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DERIVATIONS OF SIMPLE C^*-ALGEBRAS, II

BY

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1. Introduction.

In the previous paper [8], the author proved that every derivation of a simple C^*-algebra with unit is inner.

On the other hand, there exist simple C^*-algebras without unit whose derivations are not necessarily inner (actually, the author does not know an example of a simple C^*-algebra without unit, in which all derivations are inner) — for example, the C^*-algebra $C(\mathfrak{H})$ of all compact operators on an infinite dimensional Hilbert space $\mathfrak{H}$, and the C^*-subalgebra $\mathfrak{F}$ of a II_∞-factor $\mathfrak{M}$ generated by all finite projections. All of these examples satisfy the following conditions :

Let $\mathfrak{a}$ *be a simple* C^**-algebra. Then there exists a primitive* C^**-algebra* $\mathfrak{D}$ *with unit as follows :*

(1) $\mathfrak{a}$ *is a two-sided ideal of* $\mathfrak{D}$ [*consequently, every element c in* $\mathfrak{D}$ *defines a derivation* γ *on* $\mathfrak{a}$ *with* $\gamma(x) = [c, x]$ $(x \in \mathfrak{a})$];

(2) *for every derivation* δ *on* $\mathfrak{a}$, *there exists an element d (unique modulo scalar multiples of unit) in* $\mathfrak{D}$ *such that* $\delta(x) = [d, x]$ $(x \in \mathfrak{a})$;

(3) *every derivation of* $\mathfrak{D}$ *is inner.*

For example, if $\mathfrak{a} = C(\mathfrak{H})$, then $\mathfrak{D} = B(\mathfrak{H})$, where $B(\mathfrak{H})$ is the W^*-algebra of all bounded operators on $\mathfrak{H}$; if $\mathfrak{a} = \mathfrak{F}$, then $\mathfrak{D} = \mathfrak{M}$. If $\mathfrak{a}$ has unit, then $\mathfrak{a} = \mathfrak{D}$ and so this implies that every derivation of $\mathfrak{a}$ is inner. Moreover, these paired algebras $(C(\mathfrak{H}), B(\mathfrak{H}))$ and $(\mathfrak{F}, \mathfrak{M})$ are important in the study of singular integral operators and the K-theory (*cf.* [1], [2], [3]).

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In the present paper, we shall show that this situation is generally true. More strongly, we shall show that for every simple C^*-algebra, there exists a unique primitive C^*-algebra with unit satisfying the above conditions. The proof is obtained by a modification of the discussions of the previous paper. This work is done during the author's visits to University of Paris and University of Newcastle-upon-Tyne. The author wishes to express his hearty thanks to Professors Dixmier and Ringrose for their hospitality during his stay there.

2. Theorem.

In this section, we shall show the following theorem.

Theorem. — *Let $\mathfrak{a}$ be a simple C^*-algebra. Then there exists one and only one primitive C^*-algebra $\mathfrak{D}$ with unit (called the derived C^*-algebra of $\mathfrak{a}$) satisfying the following conditions :*

(1) *$\mathfrak{a}$ is a two-sided ideal of $\mathfrak{D}$;*

(2) *for every derivation δ on $\mathfrak{a}$, there exists an element d (unique modulo scalar multiples of unit) in $\mathfrak{D}$ such that $\delta(x) = [d, x]$ $(x \in \mathfrak{a})$;*

(3) *every derivation of $\mathfrak{D}$ is inner.*

To prove the theorem, we shall provide some considerations.

Let $\{\pi, \mathfrak{H}\}$ be an irreducible $\bigstar$-représentation of $\mathfrak{a}$ on a Hilbert space $\mathfrak{H}$. We shall identify $\mathfrak{a}$ with the image $\pi(\mathfrak{a})$. Let δ be a self-adjoint derivation on $\mathfrak{a}$ [i. e., $\delta(x^*)^* = -\delta(x)$ $(x \in \mathfrak{a})$]. Then there exists a bounded self-adjoint operator d on $\mathfrak{H}$ such that $\delta(x) = [d, x]$ ([5], [7]).

