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INTEGER-VALUED CONTINUOUS FUNCTIONS

BY

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Let $C(X, Z)$ denote the $\mathfrak{f}$-ring of all integer-valued continuous functions on a topological space X. $C^*(X, Z)$ stands for the sub-$\mathfrak{f}$-ring of all bounded functions in $C(X, Z)$. PIERCE [7] and ALLING [1] study $C(X, Z)$ from the point of view of the algebra of clopen sets of X. We investigate $C(X, Z)$ as an $\mathfrak{f}$-ring, and, therefore, from the point of view of its maximal l-ideals.

We prove the equivalence of the ring, the lattice and the (multiplicative) semigroup structures in $C^*(X, Z)$. We also give characterizations of $C^*(X, Z)$ as a lattice-ordered (l.-o.) ring, as a lattice-ordered (l.-o.) group and as a ring. We obtain these characterizations as soon as we observe that there exist sufficiently many characteristic functions with which every function in $C^*(X, Z)$ is expressed in a natural way. But, it is not clear, at present, how to characterize $C^*(X, Z)$ as a lattice or as a semigroup. Also, all these problems for $C(X, Z)$ remain open.

We consider only commutative rings with unit element. The notion of *hull-kernel* topology in any given collection of prime ideals of a ring is assumed to be known [1], [4], [5], [6], [7], [9], [10]. An $\mathfrak{f}$-ring is a l.-o. ring which is a subdirect product of totally-ordered (t.-o.) rings. An l-ideal I of a l.-o. ring is a (ring) ideal satisfying the property : $|x| \leq |y|$, $y \in I$ implies that $x \in I$. A maximal l-ideal of an $\mathfrak{f}$-ring is always prime [2]. If the intersection of all maximal l-ideals of a l.-o. ring is zero, the l.-o. ring is said to be l-semisimple. Thus an l-semisimple $\mathfrak{f}$-ring has no nonzero nilpotent elements.

We fix some notations for the rest of the paper : X and Y for arbitrary topological spaces; R for a ring or an $\mathfrak{f}$-ring; Z for the t.-o. ring of integers; $\mathfrak{X}$ for the hull-kernel space of all minimal prime ideals of the ring R; $\mathfrak{M}$ for the hull-kernel space of all maximal l-ideals of the $\mathfrak{f}$-ring R; δX for the Boolean space of the algebra of all clopen sets of X and σX for the space $\mathfrak{M}$ of $C(X, Z)$.

We lose nothing in the study of the ring-lattice structure of a $C(X, Z)$ if we assume that X is Hausdorff and has a base of clopen sets [7]. Therefore we consider only such spaces in this paper.

Remark 1. — For any point $x \in X$, let M_x denote $\{ f \in C(X, Z) \mid f(x) = 0 \}$. It is easy to verify that M_x is a maximal l-ideal of $C(X, Z)$. Since $\bigcap_{x \in X} M_x$ is the zero ideal, $\{ M_x \}_{x \in X}$ is a dense subspace of σX. Alling [1] has noted that X is homeomorphic to $\{ M_x \}_{x \in X}$ under the correspondence $x \to M_x$. Obviously, X is compact if, and only if, every maximal l-ideal of $C(X, Z)$ is of the form M_x for $x \in X$; that is, X is homeomorphic to σX. Thus $\sigma(\sigma X) = \sigma X$.

Theorem 1. — *A subset P of $C(X, Z)$ is a maximal l-ideal if, and only if, it is a minimal prime ideal.*

Proof. — For any $f \in C(X, Z)$, we denote by $\zeta(f)$ the zero set $\{ x \in X \mid f(x) = 0 \}$ of f. If F is a clopen set of X, χ_F stands for the characteristic function on F. Now, if P is a maximal l-ideal, it is necessarily prime [2]. To show that it is also minimal prime, consider any $f \in P$. $\zeta(f)$ is a clopen set of X, and $f . \chi_{\zeta(f)} = 0$. Since $1 \leq | f + \chi_{\zeta(f)} |$, it follows that $\chi_{\zeta(f)} \notin P$. Therefore, P is minimal prime [5], [6]. Conversely, let P be minimal prime. Surely then, P is an l-ideal [10]; so, it is contained in a maximal l-ideal, but which is also minimal prime. The desired result follows.

