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A GENERALIZED COMMUTATION RELATION FOR THE REGULAR REPRESENTATION

BY

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Introduction. — To a locally compact group G with a left invariant Haar measure ds, there correspond three von Neumann algebras on the Hilbert space $L^2(G)$ of all square integrable functions on G with respect to ds. Namely, the first one is the algebra $A(G)$ of all multiplication operators $\pi(f)$ defined by functions f of $L^\infty(G)$; the second one is the von Neumann algebra $M(G)$ generated by the left regular representation U of G; the last one is the von Neumann algebra $M'(G)$ generated by the right regular representation V of G. These three algebras satisfy the following commutation relation :

$$A(G)' = A(G); \qquad M(G)' = M'(G).$$

Usually one's attention has been concentrated mainly on $M(G)$ or $M'(G)$. But, as the author pointed out in the previous paper [17], it is very important, for the study of the group, to attack the relation of these three algebras. Indeed, a great deal of work has been done consciously or unconsciously on this subject. We can pursue its history back to the study of the Heisenberg commutation relation :

$$[p, q] = pq - qp = - i\, 1.$$

Among various results, von Neumann's unicity theorem for Schrödinger operators (*see* [8], [9], [13] and [14]) and Mackey's theory of induced representations (*see* [6], [10], [11] and [12]) are remarkable.

In this paper, we shall show a firm interdependence of these three algebras by proving the following generalized commutation relation

$$M(G/H, U(K))' = M(K\backslash G, V(H))$$

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for two closed subgroups H and K of a separable locally compact group G, where $M(G/H, U(K))$ [resp. $M(K\backslash G, V(H))$] denotes the von Neumann algebra generated by the $U(s)$, $s\in K$ [resp. $V(s)$, $s\in K$] and the $\pi(f)$, $f\in L^\infty(G/H)$ [resp. $\pi(f)$, $f\in L^\infty(K\backslash G)$]. Our method of arguments requires that we restrict ourselves to separable groups. (*See* the remark at the end of § 2.)

Before going into discussion, the author would like to express here his hearty thanks to Professor J. M. G. FELL who read the manuscript of this article and made several illuminating comments. Also, the author is indebted to Professor R. KADISON for his kind hospitality at the University of Pennsylvania.

1. Imprimitivity algebras.

Let Γ be a locally compact space and G be a locally compact transformation group on Γ. The action of G on Γ is denoted by γs, or $s\gamma$, $\gamma\in\Gamma$, $s\in G$. Naturally, we call G a right or left transformation group on Γ according to the side of the action and write the pair as (Γ, G) or (G, Γ). In this section, suppose G is a right transformation group. Of course, every result of this section can be translated into a result on left transformation groups. Let ds denote a left invariant Haar measure of G. Suppose μ is a fixed quasi-invariant Radon measure on Γ. For the sake of simplicity, suppose there exists a continuous positive function $\lambda(\gamma, s)$ on $\Gamma\times G$ such that

$$\int_\Gamma f(\gamma s)\,\lambda(\gamma, s)\,d\mu(\gamma) = \int_\Gamma f(\gamma)\,d\mu(\gamma), \qquad s\in G, \tag{1}$$

for every continuous function f on Γ with compact support

$$\begin{cases} \lambda(\gamma, rs) = \lambda(\gamma, r)\,\lambda(\gamma r, s), & \gamma\in\Gamma, \quad r, s\in G; \\ \lambda(\gamma, e) = 1. & \end{cases} \tag{2}$$

Let $\mathcal{K}$ denote the space of all continuous functions on $\Gamma\times G$ with compact support. Define a $\star$-algebra structure in $\mathcal{K}$ as follows:

$$\begin{cases} f\star g(\gamma, s) = \displaystyle\int_G f(\gamma, r)\,g(\gamma r, r^{-1}s)\,dr; \\ f^\star(\gamma, s) = \Delta_G(s)^{-\frac{1}{2}}\lambda(\gamma, s)^{\frac{1}{2}}\overline{f(\gamma s, s^{-1})}, \end{cases} \tag{3}$$

where Δ_G means the modular function of G. Furthermore, we shall define

$$\begin{cases} f^\wedge(\gamma, s) = \Delta_G(s)^{\frac{1}{2}}\lambda(\gamma, s)^{-\frac{1}{2}}f(\gamma, s); \\ f^\vee(\gamma, s) = \Delta_G(s)^{-\frac{1}{2}}\lambda(\gamma, s)^{\frac{1}{2}}f(\gamma, s), \qquad f\in\mathcal{K}. \end{cases} \tag{4}$$

