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**SYMPLECTIC STRUCTURE**  
**IN THE ENVELOPING ALGEBRA OF A LIE ALGEBRA**

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ABSTRACT. — It is shown that the enveloping algebra of a Lie algebra satisfies a condition which implies a weakened form of the Gel'fand-Kirillov conjecture. This condition leads to a generalization of a commutant property previously derived for the Weyl algebra, which has its origins in a classical theorem on function groups. This provides a dimensionality estimate which is central to a proof of the Gel'fand-Kirillov conjecture for solvable algebraic Lie algebras.

RÉSUMÉ. — Il est démontré que l'algèbre enveloppante d'une algèbre de Lie satisfait une condition qui implique une forme affaiblie de la conjecture Gel'fand-Kirillov. Cette condition amène à une généralisation d'une propriété commutante précédemment dérivée pour l'algèbre de Weyl, qui a ses origines dans un théorème classique en groupes fonctionnels. Ceci fournit une estimation dimensionnelle qui est centrale pour la preuve de la conjecture Gel'fand-Kirillov pour les algèbres de Lie algébriques résolubles.

## 1. Introduction

Let  $g$  be a finite dimensional Lie algebra over a commutative field  $K$  of characteristic zero. Let  $Ug$  denote the enveloping algebra of  $g$  and  $Dg$  the quotient field of  $Ug$ . Let  $D_{n,k}$  denote the quotient field of the Weyl algebra  $A_{n,k}$  of degree  $n$  over  $K$  and extended by  $k$  indeterminates. GEL'FAND and KIRILLOV ([2]-[4]) have suggested that  $Dg$  should depend rather weakly on  $g$  and for  $g$  algebraic have conjectured that  $Dg$  is isomorphic to one of the standard fields  $D_{n,k}$ .  $A_{n,k}$  is itself related to a polynomial algebra over the Poisson bracket (essentially equivalent to a manifold with symplectic structure) which has been subjected to considerable analysis. We wish to exploit these interrelationships in studying  $Ug$ . In this it is often sufficient to establish a correspondence of leading order terms. This is illustrated by Theorem 2.3, the second part of which represents a weak form of the Gel'fand-Kirillov

conjecture. The first part leads to an important dimensionality estimate contained in the theorem stated below.

Let  $g^*$  denote the dual of  $g$ . To each  $f \in g$  define an antisymmetric bilinear form  $B_f$  on  $g \times g$  through

$$(1.1) \quad B_f(x, y) = (f, [x, y]).$$

Recall that  $B_f$  must have even rank and set

$$(1.2) \quad m = \dim g, \quad n = \frac{1}{2} \sup_{f \in g^*} \text{rank } B_f.$$

Let  $\text{Dim}_K$  denote the dimensionality introduced by GEL'FAND and KIRILLOV [2]. It is shown in section 3 that :

**THEOREM 1.1.** — *Let  $A$  be a subalgebra of  $Ug$  and denote by  $A'$  its commutant in  $Ug$ . Then*

$$\text{Dim}_K A + \text{Dim}_K A' \leq 2(m - n),$$

with  $m, n$  given by (1.2).

We remark that  $\text{Dim}_K Ug = \dim g$  for all  $g$ . If further  $g$  is either nilpotent or semisimple  $\text{Dim}_K C(Ug) = m - 2n$  (where  $C$  denotes centre). It follows that the above bound is saturated in either of these two cases. This is also true if  $g$  is solvable and algebraic. Indeed for  $g$  solvable NGHIÊM [11] has constructed a maximal commutative subalgebra  $A$  of  $Ug$  and it is shown in [9] by use of the above theorem that  $\text{Dim}_K A = m - n$ . This equality motivated the proof of the Gel'fand-Kirillov conjecture for  $g$  solvable given in [9]. We remark that Theorem 1.1 does not follow in any obvious fashion from the truth of this conjecture. This is because the corresponding dimensionality estimates are more difficult to make in  $Dg$ .