Remark 2. — The space $\mathfrak{M}$ of any f-ring is always compact Hausdorff [4], [9], and the space $\mathfrak{L}$ of any ring is always totally-disconnected [5], [6]. Thus σX is compact Hausdorff totally-disconnected (abbreviated in the sequel at CHT). Alling [1] shows that the space $\mathfrak{L}$ of $C(X, Z)$ is homeomorphic to δX. Therefore, σX is homeomorphic to δX. Pierce [7] has shown that $C^*(X, Z)$ and $C(\delta X, Z)$ are isomorphic as rings. They are indeed isomorphic as f-rings. It follows that any general theorem concerning $C(X, Z)$'s as f-rings will also be true of $C^*(X, Z)$'s. Theorem 1 above is a case in instance. We also note that the space $\mathfrak{L}$ or $\mathfrak{M}$ of $C^*(X, Z)$ ($\approx C(\delta X, Z) \approx C(\sigma X, Z)$) is σX.

The effective content of the following theorem has been announced by Sankaran [8], but no proof seems to have been published so far.

Theorem 2. — *The following are equivalent :*

(1) *$C^*(X, Z)$ and $C^*(Y, Z)$ are isomorphic as rings;*

(2) *$C^*(X, Z)$ and $C^*(Y, Z)$ are isomorphic as lattices;*

(3) *$C^*(X, Z)$ and $C^*(Y, Z)$ are isomorphic as p.-o. groups;*

(4) *$C^*(X, Z)$ and $C^*(Y, Z)$ are isomorphic as semigroups;*

(5) *σX and σY are homeomorphic.*

Proof. — A ring (resp. lattice) isomorphism between two l-semisimple $\mathfrak{f}$-rings induces a homeomorphism between their spaces of maximal l-ideals [4] (resp. [9]). Hence each of (1) and (2) implies (5). (3) implies(2) because any order-group isomorphism between two l.-o. groups preserves the lattice structures also. A semigroup ideal in a ring is minimal prime if, and only if, it is a minimal prime (ring) ideal [6]. Thus, by Remark 2, (4) implies (5). Finally, (5) implies that $C(\sigma X, Z)$ and $C(\sigma Y, Z)$ are isomorphic as $\mathfrak{f}$-rings. Remark 2 completes the proof.

REMARK 3. — Whether the above theorem is true with $C^*(X, Z)$ and $C^*(Y, Z)$ replaced respectively by $C(X, Z)$ and $C(Y, Z)$ is not known. We may refer the analogous case for $C(X)$ shown to be true by HENRIKSEN [3]. Similar to the realcompact spaces (the crucial point in [3]) in the theory of $C(X)$, ALLING [1] considers the space

$$\delta_0 X = \{ P \in \sigma X \mid C(X, Z)/P \approx Z \}.$$

It can be proved that $C(X, Z)$ and $C(\delta_0 X, Z)$ are isomorphic as $\mathfrak{f}$-rings. Only, we have to observe that, in $\delta_0 X$, the hull-kernel topology and the weak topology generated by the functions in $C(X, Z)$ are same. Now $C(X, Z)$ and $C(Y, Z)$ are isomorphic as rings if, and only if, $\delta_0 X$ and $\delta_0 Y$ are homeomorphic. Such a result with respect to the lattice (resp. semigroup) structure can be brought about if only we can show that any $P \in \delta_0 X$ can be obtained purely from the lattice (resp. semigroup) structure of $C(X, Z)$. The following examples counter any hasty predictions on a global generalization of Theorem 2 to all $\mathfrak{f}$-rings.

EXAMPLE 1 [9]. — The t.-o. field Q of rational numbers and the non-archimedean ordered ring $Q[x]$ of polynomials over Q are order-isomorphic, but not ring-isomorphic (not semigroup isomorphic also).

EXAMPLE 2 [9]. — The non-archimedean ordered ring $Z[x]$ of polynomials over Z and the subring $Z[\theta]$ of the real number field generated by Z and a transcendental number θ (with induced order) are ring-isomorphic, but not order isomorphic.

EXAMPLE 3. — Z and $Z[x]$ are semigroup isomorphic, but not ring isomorphic.