By considering $1 + \| d \| . 1 + d$, we can assume that $d \geqq 1$.

Let $\mathfrak{D}_\delta$ be the C^*-algebra on $\mathfrak{H}$ generated by $\mathfrak{a}$ and d. Let $(\mathfrak{d}_\alpha)_{\alpha \in I}$ be a directed set of closed two-sided ideals of $\mathfrak{D}_\delta$ such that $\alpha \leqq \beta$ implies $\mathfrak{d}_\alpha \subseteqq \mathfrak{d}_\beta$ and $\mathfrak{a} \cap \mathfrak{d}_\alpha = (0)$ for all $\alpha \in I$. Put $\mathfrak{d}_1 =$ the uniform closure of $\bigcup_{\alpha \in I} \mathfrak{d}_\alpha$; then $\mathfrak{d}_1$ is a two-sided ideal for $\mathfrak{D}_\delta$. Suppose that $\mathfrak{a} \cap \mathfrak{d}_1 \neq (0)$. Then for some a in $\mathfrak{a}$ with $\| a \| = 1$, there exists an element b in $\mathfrak{d}_{\alpha_0}$ such that $\| a - b \| < 1/2$. On the other hand, $\mathfrak{a} \cap \mathfrak{d}_{\alpha_0} = (0)$ and so $\mathfrak{a} = \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{d}_{\alpha_0}) = (\mathfrak{a} + \mathfrak{d}_{\alpha_0})/\mathfrak{d}_{\alpha_0}$, so that $\| a - b \| \geqq 1$, a contradiction. Hence $\mathfrak{a} \cap \mathfrak{d}_1 = (0)$.

By Zorn's lemma, there exists a closed two-sided ideal $\mathfrak{d}_0$ in $\mathfrak{D}_\delta$ which is maximal in all closed two-sided ideals $\mathfrak{d}$ with $\mathfrak{a} \cap \mathfrak{d} = (0)$. Consider the quotient algebra $\mathfrak{D}_\delta/\mathfrak{d}_0$ (say $\mathfrak{E}_\delta$). We shall identify $\mathfrak{a}$ with the image in $\mathfrak{E}_\delta$ under the cannonical mapping. Then for any non-zero closed two-sided ideal J of $\mathfrak{E}_\delta$, $\mathfrak{a} \cap J \neq (0)$.

We have the following situation : there exists an element r in $\mathfrak{E}_\delta$ such that $r \geqq 1$, $\delta(x) = [r, x]$ $(x \in \mathfrak{a})$ and $\mathfrak{E}_\delta$ is generated by $\mathfrak{a}$ and r. Let $\mathfrak{a}_1$

be the C^*-subalgebra of $\mathfrak{C}_\delta$ generated by $\mathfrak{a}$ and the unit, and let J_0 be the least closed two-sided ideal of $\mathfrak{C}_\delta$ such that $J_0 \supseteqq \mathfrak{a}$.

Suppose that $J_0 \underset{\neq}{\supset} \mathfrak{a}$; then there exists a self-adjoint element s in J_0 such that $s \notin \mathfrak{a}_1$. In fact, otherwise $\mathfrak{a}_1 \supseteqq J_0 \underset{\neq}{\supset} \mathfrak{a}$ and so $\mathfrak{a}_1 = J_0$; then $\mathfrak{a}_1 = \mathfrak{C}_\delta$, so that $\mathfrak{a}$ is a two-sided ideal of $\mathfrak{C}_\delta$; hence $\mathfrak{a} = J_0$, a contradiction.

Let S be the set of all self-adjoint linear functionals f on $\mathfrak{C}_\delta$ with $f(\mathfrak{a}_1) = 0$ and $\|f\| \leqq 1$. S is $\sigma(\mathfrak{C}_\delta^*, \mathfrak{C}_\delta)$-compact, where $\mathfrak{C}_\delta^*$ is the dual Banach space of $\mathfrak{C}_\delta$.