## 2. Weighted filtrations

Let  $n, k$  be integers with  $n$  non-negative and  $k$  positive. Let  $g_{n,k}$  denote the Lie algebra over  $K$  with basis  $\{x_i, y_i, z_j; i = 1, 2, \dots, n; j = 0, 1, \dots, k - 1\}$  where  $[x_i, y_i] = z_0$  and all other brackets vanish. Let  $I$  denote the two-sided ideal in  $Ug_{n,k}$  generated by  $z_0 - 1$ . Set  $A_{n,k} = Ug_{n,k+1}/I$ . Observe that  $Ug_{n,k}$  is isomorphic to a subalgebra of  $A_{n,k}$  (divide the  $x_i$  by  $z_0$ ) and that  $Dg_{n,k} = D_{n,k}$  [2].

For arbitrary  $g$ , let the subspaces  $\{U^{(i)}; i = 0, 1, 2, \dots\}$  define a filtration of  $Ug$ . Set  $U_i = U^{(i)}/U^{(i-1)}$  and  $G(Ug) = \bigoplus_{i=0}^{\infty} U_i$ .

Only filtrations making  $G(Ug)$  integral are considered.

In the remainder of this section we assume  $K$  algebraically closed.

LEMMA 2.1. — *Suppose  $g$  is either nilpotent or semisimple. Define  $m, n$  by (1.2) and set  $k = m - 2n$ . Then  $Ug$  admits a filtration such that  $G(Ug) = U g_{n,k}$ .*

*Proof.* — Take  $g$  nilpotent. Recalling (1.1) and (1.2) choose  $f \in g^*$  such that  $\text{rank } B_f = 2n$ . Set  $g_0 = \{x \in g; f(x) = 0\}$ .

Let  $B'_f$  denote the restriction of  $B_f$  to  $g_0$ . We wish to show that  $\text{rank } B'_f = 2n$ . Let  $N_B, N_{B'}$  respectively denote the null spaces of  $B_f$  and  $B'_f$ . By [1], Lemma 5, it suffices to show that  $N_{B'} \subset N_B$ . Now given  $x \in N_{B'}$ ,

$$(f, [x, y]) = B_f(x, y) = 0, \quad \text{for all } y \in g_0.$$

Hence  $(\text{ad } x)g_0 \subset g_0$ . Let  $z_0 \in g, z_0 \notin g_0$ . Since  $\dim g - \dim g_0 = 1$ , we may write  $(\text{ad } x)z_0 = \alpha z_0 + y$ , for some  $\alpha \in K, y \in g_0$ . Then for each positive integer  $r$ ,

$$(\text{ad}^r x)z_0 = \alpha^r z_0 + y_r; \quad y_r \in g_0.$$

Since  $g$  is nilpotent;  $\alpha^r z_0 + y_r = 0$  for some  $r$  and hence  $\alpha = 0$ . It follows that  $(\text{ad } x)g \subset g_0$  which implies that  $x \in N_B$ , as required.

Define a filtration on  $Ug$  by setting  $U^{(0)} = K, g_0 \subset U^{(1)}, z_0 \in U^{(2)}, z_0 \notin U^{(1)}$ . To show that  $G(Ug)$  has the asserted property it suffices to show that the generators of  $g$  satisfy the commutation relations of  $g_{n,k}$  in  $G(Ug)$ . Scale  $z_0$  so that  $f(z_0) = 1$ . Then for all  $x, y \in g$ , we have

$$xy - yx = (f, [x, y])z_0 \quad \text{mod } g_0$$

in  $Ug$ . Hence by choice of filtration we obtain, for all  $x, y \in g_0$ ,

$$\begin{aligned} xy - yx &= B'_f(x, y)z_0, \\ xz_0 - z_0x &= 0 \end{aligned}$$

in  $G(Ug)$ . Finally bringing  $B'_f$  to canonical form exhibits the defining basis for  $g_{n,k}$ .