A semigroup isomorphism between Z and $Z[x]$ can be constructed as follows : Consider the set of all nonzero irreducible polynomials in x over Z, the coefficients of whose highest degree are positive. This is a countable set which can be put in one-to-one correspondence with the set of all prime numbers. This one-to-one correspondence is extended to a semigroup isomorphism between Z and $Z[x]$ in the natural way, making use of the fact that both Z and $Z[x]$ are unique factorization domains with exactly two unit elements, viz. 1 and — 1. But, in any ring isomorphism between Z and $Z[x]$, x should correspond to some integer, which means x equals the same integer, a contradiction.

However, the following theorem and corollaries are interesting in the face of the above remark and " warning post " examples. Before stating the theorem, we recall from [9] a definition.

DEFINITION. — Let R be an $\mathfrak{f}$-ring, M a maximal l-ideal of R and P a (proper) lattice-prime ideal of R. P is said to be associated with M if $y \in P$, $x(M) < y(M)$ imply that $x \in P$. [$x(M)$ denotes the homomorphic image of $x \in R$ in R/M.]

THEOREM 3. — *Let R be an $\mathfrak{f}$-ring, and M a maximal l-ideal of R. R/M is non-archimedean if, and only if, there exists a lattice-prime ideal P of R, associated with M, such that P contains all of* $\{1, 2, 3, \ldots\}$.

Proof. — If R/M is non-archimedean, there is an $f \in R$ such that $f(M) > n$ for every natural number n. Consider now

$$P = \{ g \in R \mid g(M) < f(M) \}.$$

P is a lattice-prime ideal of R, associated with M [9]. Evidently, P contains all of $\{1, 2, 3, \ldots\}$. Conversely, if such a P exists, choose $f \notin P$. Since P is associated with M, $f(M) > n$ for every natural number n. Surely then, R/M is non-archimedean.

COROLLARY 1. — *$C(X, Z)$ and $C(Y, Z)$ are isomorphic as rings if, and only if, they are Z-isomorphic* (i. e. *mapping constant functions into the same constant functions*) *as lattices.*

Proof. — The lattice structure of an $\mathfrak{f}$-ring R determines the space $\mathfrak{M}$ of R [9]. If $M \in \sigma X$, $C(X, Z)/M$ is either Z or a near η_1-set (which is not archimedean) [1]. Therefore, by Theorem 3, a Z-isomorphism between $C(X, Z)$ and $C(Y, Z)$ induces a homeomorphism between $\delta_0 X$ and $\delta_0 Y$. The result follows by Remark 3.

COROLLARY 2. — *$C(X)$ and $C(Y)$ are isomorphic as rings if, and only if, they are R-isomorphic as lattices* [*$C(X)$ denotes the $\mathfrak{f}$-ring of all real-(R)-valued continuous functions on X*].

Proof. — The same as in Corollary 1, with $\delta_0 X$ and $\delta_0 Y$ replaced respectively by νX and νY (νX stands for the realcompactification of X).

For use in the following two lemmas, we fix some terminology. If R is an $\mathfrak{f}$-ring, let us suppose that R is the subdirect product of the t.-o. rings R_α; that is, if we denote the projection from R onto R_α by φ_α with the l-ideal P_α as the kernel $\bigcap P_\alpha$ is (o).

LEMMA 1. — *Any idempotent e in a t.-o. ring R is either* o *or* 1.

Proof. — Since $e = e^2$, $e \geqq 0$. Either $e \leqq (1-e)$ or $(1-e) \leqq e$. In the first case, $e = e^2 \leqq e(1-e) = 0$. Thus $e = 0$. Since also $(1-e)^2 = (1-e)$, the second case shows that $e = 1$.

COROLLARY. — *If e is any idempotent in an $\mathfrak{f}$-ring, R, $e \wedge (1-e) = 0$.*

Proof. — Follows by observing that $\varphi_\alpha(e) = 0$ or 1 in R_α.

LEMMA 2. — *An $\mathfrak{f}$-ring R in which every element is a finite integral combination of idempotents is l-semisimple.*

Proof. — If $\{M_\beta\}$ is the collection of all maximal l-ideals of R, consider any $x \in \bigcap M_\beta$. Let $y = |x| \wedge 1$. Since y is an integral combination of idempotents, by Lemma 1, $\varphi_\alpha(y)$ is an integer in R_α. Since also $0 \leqq y \leqq 1$, $\varphi_\alpha(y)$ is 0 or 1; and, every P_α contains either y or $(1-y)$. If some P_{α_0} contains $(1-y)$, we get $(1-y) \in M_{\beta_0}$, where M_{β_0} is a maximal l-ideal containing P_{α_0}. Since also $y \in M_{\beta_0}$, this is a contradiction. Therefore $y \in \bigcap P_\alpha = (0)$. Now $|x| \wedge 1 = 0$ implies that $x = 0$, because $|\varphi_\alpha(x)| \wedge 1 = 0$.