Since $s \notin \mathfrak{a}_1$, there exists an extreme point g in S such that $g(s) \neq 0$. Let $g = g_1 - g_2$ be the orthogonal decomposition of g with $g_1, g_2 \geqq 0$, $\|g\| = \|g_1\| + \|g_2\|$. Put $\xi = g_1 + g_2$ and let $\{\pi_\xi, \mathfrak{H}_\xi\}$ be the $\star$-representation of $\mathfrak{C}_\delta$ on a Hilbert space $\mathfrak{H}_\xi$ constructed via ξ.

LEMMA 1. — *Let $\overline{\pi_\xi(\mathfrak{a}_1)}$ be the weak closure of $\pi_\xi(\mathfrak{a}_1)$ on $\mathfrak{H}_\xi$. Then $\overline{\pi_\xi(\mathfrak{a}_1)}$ is a factor.*

The proof of this lemma is exactly same with the proof of lemma 1 in the previous paper [8].

LEMMA 2. — $\pi_\xi(\mathfrak{a}) = (0)$.

Proof. — Suppose that $\pi_\xi(\mathfrak{a}) \neq (0)$. Then $\overline{\pi_\xi(\mathfrak{a})} = \overline{\pi_\xi(\mathfrak{a}_1)}$, since $\overline{\pi_\xi(\mathfrak{a})}$ is an ideal of $\overline{\pi_\xi(\mathfrak{a}_1)}$, and $\overline{\pi_\xi(\mathfrak{a}_1)}$ is a factor.

Let $\pi_\xi(r) = p + (\pi_\xi(r) - p)$ with $p \in \overline{\pi_\xi(\mathfrak{a}_1)}$ and $\pi_\xi(r) - p \in \pi_\xi(\mathfrak{a}_1)'$ (*cf.* the proof of lemma 2 in [8]). Let C be the commutative C^*-algebra generated by $\pi_\xi(r) - p$ and $1_{\mathfrak{H}_\xi}$, and let R be the C^*-*algebra* generated by $\overline{\pi_\xi(\mathfrak{a}_1)}$ and C.

Then R can be canonically identified with $\overline{\pi_\xi(\mathfrak{a}_1)} \otimes C$ (*cf.* [10]). $\dim(C) \geqq 2$, for if $\dim(C) = 1$, then $\overline{\pi_\xi(\mathfrak{C}_\delta)} = \overline{\pi_\xi(\mathfrak{a}_1)}$ and so $g_1(s) = g_2(s)$, a contradiction.

Now let ξ_1 be a state on $\overline{\pi_\xi(\mathfrak{a}_1)}$ with $\xi_1(w) = (w1_\xi, 1_\xi)$ $(w \in \overline{\pi_\xi(\mathfrak{a}_1)})$, and let χ_1, χ_2 be two different characters of C with

$$\chi_1(\pi_\xi(r) - p) \neq \chi_2(\pi_\xi(r) - p).$$

Then $\xi_1 \otimes \chi_1$ and $\xi_1 \otimes \chi_2$ are two different states on R.

Put $\varphi_i(y) = \xi_1 \otimes \chi_i(\pi_\xi(y))$ $(y \in \mathfrak{C}_\delta)$ $(i = 1, 2)$.

Then $\varphi_1(r) \neq \varphi_2(r)$, Now it is easy to see that φ_1, φ_2 are factorial states on $\mathfrak{C}_\delta$ (*cf.* the proof of lemma 2 in [8]).