Take  $g$  semisimple. As is well-known,  $k = \text{rank } g$ , and  $n$  is the number of positive roots. Let  $h$  be a Cartan subalgebra for  $g$ , and  $\Delta$  the set of all non-zero roots. Each root subspace  $g^\alpha$  is one-dimensional, and  $g$  is a direct sum of  $h$  and the  $g^\alpha; \alpha \in \Delta$ . Let  $B$  denote the Killing form. To each  $\alpha \in \Delta$  define  $H_\alpha \in h$  (cf. [5], Theorem 4.2) through  $B(H, H_\alpha) = \alpha(H)$  for all  $H \in h$ . Let  $H_0$  be a regular element ([5], p. 137) of  $h$ . Then  $\alpha(H_0) \neq 0$  for all  $\alpha \in \Delta$ . Define  $f \in g^*$  through  $f(H) = B(H, H_0)$  for all  $H \in h$  and the condition that it vanish on each

root subspace. Set  $g_0 = \{x \in g; f(x) = 0\}$ . For each  $\alpha \in \Delta$  choose  $E_\alpha \in g^\alpha$  such that  $B(E_\alpha, E_{-\alpha}) = 1$ . Then through [5], Theorem 5.5,

$$E_\alpha E_{-\alpha} - E_{-\alpha} E_\alpha = \frac{\alpha(H_0)}{B(H_0, H_0)} H_0 \quad \text{mod } g_0$$

in  $Ug$  and all other commutators vanish mod  $g_0$ . Setting  $H_0 = z_0$  the proof is completed with the filtration defined in the nilpotent case.

The conclusion of the Lemma fails on general  $g$ . For example, consider the two dimensional (solvable) Lie algebra with relation  $[x, z] = z$ . Yet through [1], Lemma 5,  $\text{rank } B'_{f'} \geq \text{rank } B_f - 2$ ;  $B'_{f'}$ ,  $B_f$  as above. It follows that we always have the weaker result, namely that

$$G(Ug) = Ug_{n-1, k+2},$$

for a suitable filtration of  $Ug$ .

LEMMA 2.2. — Let  $B$  a non-degenerate antisymmetric bilinear form on  $V \times V$ . Let  $a$  be a linear transformation on  $V$  such that

$$(2.1) \quad B(ax, y) + B(x, ay) = 0$$

for all  $x, y \in V$ . Then there exists a basis  $\{x_i; i = 1, 2, \dots, 2l\}$  for  $V$  such that

$$(2.2) \quad B(x_i, x_{2l-j}) = \delta_{ij} (-1)^j,$$

$i, j = 1, 2, \dots, 2l$ , and  $a$  is upper triangular.

*Proof.* — Recalling [10], p. 398, choose a basis  $\{y_i\}$  for  $V$  such that

$$(2.3) \quad ay_i = \alpha_i y_i + \beta_i y_{i+1},$$

$\alpha_i, \beta_i \in K$  with  $\alpha_i \leq \alpha_j$  for  $i \leq j$ . Since  $B$  is non-degenerate there exists a second basis  $\{z_i\}$  for  $V$  such that  $B(y_i, z_j) = \delta_{ij}$ , for all  $i, j$ . Substitution in (2.1) and (2.3) gives

$$(2.4) \quad az_i = -\alpha_i z_i - \beta_{i-1} z_{i-1}.$$

Let  $V_i$  denote the eigenspace belonging to eigenvalue  $\alpha_i$ . By (2.2),  $B(V_i, V_j) = 0$ , unless  $\alpha_i + \alpha_j = 0$ . Further when this holds  $B$  non-degenerate implies  $\dim V_i = \dim V_j$ . Let  $V_0$  denote the zero eigenspace and  $V'$  the direct sum of the  $V_i$  omitting  $V_0$ . On  $V'$  set

$$x_i = \begin{cases} y_i; & \alpha_i < 0, \\ (-1)^i z_{2l-i}; & \alpha_i > 0. \end{cases}$$

By (2.3) and (2.4), this determines the required basis on  $V'$ . It remains to determine a basis on  $V_0$ . Equivalently we can assume a of the lemma nilpotent.