THEOREM 4. — *An $\mathfrak{f}$-ring R as described in Lemma 2 is ring-lattice isomorphic to a $C(X, Z)$ for some (unique upto homeomorphism)* CHT *space X.*

Proof. — The obvious choice for the space X should be the space $\mathfrak{M}$ of R. Given $x \in R$ and $M \in \mathfrak{M}$, we have

$$x = n_0 . 1 + n_1 e_1 + \ldots + n_r e_r$$

where $n_k \in Z$ and $e_k^2 = e_k \in M$. For,

$$x = \sum m_k i_k, \qquad m_k \in Z, \qquad i_k^2 = i_k$$

can be rewritten by changing an i_k into $1-(1-i_k)$ whenever $i_k \notin M$. If possible, let also

$$x = n'_0\, 1 + n'_1 e'_1 + \ldots + n'_s e'_s, \qquad n'_k \in Z, \qquad e'^2_k = e'_k \in M.$$

If $n_0 > n'_0$, then $1 \leqq (n_0 - n'_0) 1 \in M$; thus $1 \in M$, which is a contradiction. $n_0 < n'_0$ is ruled out likewise. Thus $n_0 = n'_0$. Every $M \in \mathfrak{M}$ now induces a map $\psi_M : R \to Z$, $\psi_M(x)$ being the integer n_0 as we have just obtained.

It is easily seen that ψ_M is a ring homomorphism from R onto Z with M as the kernel. Since M is a maximal l-ideal, there is a canonical total order in the ring R/M, i. e., Z. The uniqueness of a compatible total order in the ring Z shows that ψ_M preserves lattice structure also. Because of l-semisimplicity, by Lemma 2, R is thus lattice-ring isomorphic [by the transform $x \to \tilde{x}$; $\tilde{x}(M) = \psi_M(x)$] to a sublattice-subring of the $\mathfrak{f}$-ring of all integer-valued functions on $\mathfrak{M}$.

If $x = \sum n_k e_k$, $n_k \in Z$ and $e_k^2 = e_k$, then $|\tilde{x}| \leq \sum |n_k|$. So x has finite range which effects a finite partition of the space $\mathfrak{M}$. Each coset, x being a constant in it, is exactly the collection of all maximal l-ideals containing $x - k.1$ for some $k \in Z$; and so it is closed. It follows that each coset is open. Hence $\tilde{x}$ is continuous because the preimage of every single point open set in the discrete space Z is open.

The map $x \to \tilde{x}$ from R to $C(\mathfrak{M}, Z)$ is also onto. Since $\mathfrak{M}$ is always compact Hausdorff [4], [9], any $f \in C(\mathfrak{M}, Z)$ is a finite sum $\sum n_k \chi_{E_k}$, where each E_k is a clopen set of $\mathfrak{M}$. But E_k is the hull of a direct summand of R, and thereforet he hull of an idempotent $i_k \in R$. Obviously, $\tilde{j}_k = \chi_{E_k}$, where $j_k = 1 - i_k$. Thus if $x = \sum n_k j_k$, then $\tilde{x} = f$.

The proof is complete on showing that $\mathfrak{M}$ is a totally disconnected space. Every $x \in M \in \mathfrak{M}$ is a finite integral combination of idempotents within M; so distinct elements of $\mathfrak{M}$ contain in them different collections of idempotents. Therefore any two points of $\mathfrak{M}$ are separated by a clopen set, and $\mathfrak{M}$ is totally disconnected. The uniqueness follows from Theorem 2 and Remark 1.