Let $\{\pi_{\varphi_i}, \mathfrak{H}_{\varphi_i}\}$ $(i = 1, 2)$ be the $\star$-representation of $\mathfrak{C}_\delta$ on a Hilbert space $\mathfrak{H}_{\varphi_i}$ constructed via φ_i. Then by the discussions of the proof of the theorem in [8], there exist a $\bigstar$-isomorphism Φ of $\overline{\pi_{\varphi_1}(\mathfrak{E}_\delta)}$ onto $\overline{\pi_{\varphi_2}(\mathfrak{E}_\delta)}$ such that

$$\Phi(\pi_{\varphi_1}(x)) = \pi_{\varphi_2}(x)\ (x \in \mathfrak{a}_1) \qquad \text{and} \qquad \Phi(\pi_{\varphi_1}(r)) = \pi_{\varphi_2}(r) + \lambda 1_{\mathfrak{H}_{\varphi_2}},$$

where λ is a real number. Since $\varphi_1(r) \neq \varphi_2(r)$, $\lambda \neq 0$ (*cf.* the proof of the theorem in [8]); hence $\| \pi_{\varphi_1}(r) \| > \| \pi_{\varphi_2}(r) \|$ or $\| \pi_{\varphi_1}(r) \| < \| \pi_{\varphi_2}(r) \|$. This implies that the kernel J of π_{φ_1} or π_{φ_2} in $\mathfrak{E}_\delta$ is not zero.

Since $\mathfrak{a}$ is simple, $\mathfrak{a} \cap J = (0)$, a contradiction. This completes the proof.

Therefore $\pi_\xi(\mathfrak{a}) = (0)$ and so $\pi_\xi(s) = 0$, since s belongs to the least closed two-sided ideal J_0 of $\mathfrak{E}_\delta$ containing $\mathfrak{a}$. This contradicts to $g(s) \neq 0$ and so $\mathfrak{a} = J_0$, i. e. $\mathfrak{a}$ is a two-sided ideal of $\mathfrak{E}_\delta$. Now we shall show the theorem.

Proof of theorem. — Let D be the set of all self-adjoint derivations on $\mathfrak{a}$. By the preceding discussions, for each $\delta \in D$, there exist a C^*-algebra $\mathfrak{E}_\delta$ with unit and a positive element d_δ in $\mathfrak{E}_\delta$ such that $d_\delta \geqq 1$, $\mathfrak{E}_\delta$ is generated by $\mathfrak{a}$ and d_δ, $\mathfrak{a}$ is a two-sided ideal of $\mathfrak{E}_\delta$, and $\delta(x) = [d_\delta, x]\ (x \in \mathfrak{a})$. Now let $\{\pi_1, \mathfrak{H}_1\}$ be an irreducible $\bigstar$-representation of $\mathfrak{a}$ on a Hilbert space $\mathfrak{H}_1$. Let $\mathfrak{E}_\delta^{**}$ be the second dual of $\mathfrak{E}_\delta$; then it is a W^*-algebra (*cf.* [9]). Let $\mathfrak{a}^{00}$ be the bipolar of $\mathfrak{a}$ in $\mathfrak{E}_\delta^{**}$; then $\mathfrak{a}^{00}$ is a σ-closed two-sided ideal of $\mathfrak{E}_\delta^{**}$. Hence there exists a central projection z in $\mathfrak{E}_\delta^{**}$ such that $\mathfrak{a}^{00} = \mathfrak{E}_\delta^{**} z$. Since $\mathfrak{a}^{00}$ can be considered as the second dual of $\mathfrak{a}$, $\{\pi_1, \mathfrak{H}_1\}$ can be uniquely extended to a W^*-representation $\{\pi_1^W, \mathfrak{H}_1\}$ of $\mathfrak{a}^{00}$ on $\mathfrak{H}_1$ (*cf.* [4], [6]). Since $\mathfrak{a}^{00} = \mathfrak{E}_\delta^{**} z \supset \mathfrak{E}_\delta z$, we can define a $\bigstar$-representation, $\{\pi_1^\delta, \mathfrak{H}_1\}$ of $\mathfrak{E}_\delta$ such that $\pi_1^\delta(y) = \pi_1^W(yz)\ (y \in \mathfrak{E}_\delta)$. Then clearly, $\pi_1^\delta(x) = \pi_1(x)\ (x \in \mathfrak{a})$.