Let  $r$  be the least positive integer such that  $\alpha^{r+1} V_0 = \{0\}$ . Set

$$W^{(s)} = \{x \in V_0; \alpha^{s+1} x = 0\} \quad \text{and} \quad W_s = W^{(s)}/W^{(s-1)}.$$

We have

$$V_0 = \bigoplus_{i=0}^r W_i; \quad \alpha W_i \subset W_{i-1} \quad \text{for all } i.$$

Hence to prove the lemma it suffices to exhibit a basis for  $V_0$  on which  $B$  is antidiagonal and which, for each  $i$ , contains as a subspace a basis for  $W_i$ .

Set  $U = \alpha^r V_0$ . Then  $\dim U = \dim W_r$ , and by (2.2) :

$$(2.5) \quad B(U, W_s) = 0; \quad s < r.$$

Hence there exists a basis  $\{y_i; i = 1, 2, \dots, t\}$  for  $W_r$  and a basis  $\{y'_i; i = 1, 2, \dots, t\}$  for  $U$  such that  $B$  is antidiagonal on their linear span  $U'$ . Further since  $B$  is non-degenerate we may assume that  $B(y_j, y'_{t-j}) = (-1)^j$ . Set

$$x_j = y_j, \quad x_{2t-j} = y'_{t-j}; \quad j = 1, 2, \dots, t.$$

Observe that  $U \subset W_0$  and set  $V'_0 = W^{(r-1)}/U$ . Let  $\{z_i\}$  be a basis for  $W^{(r-1)}$ . Set

$$z'_i = z_i - \sum_{j=1}^t (-1)^j B(y_j, z_i) y'_{t-j}.$$

Recalling (2.5) it follows that  $B(x, z'_i) = 0$ , for all  $x \in U'$  and all  $i$ . On the other hand  $z'_i = z_i$  on  $V'_0$ . Induction provides the required basis. The lemma is proved.

**THEOREM 2.3.** — *Define  $m, n$  by (1.2) and set  $k = m - 2n$ . Then  $Ug$  admits a filtration such that  $G(Ug)$  is isomorphic to a subalgebra of  $A_{n,k}$  and  $D(G(Ug)) = D_{n,k}$ .*

*Proof.* — Let  $f, B_f, B'_f, g_0, N_B, N_{B'}$ , be as in the proof of lemma 2.1. Given  $\text{rank } B'_f = \text{rank } B_f$ , the conclusion of lemma 2.1 holds and the theorem follows easily. Otherwise by [1], Lemma 5,  $N_B$  is of codimension 1 in  $N_{B'}$ . Choose  $x \in N_{B'}, x \notin N_B$ . Then as before  $(\text{ad } x)g_0 \subset g_0$  and given  $z_0 \in g, z \notin g_0$ ,

$$(\text{ad } x)z_0 = \alpha z_0 + y; \quad y \in g_0; \quad \alpha \neq 0.$$

By definition of  $B_f$  and the Jacobi identity :

$$(2.6) \quad B_f([x, y], z) = B_f([x, z], y) + B_f(x, [y, z])$$

for all  $x, y, z \in g$ . Choosing  $x$  as above,  $y \in N_B, z \in g_0$ , it follows from (2.6) that  $(\text{ad } x)N_B \subset N_B$ .

Suppose that there exists  $z \in N_B$ , such that  $(\text{ad } x)z = \beta z; \beta \neq \alpha, 0$ .