Then we have $(f^\wedge)^* = (f^*)^\vee$. Now, define an inner product in $\mathcal{K}$ by

$$(5)\qquad (f \mid g) = \iint_{\Gamma\times G} f(\gamma, s)\,\overline{g(\gamma, s)}\, d\mu(\gamma)\, ds.$$

Then we can easily show that

$$(6)\qquad \begin{cases} (f \mid g) = (g^* \mid f^*); \\ (f \star g \mid h) = (g \mid (f^\wedge)^* \star h), \qquad f, g, h \in \mathcal{K}. \end{cases}$$

The completion of the pre-Hilbert space $\mathcal{K}$ is nothing but $L^2(\Gamma\times G, \mu\otimes ds)$.

Then by DIXMIER's result ([4], prop. 9), we get the following theorem.

THEOREM 1. — *$\mathcal{K}$ is a quasi-Hilbert algebra.*

Therefore, the left von Neumann algebra $\mathfrak{U}(\mathcal{K})$ and the right one $\mathfrak{V}(\mathcal{K})$ of $\mathcal{K}$ are the commutants of each other. We shall call them the left imprimitivity von Neumann algebra or the right imprimitivity von Neumann algebra of a topological transformation group (Γ, G) with the quasi-invariant measure μ. For each f of $\mathcal{K}$, let $u(f)$ and $v(f)$ denote the operators defined by $u(f)\xi = f\star\xi$ and $v(f)\xi = \xi\star f$, $\xi\in\mathcal{K}$, respectively.

To each $f\in L^\infty(\Gamma, \mu)$, there corresponds a bounded operator $\tilde{\pi}(f)$ on $L^2(\Gamma\times G, \mu\otimes ds)$ defined by the equality

$$(\tilde{\pi}(f)\,\xi)(\gamma, s) = f(\gamma s)\,\xi(\gamma, s), \qquad \xi\in L^2(\Gamma\times G, \mu\otimes ds).$$

Also, there is a unitary representation $\tilde{V}$ of G on $L^2(\Gamma\times G, \mu\otimes ds)$ defined by

$$(\tilde{V}(s)\xi)(\gamma, r) = \Delta_G(s)^{\frac{1}{2}}\xi(\gamma, rs), \qquad s\in G.$$

Then a direct calculation shows that $\tilde{\pi}(f)$ and $\tilde{V}(s)$ commutes with $u(g)$, $g\in\mathcal{K}$, so that $\tilde{\pi}(f)$ and $\tilde{v}(s)$ are in $\mathfrak{V}(\mathcal{K})$. Then we have the following by [4], prop. 10 :

THEOREM 2. — *$\mathfrak{V}(\mathcal{K})$ is generated by the $\tilde{\pi}(f)$ and $\tilde{V}(s)$, $f\in L^\infty(\Gamma, \mu)$ and $s\in G$.*

2. **Generalized commutation relation.**

Suppose G is a separable locally compact group with a left invariant Haar measure ds. Take and fix two arbitrary closed subgroups H and K of G, whose left invariant Haar measures are denoted by $d_H r$ and $d_K t$ respectively. Let ${}_K\Gamma$ and Γ_H denote the right homogeneous space $K\diagdown G$ and the left one G/H respectively. For each $s\in G$, let ${}_K\gamma(s)$ and $\gamma_H(s)$ denote the right coset Ks and the left one Hs respectively. Then both ${}_K\Gamma$ and Γ_H become locally compact spaces with respect to the quotient topologies. G acts on both spaces ${}_K\Gamma$ and Γ_H as a topological transformation group. The action of G on ${}_K\Gamma$ is given by

$$ {}_K\gamma(s)t = {}_K\gamma(st), \qquad s, t \in G; $$

the action of G on Γ_H is also given by

$$ t\gamma_H(s) = \gamma_H(ts), \qquad s, t \in G. $$

By [1] (Chap. VII, § 2, Théorème 2), there exist essentially unique quasi-invariant measure μ on Γ_H and ν on ${}_K\Gamma$ respectively. By [1] (Chap. VII, § 2, Théorème 2), there exists a continuous function $\lambda_H(s, \gamma)$ [resp. ${}_K\lambda(\gamma, s)$] of $G \times \Gamma_H$ (resp. ${}_K\Gamma \times G$) with the properties

$$ \int_{\Gamma_H} f(s^{-1}\gamma)\, \lambda_H(s, \gamma)\, d\mu(\gamma) = \int_{\Gamma_H} f(\gamma)\, d\mu(\gamma), $$

resp.