Let $\mathfrak{D}$ be the C^*-algebra on $\mathfrak{H}_1$ generated by $\pi_1(\mathfrak{a})$ and all $\pi_1^\delta(d_\delta)$ $(\delta \in D)$. Then $\pi_1^\delta(d_\delta)\ \pi_1(\mathfrak{a})$, $\pi_1(\mathfrak{a})\ \pi_1^\delta(d_\delta) \subset \pi_1(\mathfrak{a})$ and so $\pi_1(\mathfrak{a})$ is a two-sided ideal of $\mathfrak{D}$. Since arbitrary derivation of $\mathfrak{a}$ can be written as a linear combination of self-adjoint derivations of $\mathfrak{a}$, for every derivation δ of $\pi_1(\mathfrak{a})$, there exists an element d_δ in $\mathfrak{D}$ such that $\delta(x) = [d_\delta, x]$ $(x \in \pi_1(\mathfrak{a}))$. Since $\pi_1(\mathfrak{a})$ is irreducible, such a d_δ is unique modulo scalar multiples of unit. Moreover, let $\tilde{\delta}$ be a derivation of $\mathfrak{D}$; then there exists an element t in $\overline{\mathfrak{D}}$ such that $\tilde{\delta}(y) = [t, y]\ (y \in \mathfrak{D})$ ([5], [7]).

On the other hand, for arbitrary positive element q of $\pi_1(\mathfrak{a})$,

$$\tilde{\delta}(q^{1/2} q^{1/2}) = \tilde{\delta}(q^{1/2})\, q^{1/2} + q^{1/2}\, \tilde{\delta}(q^{1/2}) \in \pi_1(\mathfrak{a});$$

hence $\tilde{\delta}(\pi_1(\mathfrak{a})) \subset \pi_1(\mathfrak{a})$.

Therefore there is an element t_1 in $\mathfrak{D}$ such that $\tilde{\delta}(x) = [t, x] = [t_1, x]$ $(x \in \pi_1(\mathfrak{a}))$, and so $t = t_1 + \lambda 1_{\mathfrak{H}_1}$.

Hence every derivation of $\mathfrak{D}$ is inner. Since $\mathfrak{D}$ is irreducible, it is primitive. Now we shall identify $\mathfrak{a}$ with $\pi_1(\mathfrak{a})$. Then we establish the existence of a derived C^*-algebra of $\mathfrak{a}$.

Finally, we shall show the uniqueness of $\mathfrak{D}$. Let $\mathfrak{D}_1$ be another derived C^*-algebra of $\mathfrak{a}$; since $\mathfrak{a}$ is a two-sided ideal of $\mathfrak{D}_1$, by the preceding considerations, we have a $\star$-representation $\{\pi_2, \mathfrak{H}_1\}$ of $\mathfrak{D}_1$ on the Hilbert space $\mathfrak{H}_1$ such that $\pi_2(x) = \pi_1(x)$ $(x \in \mathfrak{a})$.

For $\delta \in D$, there are d_δ, d'_δ such that $d_\delta \in \mathfrak{D}$, $d'_\delta \in \mathfrak{D}_1$, and

$$\delta(x) = [d_\delta, x] = [d'_\delta, x] \qquad (x \in \mathfrak{a}).$$

Hence $d_\delta - \pi_2(d'_\delta) = \lambda 1_{\mathfrak{H}_1}$ and so $\pi_2(\mathfrak{D}_1) = \mathfrak{D}$. Therefore π_2 will give a $\star$-homomorphism of $\mathfrak{D}_1$ onto $\mathfrak{D}$ such that $\pi_2(x) = x$ $(x \in \mathfrak{a})$.

Let K be the kernel of π_2. For $u \in K$,

$$\pi_1([u, x]) = [\pi_2(u), \pi_2(x)] = [\pi_2(u), x] = 0 \qquad (x \in \mathfrak{a})$$

and so $[u, x] = 0$ for all $x \in \mathfrak{a}$.

Hence $u = 0$ and so π_2 is a $\star$-isomorphism of $\mathfrak{D}_1$ into $\mathfrak{D}$. This completes the proof.

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