Then for all  $\gamma \in K$ ,

$$(2.7) \quad (\text{ad } x)(z + \gamma z_0) = \beta(z + \gamma z_0) + (\alpha - \beta)\gamma z_0 + \gamma y.$$

Since  $\beta \neq 0$ ;  $x, z$  are linearly independent. Let  $\{x_i\}$  be a basis for  $g_0$  with  $x_1 = z, x_2 = x$ . Set  $g_0(\gamma) = \text{lin span}\{z + \gamma z_0, x_2, x_3, \dots, x_{m-1}\}$ . Define  $f_\gamma \in g^*$  such that

$$g_0(\gamma) = \{x \in g; f_\gamma(x) = 0\}.$$

Denote by  $g_{00}$  a maximal subspace of  $g_0$  on which  $B'_f$  is non-degenerate. Since  $z \in N_B$ , we may choose a fixed  $g_{00}$  such that  $g_{00} \subset g_0(\gamma)$  for all  $\gamma$ . Let  $B''_{f_\gamma}$  denote the restriction of  $B'_{f_\gamma}$  to  $g_{00} \times g_{00}$ . By choice of  $g_{00}$ ,

$$\text{rank } B''_{f_\gamma} = \text{rank } B'_f = \text{rank } B_f - 2.$$

Hence except for finitely many values of  $\gamma$ ,

$$\text{rank } B''_{f_\gamma} = \text{rank } B_f - 2.$$

Since  $z, x \in N_B$ , there exists, for all but finitely many values of  $\gamma$ ,  $y_\gamma \in g_{00}$  such that  $B'_{f_\gamma}(x, y_\gamma) = O(\gamma)$  and setting  $z_\gamma = z + \gamma z_0 + \gamma y_\gamma$  that  $B'_{f_\gamma}(a, z_\gamma) = 0$  for all  $a \in g_{00}$ . Again  $f_\gamma(y) = O(\gamma)$  so from (2.7) we obtain

$$B'_{f_\gamma}(x, z_\gamma) = (\alpha - \beta)\gamma + O(\gamma^2).$$

Since  $\alpha \neq \beta$ , it follows by [1], Lemma 5 and the above that we may choose  $\gamma$  such that  $\text{rank } B'_{f_\gamma} = \text{rank } B_f$ . Then the conclusion of Lemma 2.1 holds and the theorem is proved in this case. We conclude that there is no loss of generality in assuming that  $N_B$  admits a basis  $\{z_i\}$  such that

$$(2.8) \quad (\text{ad } x)z_i = \alpha'_i z_i + \beta'_i z_{i+1},$$

where  $\alpha'_i = \alpha, 0$ , for all  $i = 1, 2, \dots, k$ , with  $x = z_k$ .

Set  $V = g_0/N_B$ , and let  $B$  denote the restriction of  $B'_f$  to  $V$ . Use of (2.6) shows that  $\text{ad } x - (\alpha/2)$  is a linear transformation on  $V$  satisfying (2.1). Further  $B$  is non-degenerate on  $V$ , so Lemma 2.2 applies. Let  $\{x_i; i = 1, 2, \dots, 2l\}$  be a basis satisfying its conclusion. Since  $\text{rank } B = \text{rank } B_f - 2$ , we have  $l = n - 1$ . Define a filtration on  $Ug$  as follows.

Set  $U^{(0)}$  equal the tensor algebra generated by  $x$ . Let

$$\begin{aligned} z_0 &\in U^{(4m)}, & z_0 &\notin U^{(4m-1)}, \\ x_i &\in U^{(2m+n-i-1)}, & x_i &\notin U^{(2m+n-i-2)}, \\ z_j &\in U^{(2m-n+1-j)}, & z_j &\notin U^{(2m-n-j)}; \\ i &= 1, 2, \dots, 2n - 2; & j &= 1, 2, \dots, k - 1. \end{aligned}$$

