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NON-VANISHING
OF THE FIRST COHOMOLOGY

BY

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RÉSUMÉ. — On démontre que, pour les réseaux Γ du type fini dans les groupes semi simples sur les corps locaux de caractéristique positive, $H^1(\Gamma, \text{Ad})$ ne s'annule pas; ceci est bien différent de ce que passe dans le cas de caractéristique zéro.

ABSTRACT. — It is shown here that, for any finitely generated lattice Γ in certain semi simple groups over local fields of positive characteristics, $H^1(\Gamma, \text{Ad})$ is non-vanishing; this is in sharp contrast with the situation in characteristic zero.

Let K be a local field (i. e. a non-discrete locally compact field), and let \mathbf{G} be a connected semi simple algebraic group defined over K . Let $G = \mathbf{G}(K)$, and let $r = K$ -rank \mathbf{G} . The topology on K induces a locally compact Hausdorff topology on G ; in the sequel, we assume G endowed with this topology. G is then a K -analytic group. Let Γ be a lattice in G i.e., a discrete subgroup of G such that G/Γ carries a finite G -invariant Borel measure. We assume that Γ is *irreducible*, i.e. no subgroup of Γ of finite index is a direct product of two infinite normal subgroups.

In case $K = \mathbf{R}$ and G is not locally isomorphic to either $SL(2, \mathbf{R})$ or $SL(2, \mathbf{C})$, it is known that $H^1(\Gamma, \text{Ad}) = 0$; where, as usual, Ad denotes the adjoint representation of G on its Lie algebra (see WEIL [9], [10] for uniform lattices; for non-uniform lattices in groups of \mathbf{R} -rank > 1 , this vanishing theorem follows from the results of RAGHUNATHAN [8], combined with the results of MARGULIS [4] on arithmeticity; for non-uniform lattices in groups of \mathbf{R} -rank 1, it is contained in GARLAND-RAGHUNATHAN [2]).

It is also known, in view of a recent result of MARGULIS ([5], theorem 8), that in case K is non-archimedean but of characteristic zero, $H^1(\Gamma, \text{Ad}) = 0$ when $r > 1$.

The object of this note is to show that *when K is of positive characteristic, then it is not in general true that $H^1(\Gamma, \text{Ad}) = 0$.*

We shall in fact prove the following theorem.

THEOREM. — *Let F be a finite field, and let K be the local field $F((t))$. Let \mathbf{G} be a connected semi simple algebraic group, with trivial center, defined over F . Let $G = \mathbf{G}(K)$, let Γ be a finitely generated lattice in G . Then $H^1(\Gamma, \text{Ad}) \neq 0$.*

Remark. — If \mathbf{G} has no K -rank 1 factors, then according to a well-known theorem of D.A. KAZHDAN (see [1]), every lattice in G is finitely generated.

For the proof of the theorem, we need to recall a result of WEIL [10].

We introduce some notation and a definition.

Let Λ be a finitely generated abstract group. We shall let $\mathcal{A}(\Lambda, G)$ denote the space of all homomorphisms of Λ in G with the topology of pointwise convergence. There is a natural action of G on $\mathcal{A}(\Lambda, G)$ induced by the inner automorphism.

Now assume that Λ is a finitely generated subgroup of G , and let $\iota : \Lambda \rightarrow G$ be the natural inclusion. Then Λ is said to be *locally* (or *infinitesimally*) *rigid* if the orbit of ι under G is open in $\mathcal{A}(\Lambda, G)$. According to a result of WEIL [10], vanishing of $H^1(\Lambda, \text{Ad})$ implies local rigidity of Λ .

Proof of the theorem. — In view of the above result of WEIL, to prove that $H^1(\Gamma, \text{Ad}) \neq 0$, it suffices to show that Γ is not infinitesimally rigid.

For $i > 1$, $t \mapsto t + t^i$ extends uniquely to give a continuous automorphism a_i of $F((t))/F$. It is evident that, for any fixed $x \in F((t))$, the sequence $\{a_i(x)\}$ converge to x .

Now since \mathbf{G} is defined over F , a_i induces a continuous automorphism α_i of G . Therefore, for all i , $\alpha_i \cdot \iota$ is an embedding of Γ in G ; where $\iota : \Gamma \rightarrow G$ is the natural inclusion of Γ in G . It is also obvious that the sequence $\{\alpha_i \cdot \iota\}$ converges to ι in $\mathcal{A}(\Gamma, G)$. We shall show that none of the $\alpha_i \cdot \iota$ lie in the G -orbit of ι . This will prove that Γ is not locally rigid and hence $H^1(\Gamma, \text{Ad}) \neq 0$.

If possible, assume that, for some i , $\alpha_i \cdot \iota = \text{Int } g_i \cdot \iota$. Then $(\text{Int } g_i^{-1} \cdot \alpha_i) \cdot \iota = \iota$, and the main theorem of PRASAD [6] implies that $\text{Int } g_i^{-1} \cdot \alpha_i$ is the identity automorphism of G . Hence, $\alpha_i = \text{Int } g_i$.

We now fix a 1-dimensional torus $\mathbf{T} (\subset \mathbf{G})$ which is defined and split over the finite field F (existence of such a torus follows from Lang's theorem [3]). Let $T = \mathbf{T}(K)$. Then since \mathbf{T} is defined over F , $\alpha_i(T) = T$. Moreover,

for any rational character χ on \mathbf{T} and all $t \in T$,

$$\chi(\alpha_i(t)) = a_i(\chi(t)).$$

Since $\alpha_i = \text{Int } g_i$ and $\alpha_i(T) = T$, it follows that g_i normalizes T and hence also \mathbf{T} . Therefore, for any rational character χ on \mathbf{T} :

$$\chi(\alpha_i(t)) = \chi(g_i t g_i^{-1}) = \chi^d(t),$$

where $d = +1$ or -1 . Hence,

$$(\star) \quad a_i(\chi(t)) = \chi^d(t), \quad \text{where } d = +1 \quad \text{or} \quad -1.$$

Now take χ to be one of the generators of the group of rational characters on \mathbf{T} . Then it follows from (\star) that, for all $k \in K$, either

$$a_i(k) = k \quad \text{or} \quad a_i(k) = k^{-1}.$$

But it is obvious from the definition of a_i , that this is not the case. Hence, none of the $\alpha_i \cdot \mathfrak{t}$ lie in the G -orbit of \mathfrak{t} . This proves that $H^1(\Gamma, \text{Ad}) \neq 0$.

Remark. — As the above proof shows, Γ is not locally rigid. However, in case K -rank $\mathbf{G} > 1$ and Γ is an irreducible uniform lattice, it is *strongly rigid* (see PRASAD [7], § 8).

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