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THE LAGRANGE COMPLEX

BY

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RÉSUMÉ. — Nous définissons le complexe de co-chaînes (Λ, δ) , et nous prouvons le lemme de Poincaré pour l'opérateur δ . L'opérateur δ est utilisé dans le calcul des variations en vue de déduire les équations d'Euler-Lagrange. Le lemme de Poincaré fournit alors le critère suivant lequel un système d'équations est un système d'Euler-Lagrange.

ABSTRACT. — A cochain complex (Λ, δ) is defined, and the δ -Poincaré lemma is proved. The work is motivated by applications to the calculus of variations. The operator δ is used in the calculus of variations to construct the Euler-Lagrange equations, and the δ -Poincaré lemma provides criteria for partial differential equations to be Euler-Lagrange equations.

The present paper generalizes results contained in earlier publications ([6], [8]) which were applicable to ordinary differential equations of the Euler-Poisson type.

1. Jets and tangent vectors

Let M be a C^∞ -manifold. We denote by $T^{(k)} M$ the manifold $J_0^k(\mathbf{R}^p, M)$ of jets of order k from \mathbf{R}^p to M with source 0 called by EHRESMANN [1] *p^k-vitesses* in M . Elements of $T^{(k)} M$ are equivalence classes of smooth mappings of \mathbf{R}^p into M . Two mappings γ and γ' are equivalent if $D^n(f \circ \gamma)(0) = D^n(f \circ \gamma')(0)$ for each C^∞ -function f on M and each $n = (n_1, \dots, n_p) \in \mathbf{N}^p$ such that $|n| = n_1 + \dots + n_p \leq k$. The symbol $D^n g(0)$ is used to denote the partial derivative of a function g :

$$\mathbf{R}^p \rightarrow \mathbf{R} : (t_1, \dots, t_p) \mapsto g(t_1, \dots, t_p)$$

of orders n_1, \dots, n_p with respect to the arguments t_1, \dots, t_p respectively at $(t_1, \dots, t_p) = (0, \dots, 0)$. We denote by $j_0^k(\gamma)$ the jet of the mapping γ . For each $k \in \mathbf{N}$, there is the projection

$$\tau_{(k)} : T^{(k)} M \rightarrow M : j_0^k(\gamma) \mapsto \gamma(0)$$

and, if $k' \leq k$, then there is the projection

$$\rho_{(k')(k)} : T^{(k)} M \rightarrow T^{(k')} M : j_0^k(\gamma) \mapsto j_0^{k'}(\gamma).$$

The manifold $T^{(0)} M$ is identified with M , and $T^{(1)} M$ is the tangent bundle TM of M . For each $n \in \mathbb{N}^p$ such that $|n| \leq k$ and each C^∞ -function f on M there is a C^∞ -function f_n defined on $T^{(k)} M$ by $f_n(j_0^k(\gamma)) = D^n(f \circ \gamma)(0)$.

For each $k \in \mathbb{N}$, we introduce an equivalence relation in the set of smooth mappings of \mathbb{R}^{p+1} into M . Two mappings χ and χ' will be considered equivalent if $D^{(r,n)}(f \circ \chi)(0) = D^{(r,n)}(f \circ \chi')(0)$ for each C^∞ -function f on M , each $n \in \mathbb{N}^p$ such that $|n| \leq k$ and $r = 0, 1$. The symbol $D^{(r,n)}g(0)$ denotes the partial derivative of a function g :

$$\mathbb{R}^{p+1} \rightarrow \mathbb{R} : (s, t_1, \dots, t_p) \mapsto g(s, t_1, \dots, t_p)$$

of orders r, n_1, \dots, n_p with respect to the arguments s, t_1, \dots, t_p respectively at $(s, t_1, \dots, t_p) = (0, 0, \dots, 0)$. We denote the equivalence class of the mapping χ by $j_0^{(1,k)}(\chi)$. The set of equivalence classes can be canonically identified with the tangent bundle $TT^{(k)} M$ in such a way that

$$\langle j_0^{(1,k)}(\chi), df_n \rangle = D^{(1,n)}(f \circ \chi)(0)$$

for each function f on M and each $n \in \mathbb{N}^p$ such that $|n| \leq k$ and also

$$\tau_{T^{(k)}M}(j_0^{(1,k)}(\chi)) = j_0^k(\chi_0),$$

where $\tau_{T^{(k)}M} : TT^{(k)} M \rightarrow T^{(k)} M$ is the tangent bundle projection, and χ_0 is the mapping

$$\chi_0 : \mathbb{R}^p \rightarrow M : (t_1, \dots, t_p) \mapsto \chi(0, t_1, \dots, t_p) \quad [7].$$

