** — Let \( R \) be the subdirect product of subdirectly irreducible rings \( R_a \). If we denote by \( \varphi_a \) the projection \( R \rightarrow R_a \) with kernel \( K_a \), \( \bigcap K_a \) is \((0)\). Consider any \( x \in R \) such that \( x^n = 0, n \neq 1 \). Let

\[
x = n_1 e_1 + \ldots + n_k e_k,
\]

where \( n_k \in \mathbb{Z} \) and \( e_k^2 = e_k \). Since a subdirectly irreducible ring cannot contain any idempotent except 0 and 1, we have either \( x \in K_a \).
or $x \equiv m_a \pmod{K_a}$ ($m_a \neq 0$) for each $K_a$, where $m_a \in \mathbb{Z}$. But each $m_a$ so obtained is a sum of some or all of $n_1, \ldots, n_k$. Thus there are only finitely many among the $m_a$'s which are distinct; if they are $m_1, m_2, \ldots, m_r$, we see that $m_1 \ldots m_r \cdot x \in \bigcap K_a = (0)$. This means $x$ is $0$, because the additive group is torsion-free.

In all the following lemmas, let $R$ denote a ring as described in Lemma 3. We proceed to prove that such a ring $R$ characterizes a $C(X, \mathbb{Z})$ for some CHT space $X$.

**Lemma 4.** — $R$ is isomorphic to a subring of a $C(X, \mathbb{Z})$.

**Proof.** — Choose $X$ to be the space $\mathcal{X}$ of $R$. For any $x \in R$ and $P \in \mathcal{P}$, obtain $n_0 \in \mathbb{Z}$ exactly as in Theorem 4. If this $n_0$ is not unique, there exists $n \in \mathbb{Z}^+$ such that $n \cdot 1 \in P$. $P$ being minimal prime, $nx = (n \cdot 1)x = 0$ for some $x \in P$ [5], [6]. This contradicts the assumption that the additive group of $R$ is torsion-free. The result of the proof is exactly like in Theorem 4 using Lemma 3 in place of Lemma 2.

**Lemma 5.** — If $P, Q \in \mathcal{P}$ are distinct, then $P + Q = R$.

**Proof.** — Choose $x \in P \sim Q$. Express $x$ as a sum of idempotents within $P$, as in Theorem 4. One of these idempotents, say $e_1$, is necessarily not in $0$. Thus $(1 - e) \in Q$, and $P + Q = R$.

**Corollary.** — Any maximal ideal of $R$ contains a unique minimal prime ideal.

For every maximal ideal $M$ of $R$, let $\pi(M)$ denote

$$\{x \in R \mid xy = 0 \text{ for some } y \notin M\}.$$ 

It is easily seen that $\pi(M)$ is an ideal of $R$ and $\pi(M) \supset M$.

**Lemma 6.** — For any $P \in \mathcal{P}$, $P = \pi(M)$ for any maximal ideal $M$ of $R$ such that $P \subset M$.

**Proof.** — If $x \in \pi(M)$, let $xy = 0$, $y \notin M$. Then $y \notin P$, implying that $x \in P$. Conversely, let $x \in \pi(M)$. Consider the multiplicatively closed subset $S = \{x \cdot y \mid y \notin M \} \cup \{R \sim M\}$. It is easily checked that $\pi(M)$ is disjoint with $S$. Therefore, by Zorn's Lemma, there exists a prime ideal $Q$ containing $\pi(M)$ and maximally disjoint with $S$. Since $Q \subset M$, $P$ should be, by Corollary to Lemma 5, the only minimal prime ideal contained in $Q$. Since $x \in S$, $x \notin P$.

**Corollary.** — For any $P \in \mathcal{P}$, $x \in P$ if, and only if, $x^* + P = R$, where $x^*$ denotes the annihilator ideal $\{y \in R \mid yx = 0\}$ of $x$.

**Proof.** — Let $x \in P$. If $M$ is any maximal ideal containing $P$, $M$ cannot contain $x^*$ by Lemma 6. Thus the ideal generated by $x^*$ and $P$ together is $R$. The converse is obvious.
LEMMA 7. — \{h(y) \mid y \in R\} is a base for open sets in \(\mathfrak{A}\), where for any \(A \subseteq R\), \(h(A) = \{P \in \mathfrak{A} \mid A \subseteq P\}\), the hull of \(A\) in \(\mathfrak{A}\).