$$ \int_{{}_K\Gamma} f(\gamma s)\, {}_K\lambda(\gamma, s)\, d\nu(\gamma) = \int_{{}_K\Gamma} f(\gamma)\, d\nu(\gamma); $$

$$ \lambda_H(st, \gamma) = \lambda_H(s, \gamma)\, \lambda_H(t, s^{-1}\gamma); \qquad \lambda_H(e, \gamma) = 1, $$

resp.

$$ {}_K\lambda(\gamma, st) = {}_K\lambda(\gamma, s)\, {}_K\lambda(\gamma s, t); \qquad {}_K\lambda(\gamma, e) = 1, \qquad s, t \in G. $$

Since for each subset S of Γ_H (resp. ${}_K\Gamma$) to be a μ-null (resp. ν-null) set it is necessary and sufficient that $\gamma_H^{-1}(S)$ [resp. ${}_K\gamma^{-1}(S)$] is a ds-null set, every function $f \in L^\infty(\Gamma_H, \mu)$ [resp. $f \in L^\infty({}_K\Gamma, \nu)$] can be regarded as an element of $L^\infty(G, ds)$ which is constant on each coset Hs (resp. sK); so the representation π of $L^\infty(G, ds)$ on $L^2(G, ds)$, defined by

$$ \pi(f)\xi(s) = f(s)\xi(s), \qquad f \in L^\infty(G, ds) \quad \text{and} \quad \xi \in L^2(G, ds), $$

can be applied to the elements of $L^\infty(\Gamma_H, \mu)$ and of $L^\infty({}_K\Gamma, \nu)$. Namely, $\pi(f)$, $f \in L^\infty(\Gamma_H, \mu)$ [resp. $f \in L^\infty({}_K\Gamma, \nu)$], is defined by

$$ (\pi(f)\xi)(s) = f(\gamma_H(s))\,\xi(s); $$

resp.

$$ (\pi(f)\xi)(s) = f({}_K\gamma(s))\xi(s), \qquad \xi \in L^2(G, ds). $$

Remark that $L^\infty(\Gamma_H, \mu)$ [resp. $L^\infty({}_K\Gamma, \nu)$] is the fixed algebra of the automorphism group H (resp. K) of $L^\infty(G, ds)$ which acts on $L^\infty(G, ds)$ as right (resp. left) translation. Let U and V be the left and right regular representations of G respectively. Now, let $M(G/H, U(K))$ [resp. $M(K\backslash G, V(H)]$ denote the von Neumann algebra on $L^2(G, ds)$ generated by $\pi(f)$, $f \in L^\infty(\Gamma_H, \mu)$ [resp. $f \in L^\infty({}_K\Gamma, \nu)$], and $U(s)$, $s \in K$ [resp. $V(s)$, $s \in H$].

THEOREM 3. — $M(G/H, U(K))' = M(K\backslash G, V(H))$, *equivalently*

$$M(K\backslash G, V(H))' = M(G/H, U(K)).$$

Taking particular subgroups H and K, we get various commutation relations concerning G. For example, if $G = H = K$, then our theorem turns out to be the commutation relation for the left and right regular representations which was shown by DIXMIER [4]. If $H = G$ and $K = \{ e \}$, then it tells us that the covariant representation (π, V) of $(L^\infty(G, ds), G)$ is irreducible.

Proof. — Since it is clear that $M(G/H, U(K))' \supset M(K\backslash G, V(H))$, we shall prove only the reverse inclusion.

By the stage theorem on induced representations [10], the left regular representation U of G is induced by the left regular representation U_H of H. Therefore, $L^2(G, ds)$ can be regarded as the Hilbert space of all $L^2(H, d_H r)$-valued functions $\xi(s)$ on G with properties

$$\xi(sr) = U_H(r)^{-1}\xi(s), \qquad r \in H, \qquad s \in G; \tag{1}$$

$$\int_{\Gamma_H} \|\xi(s)\|^2 \, d\mu(\gamma_H(s)) < \infty, \tag{2}$$

where integration (2) is justified by the fact that $\|\xi(s)\|$ can be regarded as a function on Γ_H because of property (1). The representations $U(s)$, $s \in G$, and $\pi(f)$, $f \in L^\infty(\Gamma_H, \mu)$, are defined by the equalities

$$(U(s)\xi)(t) = \lambda_H(s, \dot{t})^{\frac{1}{2}}\xi(s^{-1}t), \qquad t \in G, \qquad \xi \in L^2(G, ds); \tag{3}$$

$$(\pi(f)\xi)(t) = f(\dot{t})\xi(t), \tag{4}$$

where $\dot{t}$ means $\gamma_H(t)$.