Let G denote a l.-o. Abelian group with strong order unit 1 (i. e. an element contained in no maximal l-subgroup). We define an *idempotent* in G to be a relatively complemented element in the interval $(0, 1)$. The property of being a strong order unit is preserved under any group-lattice homomorphism. Since G is a subdirect product of t.-o. groups, $e \in (G, 1)$ is an idempotent if, and only if, $e \wedge (1 - e) = 0$. Any maximal l-subgroup of G is the kernel of a group-lattice homomorphism of G onto a t.-o. group, and contains one, and only one, of the conjugate idempotents e and $(1 - e)$. Any clopen set in the hull-kernel space (certainly compact T_1) of maximal l-ideals of $(G, 1)$ is given by the hull of an idempotent. In Summary, we obtain the following theorem.

THEOREM 5. — *Lemma 2 and Theorem 4 are true in the set up of a l.-o. Abelian group with strong order unit in the place of an $\mathfrak{f}$-ring.*

LEMMA 3. — *A ring R, whose additive group is torsion-free, and whose elements are finite integral combinations of idempotents, does not have any nonzero nilpotent elements.*

Proof. — Let R be the subdirect product of subdirectly irreducible rings R_a. If we denote by φ_a the projection $R \to R_a$ with kernel K_a, $\bigcap K_a$ is (0). Consider any $x \in R$ such that $x^n = 0$, $n \neq 1$. Let

$$x = n_1 e_1 + \ldots + n_k e_k,$$

where $n_k \in Z$ and $e_2^k = e_k$. Since a subdirectly irreducible ring cannot contain any idempotent except 0 and 1, we have either $x \in K_a$

or $x \equiv m_a \pmod{K_a}$ $(m_a \neq 0)$ for each K_a, where $m_a \in Z$. But each m_a so obtained is a sum of some or all of $n_1, \ldots, n_k$. Thus there are only finitely many among the m_a's which are distinct; if they are $m_1, m_2, \ldots, m_r$, we see that $m_1^n \ldots m_r^n . x \in \bigcap K_a = (0)$. This means x is 0, because the additive group is torsion-free.

In all the following lemmas, let R denote a ring as described in Lemma 3. We proceed to prove that such a ring R characterizes a $C(X, Z)$ for some CHT space X.

LEMMA 4. — *R is isomorphic to a subring of a $C^*(X, Z)$.*

Proof. — Choose X to be the space $\mathfrak{P}$ of R. For any $x \in R$ and $P \in \mathfrak{P}$, obtain $n_0 \in Z$ exactly as in Theorem 4. If this n_0 is not unique, there exists $n \in Z^+$ such that $n . 1 \in P$. P being minimal prime, $nx = (n . 1)x = 0$ for some $x \notin P$ [5], [6]. This contradicts the assumption that the additive group of R is torsion-free. The result of the proof is exactly like in Theorem 4 using Lemma 3 in place of Lemma 2.

LEMMA 5. — *If $P, Q \in \mathfrak{P}$ are distinct, then $P + Q = R$.*

Proof. — Choose $x \in P \sim Q$. Express x as a sum of idempotents within P, as in Theorem 4. One of these idempotents, say e, is necessarily not in Q. Thus $(1 - e) \in Q$, and $P + Q = R$.

COROLLARY. — *Any maximal ideal of R contains a unique minimal prime ideal.*

For every maximal ideal M of R, let $\pi(M)$ denote

$$\{ x \in R \mid xy = 0 \text{ for some } y \notin M \}.$$

It is easily seen that $\pi(M)$ is an ideal of R and $\pi(M) \subset M$.

LEMMA 6. — *For any $P \in \mathfrak{P}$, $P = \pi(M)$ for any maximal ideal M of R such that $P \subset M$.*

Proof. — If $x \in \pi(M)$, let $xy = 0$, $y \notin M$. Then $y \notin P$, implying that $x \in P$. Conversely, let $x \notin \pi(M)$. Consider the multiplicatively closed subset $S = \{ x^n y \mid y \notin M \} \cup \{ R \sim M \}$. It is easily checked that $\pi(M)$ is disjoint with S. Therefore, by Zorn's Lemma, there exists a prime ideal Q containing $\pi(M)$ and maximally disjoint with S. Since $Q \subset M$, P should be, by Corollary to Lemma 5, the only minimal prime ideal contained in Q. Since $x \in S$, $x \notin P$.