Proof. — By Lemma 6, \(h(y) = h'(y')\) for any \(y \in R\), where \(h'(A)\) denotes the complement of the hull of \(A\) in \(\mathfrak{A}\). Choose any \(x \in R\) and \(P \in \mathfrak{A}\). Let \(n_1, n_2, \ldots, n_k\) be all the nonzero values of the function \(\mathfrak{A}\), defined as in Lemma 4. Consider \(y = (x - n_1) \cdots (x - n_k)\). Suppose that \(x \notin P\). Then \(y \in P\); and, for any \(Q \in \mathfrak{A}\) such that \(y \in Q, x \notin Q\). Thus \(P \in h(y) \subseteq h'(x)\). Since \(\{h'(x) \mid x \in R\}\) forms a base for open sets in \(\mathfrak{A}\), the result follows.

By Corollary to Lemma 5 and Lemma 6, \(\pi\) is a map defined from the hull-kernel space \(\mathcal{M}\) of all maximal ideals of \(R\) onto \(\mathfrak{A}\). We establish:

LEMMA 8. — \(\pi\) is continuous.

Proof. — For any \(y \in R\), \(\pi^{-1}(h(y)) = \{M \in \mathcal{M} \mid y \in \pi(M)\}\). By definition of \(\pi(M)\), it follows that \(\pi^{-1}(h(y)) = \{M \in \mathcal{M} \mid y \notin M\}\), which is an open set in \(\mathcal{M}\). Since \(h(y)\) is a basic open set in \(\mathfrak{A}\), \(\pi\) is continuous.

Since \(\mathcal{M}\) is always compact, we have:

COROLLARY. — The space \(\mathfrak{A}\) of \(R\) is compact.

LEMMA 9. — If \(F\) is any clopen set in the space \(\mathfrak{A}\) of \(R\), then \(F = h(e)\), where \(e\) is an idempotent in \(R\).

Proof. — \(\pi^{-1}(F)\) is clopen in \(\mathcal{M}\); therefore, there exists \(e \in R\) such that \(e = e^2\) and \(\pi^{-1}(F) = \{M \in \mathcal{M} \mid e \in M\}\). If \(P \in h(e)\) and \(P \subseteq M \in \mathcal{M}\), then \(P = \pi(M)\) and \(M \in \pi^{-1}(F)\). Therefore \(P \in F\). Conversely, if \(P \in F\), let \(P = \pi(M)\) for some \(M \in \mathcal{M}\). Then \(e \in M\). Since \(e = e^2\) and \(\pi^{-1}(F)\), \(e \in P\). That is, \(P \in h(e)\).

We now state:

THEOREM 6. — A ring \(R\), as described in Lemma 3 is ring isomorphic to a \(C(X, Z)\) for some (unique up to homeomorphism) CHT space \(X\).

Proof. — Choose \(\mathfrak{A}\) to be the space \(X\). Using Lemma 4, Corollary to Lemma 8, and Lemma 9, modify the proof of Theorem 4 accordingly.

REMARK 4. — The converse of each of Theorems 4, 5 and 6 are clearly true. In fact, for any \(f \in C(X, Z)\), we have

\[
f = \sum_{n \in \mathbb{N}, f - (n) \neq 0} n \chi_{f - (n)}
\]

and this is the unique representation of \(f\) as an integral combination with the fewest number of orthogonal idempotents. We could simplify the proofs of Theorems 4, 5 and 6 with such a strong representation of
elements. But a weaker hypothesis as we have assumed in these Theorems are enough.

Remark 5. — Neither Lemma 2 (hence Theorem 4) nor Lemma 3 (hence Theorem 6) extends to non-commutative cases. The necessary example to illustrate this is given below.

Example 4. — In the additive group $\mathbb{Z} \times \mathbb{Z}$, define $(a, b)(c, d) = (ac, ad)$ and give the lexicographic order. This non-commutative (f-)ring is additively generated by the idempotents $(1, b)$. $(0, 1)$ is nilpotent and generates the only maximal $l$-ideal in $\mathbb{Z} \times \mathbb{Z}$.

The author thanks John R. Isbell who provided the examples 3 and 4.

REFERENCES.


(Texte reçu le 9 janvier 1969.)

H. Subramanian,
Department of Mathematics,
State University of New-York,
Buffalo, N. Y. 14226-U. S. A.
(États-Unis).