Take a bounded operator x on $L^2(G)$ commuting with $\pi(L^\infty(\Gamma_H))$. Then there exists a bounded $\mathfrak{B}(L^2(H))$-valued Borel function $x(s)$ on G with the properties

$$x(sr) = U_H(r)^{-1}x(s)U_H(r) \tag{5}$$

for every $r \in H$ and almost every $s \in G$ and

$$(x\xi)(s) = x(s)\xi(s), \qquad \xi \in L^2(G). \tag{6}$$

For $s \in G$, we have

$$(U(s)xU(s)^{-1}\xi)(t) = x(s^{-1}t)\xi(t), \qquad \xi \in L^2(G).$$

So if x is in $M(G/H, U(K))'$, then we have

$$x(tsr) = U_H(r)^{-1}x(s)U_H(r) \tag{7}$$

for every $r \in H$, $t \in K$ and almost every $s \in G$. Therefore, we can define a normal faithful representation σ of $M(G/H, U(K))'$ on the Hilbert space $L^2({}_K\Gamma) \otimes L^2(H)$ by the direct integral

$$\sigma(x) = \int_{{}_K\Gamma}^{\oplus} x(\dot{s})\, d\nu(\dot{s}), \tag{8}$$

where $x(\dot{s})$ is the $\mathfrak{B}(L^2(H))$-valued function on ${}_K\Gamma$ which is naturally defined by the fact that $x(s)$ is constant on each right K-coset set Ks. To get the conclusion, it is sufficient to show that

$$\sigma(M(G/H, U(K))') = \sigma(M(K\backslash G, V(H))).$$

First of all, let us determine $\sigma(\pi(f))$, $f \in L^\infty({}_K\Gamma)$, and $\sigma(V(p))$, $p \in H$. If $x = \pi(f)$, $f \in L^\infty({}_K\Gamma)$, then $x(s)$ is, for almost every $s \in G$, a multiplication operator on $L^2(H)$ defined by the function: $r \in H \mapsto f(sr)$, so that we have

$$(\sigma(x)\xi)(\dot{s}, r) = f(\dot{s}r)\xi(\dot{s}, r), \qquad \xi \in L^2({}_K\Gamma \times H, \nu \otimes d_H r). \tag{9}$$

If $x = V(p)$, $p \in H$, then $x(s)$ is constantly $V_H(p)$; so $\sigma(x)$ is simply given by

$$(\sigma(x)\xi)(\dot{s}, r) = \Delta_H(p)^{\frac{1}{2}}\xi(\dot{s}, rp), \qquad \xi \in L^2({}_K\Gamma \times H), \tag{10}$$

where Δ_H means the modular function of H. Therefore, applying Theorem 2 to the topological transformation group $({}_K\Gamma, H)$, the von Neumann algebra $\sigma(M(K\backslash G, V(H))$, generated by $\sigma(\pi(f))$, $f \in L^\infty({}_K\Gamma)$, and $\sigma(V(p))$, $p \in H$, turns out to be the right imprimitivity von Neumann algebra of $({}_K\Gamma, H)$. Let $\mathcal{K}$ denote the space of all continuous functions on ${}_K\Gamma \times H$ with compact support, in which a quasi-Hilbert algebra structure is given as in paragraph 1. Then we have

$$\sigma(M(K\backslash G, V(H))) = \mathcal{V}(\mathcal{K}).$$

Therefore to complete the proof, it is sufficient to show that

$$\sigma(M(G/H, U(K))') \subset \mathcal{U}(\mathcal{K})'.$$

Regarding each element $\xi \in L^2({}_K\Gamma \times H)$ as a $L^2(H)$-valued function on ${}_K\Gamma$, let $\xi_{\dot{s}}$ denote the value of ξ in $L^2(H)$ at $\dot{s} \in {}_K\Gamma$. Then we have, for each $f \in \mathcal{K}$, $\xi \in \mathcal{K}$,

$$(u(f)\xi)_{\dot{s}} = \int_H f(\dot{s}, p)\, U_H(p)\xi_{\dot{s}p}\, d_H p.$$

Take an arbitrary x of $M(G/H, U(K))'$. Then we have, for each $f \in \mathcal{K}$, $\xi \in \mathcal{K}$,

$$\begin{aligned}(\sigma(x)\, u(f)\xi)_{\dot{s}} &= x(\dot{s})\, (u(f)\xi)_{\dot{s}} \\ &= x(\dot{s}) \int_H f(\dot{s}, p)\, U_H(p)\xi_{\dot{s}p}\, d_H p \\ &= \int_H f(\dot{s}, p)\, x(\dot{s})\, U_H(p)\xi_{\dot{s}p}\, d_H p \\ &= \int_H f(\dot{s}, p)\, U_H(p) x(\dot{s}p)\xi_{\dot{s}p}\, d_H p \quad \text{by (7)}, \\ &= \int_H f(\dot{s}, p)\, U_H(p)\, (\sigma(x)\xi)_{\dot{s}p}\, d_H p \\ &= (u(f)\, \sigma(x)\xi)_{\dot{s}}\,.\end{aligned}$$