Recalling that  $k = m - 2n$  and that  $\text{ad } x$  is upper triangular on  $V$  and on  $N_B$ , we obtain the following bracket relations in  $G(Ug)$  :

$$\begin{aligned} [x_i, x_{2n-2-j}] &= \delta_{ij} z_0, \\ [x_i, z_r] &= 0, \\ [x, x_i] &= \alpha_i x_i, \\ [x, z_r] &= \alpha'_r z_r, \\ [z_r, z_s] &= 0, \end{aligned}$$

for all  $i, j = 1, 2, \dots, 2n - 2$ ;  $r, s = 0, 1, \dots, k - 1$ , where  $\alpha'_0 = \alpha$ ,  $\alpha'_r = 0$ ,  $\alpha$ ;  $\alpha_i + \alpha_{2n-2-i} = \alpha$ . Set

$$y_i = x_{2n-1-i}; \quad i = 1, 2, \dots, n - 1$$

and

$$x'_n = (x - \sum_{i=1}^{n-1} (\alpha - \alpha_i) (x_i z_0^{-1}) y_i) z_0.$$

Then for all  $i = 1, 2, \dots, n - 1, r = 0, 1, 2, \dots, k$ ,

$$[x_i, y_i] = z_0, \quad [x'_n, z_r] = \alpha'_r z_0 z_r$$

and all remaining brackets vanish. Set  $x_n = x'_n z_0^{-1}$ ,  $y_n = z_0$ . The proof is completed by noting that  $x_i z_0^{-1}, y_i; i = 1, 2, \dots, n; z_r z_0^{-1}; r = 0, \alpha'_r = \alpha$  and  $z_r; \alpha'_r = 0$  generate  $A_{n,k}$ .

We remark that the proof and consequently the filtration simplifies should  $g$  be almost algebraic [6], p. 98. In this case  $\text{ad } x$  may be assumed semisimple.

Given  $x \in U^{(s)}$ ,  $x \notin U^{(s-1)}$ , we write  $f(x)$  for the leading term of  $x$ . Theorem 2.3 has the following easy corollary which illustrates the symplectic structure associated with the enveloping algebra of a Lie algebra.

**COROLLARY 2.4.** — *There exists a filtration of  $Ug$  with  $U^{(0)} = K$ , such that  $G(Ug)$  is isomorphic to a subalgebra of  $K[x_i, y_i, z_j]; i = 1, 2, \dots, n; j = 1, 2, \dots, k$ . Furthermore given  $x \in U^{(r)}$ ,  $x \notin U^{(r-1)}$ ,  $y \in U^{(s)}$ ,  $y \notin U^{(s-1)}$ ; then either*

$$[x, y] \in U^{(k+s-2)} \quad \text{and} \quad \{f(x), f(y)\} = 0,$$

or

$$f([x, y]) = \{f(x), f(y)\},$$

where

$$\{f(x), f(y)\} = \sum_{i=1}^n \left( \frac{\partial f(x)}{\partial x_i} \frac{\partial f(y)}{\partial y_i} - \frac{\partial f(x)}{\partial y_i} \frac{\partial f(y)}{\partial x_i} \right).$$

*Proof.* — Given  $x \in g$  suppose with respect to the filtration of  $Ug$  defined in Theorem 2.3 that  $x \in U^{(s)}$ ,  $x \notin U^{(s-1)}$ . Define a new filtration in  $Ug$  by setting  $U^{(0)} = K$  and defining  $x \in U^{(s+1)}$ ,  $x \notin U^{(s)}$ . Computation shows that the new graded algebra  $G(Ug)$  has the asserted properties.



### 3. The commutant theorem

In this section, we consider only filtrations on  $Ug$  such that  $U^{(0)} = K$  and  $G(Ug)$  is commutative. Given a subalgebra  $A$  of  $Ug$ , set  $f(A) = \{f(a); a \in A\}$ .  $f(A)$  is isomorphic to a subalgebra of polynomials in  $m$  variables:  $m = \dim g$ . Set  $df(A) = \{df(a); f(a) \in f(A)\}$ . Let  $\dim df(A)$  denote the dimension of  $df(A)$  considered as a module over  $G(Ug)$ .