COROLLARY. — *For any $P \in \mathfrak{P}$, $x \in P$ if, and only if, $x^* + P = R$, where x^* denotes the annihilator ideal $\{ y \in R \mid yx = 0 \}$ of x.*

Proof. — Let $x \in P$. If M is any maximal ideal containing P, M cannot contain x^* by Lemma 6. Thus the ideal generated by x^* and P together is R. The converse is obvious.

LEMMA 7. — *$\{h(y) \mid y \in R\}$ is a base for open sets in $\mathfrak{X}$, where for any $A \subseteq R$, $h(A) = \{P \in \mathfrak{X} \mid A \subseteq P\}$, the hull of A in $\mathfrak{X}$.*

Proof. — By Lemma 6, $h(y) = h'(y^*)$ for any $y \in R$, where $h'(A)$ denotes the complement of the hull of A in $\mathfrak{X}$. Choose any $x \in R$ and $P \in \mathfrak{X}$. Let $n_1, n_2, \ldots, n_k$ be all the nonzero values of the function $\tilde{x}$, defined as in Lemma 4. Consider $y = (x - n_1) \ldots (x - n_k)$. Suppose that $x \notin P$. Then $y \in P$; and, for any $Q \in \mathfrak{X}$ such that $y \in Q$, $x \notin Q$. Thus $P \in h(y) \subseteq h'(x)$. Since $\{h'(x) \mid x \in R\}$ forms a base for open sets in $\mathfrak{X}$, the result follows.

By Corollary to Lemma 5 and Lemma 6, π is a map defined from the hull-kernel space $\mathfrak{M}$ of all maximal ideals of R onto $\mathfrak{X}$. We establish :

LEMMA 8. — *π is continuous.*

Proof. — For any $y \in R$, $\pi^{-1}(h(y)) = \{M \in \mathfrak{M} \mid y \in \pi(M)\}$. By definition of $\pi(M)$, it follows that $\pi^{-1}(h(y)) = \{M \in \mathfrak{M} \mid y^* \nsubseteq M\}$, which is an open set in $\mathfrak{M}$. Since $h(y)$ is a basic open set in $\mathfrak{X}$, π is continuous.

Since $\mathfrak{M}$ is always compact, we have :

COROLLARY. — *The space $\mathfrak{X}$ of R is compact.*

LEMMA 9. — *If F is any clopen set in the space $\mathfrak{X}$ of R, then $F = h(e)$, where e is an idempotent in R.*

Proof. — $\pi^{-1}(F)$ is clopen in $\mathfrak{M}$; therefore, there exists $e \in R$ such that $e = e^2$ and $\pi^{-1}(F) = \{M \in \mathfrak{M} \mid e \in M\}$. If $P \in h(e)$ and $P \subseteq M \in \mathfrak{M}$, then $P = \pi(M)$ and $M \in \pi^{-1}(F)$. Therefore $P \in F$. Conversely, if $P \in F$, let $P = \pi(M)$ for some $M \in \mathfrak{M}$. Then $e \in M$. Since $e = e^2$ and $1 - e \notin P$, $e \in P$. That is, $P \in h(e)$.

We now state :

THEOREM 6. — *A ring R, as described in Lemma 3 is ring isomorphic to a $C(X, Z)$ for some (unique upto homeomorphism)* CHT *space X.*

Proof. — Choose $\mathfrak{X}$ to be the space X, Using Lemma 4, Corollary to Lemma 8, and Lemma 9, modify the proof of Theorem 4 accordingly.

REMARK 4. — The converse of each of Theorems 4, 5 and 6 are clearly true. In fact, for any $f \in C^*(X, Z)$, we have

$$f = \sum_{n \in f(x) - \{0\}} n \chi_{f^{-1}(n)}$$

and this is the unique representation of f as an integral combination with the fewest number of orthogonal idempotents. We could simplify the proofs of Theorems 4, 5 and 6 with such a strong representation of elements. But a weaker hypothesis as we have assumed in these Theorems are enough.

REMARK 5. — Neither Lemma 2 (hence Theorem 4) nor Lemma 3 (hence Theorem 6) extends to non-commutative cases. The necessary example to illustrate this is given below.

EXAMPLE 4. — In the additive group $Z \times Z$, define $(a, b)(c, d) = (ac, ad)$ and give the lexicographic order. This non-commutative (f-)ring is additively generated by the idempotents $(1, b)$. $(0, 1)$ is nilpotent and generates the only maximal l-ideal in $Z \times Z$.

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