Therefore, we have $\sigma(x)\, u(f) = u(f)\, \sigma(x)$, which implies that $\sigma(x) \in \mathfrak{U}(\mathcal{K})'$. This completes the proof.

Remark. — Because of direct integral (8), we must assume the separability for the given group G. However, it is very likely that Theorem 3 (and then Theorem 4) remains true without separability assumption. Indeed if we can define the isomorphism σ of $M(G/H, U(K))'$ without making use of direct integral, then we can get rid of the separability assumption.

3. Connection with a generalized Heisenberg commutation relation.

In general, the Heisenberg commutation relation is presented in the form

$$(1) \qquad [p, q] = pq - qp = -i\, 1$$

for self-adjoint operators p, q on a Hilbert space. On taking exponentials, equality (1) becomes

$$(2) \qquad U(s)\, V(t)\, V(s)^{-1}\, V(t)^{-1} = e^{-ist}\, 1$$

for one-parameter unitary groups $U(s)$ and $V(t)$. In this form, we can get rid of the difficulties of unboundedness of p, q or of their domains. Equality (2) is easily generalized to a commutation relation based on a locally compact abelian group, say G, as follows:

$$(3) \qquad U(s)\, \hat{V}(\hat{t})\, U(s)^{-1}\, \hat{V}(\hat{t})^{-1} = \overline{\langle s, \hat{t} \rangle}\, 1,$$

where $U(s)$ and $\hat{V}(\hat{t})$ are unitary representations of G and of its dual $\hat{G}$ on same Hilbert space and $\langle s, \hat{t} \rangle$ means the value of the character $\hat{t} \in \hat{G}$

at $s \in G$. We shall call the representations U and $\hat{V}$ covariant if U and V satisfy equality (3). As was shown by von Neumann in [14] long ago, the covariant representations of G and $\hat{G}$ are essentially unique, and are given on $L^2(G)$ by the equalities

$$(4) \qquad \begin{cases} (U(s)\xi)(x) = \xi(s^{-1}x), & s, x \in G; \\ (\hat{V}(\hat{\ })\xi)(x) = \langle x, \hat{t} \rangle \xi(x), & x \in G, \quad \hat{t} \in \hat{G}. \end{cases}$$

The above covariant representations are irreducible, that is, they have no non-trivial closed invariant subspace in common. The von Neumann algebra $M(G)$ [resp. $M(\hat{G})$], generated by $U(s), s \in G$ [resp. $\hat{V}(\hat{t}), \hat{t} \in \hat{G}$], is maximal abelian.

Now, take closed subgroups H of G and $\hat{K}$ of $\hat{G}$. Let $M(H, \hat{K})$ denote the von Neumann algebra generated by $U(s), s \in H$, and $\hat{V}(\hat{t}), \hat{t} \in \hat{K}$. As $\hat{K}$ is the dual group of the quotient group $G/\hat{K}^{\perp}$ of $\hat{G}$ by the annihilator $\hat{K}^{\perp}$ of $\hat{K}$, the representation $\hat{V}(\hat{t})$ of $\hat{K}$ induces naturally the multiplication representation π of the von Neumann algebra $L^{\infty}(G/\hat{K}^{\perp})$ of all bounded measurable functions in $G/\hat{K}^{\perp}$. Therefore, the von Neumann algebra $M(H, \hat{K})$ turns out to be the $M(G/K^{\perp}, UH)$ of the previous section. Thus we get the following :

THEOREM 4. — *If G is a separable locally compact abelian group, then we have $M(H, \hat{K})' = M(\hat{K}^{\perp}, H^{\perp})$, for each closed subgroups H of G and $\hat{K}$ of $\hat{G}$, where $H^{\perp}$ is the annihilator of H in $\hat{G}$.*

Added in proof. — For the additive group of Hilbert space, ARAKI shown, in [21], the same commutation theorem for closed vector subspaces in the Fock representation of canonical commutation relation.

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