LEMMA 3.1. — *Let  $A$  be a subalgebra of  $Ug$ . Then*

$$\text{Dim}_K A = \dim df(A).$$

*Proof.* — The proof follows that of [8], Theorem 3.3. Let  $\{x_i\}$  be a basis for  $g$ . We have  $x_i \in U^{(n_i)}$ ,  $x_i \notin U^{(n_i-1)}$ , for each  $i$ , where the  $n_i$  are positive integers. Set  $y_i = x_i^{1/n_i}$ . Let  $f(A')$  denote  $f(A)$  considered as an algebra of polynomials in the  $y_i$ . Clearly

$$\dim df(A') = \dim df(A).$$

That  $\text{Dim}_K A \leq \dim df(A')$ , follows from the dimensionality estimate of [7], Lemma 3.3. On the other hand choosing  $a_1, a_2, \dots, a_r \in A$  such that  $\{df(a_i); i = 1, 2, \dots, r\}$  is a basis for  $df(A)$  shows that  $\text{Dim}_K A \geq \dim df(A)$ .

Theorem 1.1 follows on application of [8], Lemma 2.1, and Corollary 2.4 and Lemma 3.1 to the algebraic closure of  $K$ .

#### REFERENCES

- [1] DIXMIER (J.). — Sur les représentations unitaires des groupes de Lie nilpotents, II., *Bull. Soc. math. Fr.*, t. 85, 1957, p. 325-388.
- [2] GEL'FAND (I. M.) and KIRILLOV (A. A.). — Fields associated with enveloping algebras of Lie algebras, *Soviet Mathematics*, t. 7, 1966, p. 407-409 [en russe]; *Doklady Akad. Nauk SSSR*, t. 167, 1966, p. 503-505.
- [3] GEL'FAND (I. M.) and KIRILLOV (A. A.). — *Sur les corps liés aux algèbres enveloppantes des algèbres de Lie.* — Paris, Presses universitaires de France, 1966 (*Institut des Hautes Études Scientifiques, Publications mathématiques*, 31, p. 5-19).
- [4] GEL'FAND (I. M.) and KIRILLOV (A. A.). — Structure of the Lie field connected with a split semisimple Lie algebra [en russe], *Funkcional'nyj Analiz*, t. 3, 1969, fasc. 1, p. 7-26.
- [5] HELGASON (S.). — *Differential geometry and symmetric spaces.* — New York, Academic Press, 1962 (*Pure and applied Mathematics, Academic Press*, 12).
- [6] JACOBSON (N.). — *Lie algebras.* — New York, Interscience Publishers, 1962 (*Interscience Tracts in pure and applied Mathematics*, 10).

- [7] JOSEPH (A.). — Commuting polynomials in quantum canonical operators and realizations of Lie algebras, *J. of math. Phys.*, t. 13, 1972, p. 351-357.
- [8] JOSEPH (A.). — Gel'fand-Kirillov dimension for algebras associated with the Weyl algebra, *Ann. Inst. H. Poincaré*, série A, t. 17, 1973, p. 325-336.
- [9] JOSEPH (A.). — *Proof of the Gel'fand-Kirillov conjecture for solvable Lie algebras*, *Proc. Amer. math. Soc.* (à paraître).
- [10] LANG (S.). — *Algebra*. — New York, Addison-Wesley publishing Company, 1965 (*Addison-Wesley Series in Mathematics*).
- [11] NGHIÊM (X. H.). — Sur certains sous-corps commutatifs du corps enveloppant d'une algèbre de Lie résoluble, *Bull. Sc. math.*, série 2, t. 96, 1972, p. 111-128.

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