The tangent mapping $T\rho_{(k')(k)} : TT^{(k)} M \rightarrow TT^{(k')} M$ is given by

$$T\rho_{(k')(k)}(j_0^{(1,k)}(\chi)) = j_0^{(1,k')}(\chi).$$

For each $k \in \mathbb{N}$ and each $m \in \mathbb{N}^p$ there is the mapping

$$F_m : TT^{(k)} M \rightarrow TT^{(k)} M : j_0^{(1,k)}(\chi) \mapsto j_0^{(1,k)}(\chi_m),$$

where χ_m is the mapping

$$\chi_m : \mathbb{R}^{p+1} \rightarrow M : (s, t_1, \dots, t_p) \mapsto \chi(st^m, t_1, \dots, t_p),$$

and $t^m = t_1^{m_1} \dots t_p^{m_p}$. Diagrams

$$\begin{array}{ccc} TT^{(k)} M & \xrightarrow{F_m} & TT^{(k)} M \\ \tau_{T^{(k)}M} \downarrow & & \downarrow \tau_{T^{(k)}M} \\ T^{(k)} M & = & T^{(k)} M \end{array}$$

and

$$\begin{array}{ccc} TT^{(k)} M & \xrightarrow{F_m} & TT^{(k)} M \\ \downarrow T \rho_{(k')(k)} & & \downarrow T \rho_{(k')(k)} \\ TT^{(k')} M & \xrightarrow{F_m} & TT^{(k')} M \end{array}$$

are commutative.

For each $\alpha = 1, \dots, p$ and each $k \in \mathbf{N}$, there is the mapping

$$\mathbf{T}^\alpha : T^{(k+1)} M \rightarrow TT^{(k)} M : j_0^{k+1}(\gamma) \mapsto j_0^{(1,k)}(\gamma^\alpha),$$

where γ^α is the mapping

$$\gamma^\alpha : \mathbf{R}^{p+1} \rightarrow M : (s, t_1, \dots, t_p) \mapsto \gamma(t_1, \dots, t_\alpha + s, \dots, t_p) \quad (1).$$

Diagrams

$$\begin{array}{ccc} T^{(k+1)} M & \xrightarrow{\mathbf{T}^\alpha} & TT^{(k)} M \\ \downarrow \rho_{(k)(k+1)} & & \downarrow \tau_{T^{(k)} M} \\ T^{(k)} M & = & T^{(k)} M \end{array}$$

and

$$\begin{array}{ccc} T^{(k+1)} M & \xrightarrow{\mathbf{T}^\alpha} & TT^{(k)} M \\ \downarrow \rho_{(k'+1)(k+1)} & & \downarrow T \rho_{(k')(k)} \\ T^{(k'+1)} M & \xrightarrow{\mathbf{T}^\alpha} & TT^{(k')} M \end{array}$$

are commutative.

2. Forms and derivations

Let $\Omega_k^{(q)}$ denote the \mathbf{R} -linear space of q -forms on $T^{(k)} M$, and let $\Omega_{(k)}$ be the nonnegative graded linear space $\{\Omega_{(k)}^q\}$. The exterior differential d is a collection $\{d^q\}$ of linear mappings

$$d^q : \Omega_{(k)}^q \rightarrow \Omega_{(k)}^{q+1}$$

and the exterior product \wedge is a collection $\{\wedge^{(q,q')}\}$ of operations $\wedge^{(q,q')} : \Omega_{(k)}^q \times \Omega_{(k)}^{q'} \rightarrow \Omega_{(k)}^{q+q'}$. For each $k' \leq k$ and each q , there is the cotangent mapping $\rho_{(k')(k)}^* : \Omega_{(k')}^q \rightarrow \Omega_{(k)}^q$ corresponding to the mapping $\rho_{(k')(k)} : T^{(k)} M \rightarrow T^{(k')} M$, and, if $k'' \leq k' \leq k$, then

$$\rho_{(k')(k)}^* \circ \rho_{(k'')(k')}^* = \rho_{(k'')(k)}^*.$$

(1) The mappings \mathbf{T}^α are related to the *holonomic lift* λ defined by KUMPERA [3].

Hence $(\Omega_{(k)}^q, \rho_{(k')(k)}^*)$ is a directed system. Let Ω^q denote the direct limit of this system, and let Ω be the graded linear space $\{\Omega^q\}$. The underlying set of Ω^q is the quotient set of $\bigcup_k \Omega_{(k)}^q$ by the equivalence relation according to which two forms $\mu \in \Omega_{(k)}^q$ and $\nu \in \Omega_{(k')}^q$ are equivalent if $k' \leq k$ and $\mu = \rho_{(k')(k)}^* \nu$, or $k' \geq k$ and $\nu = \rho_{(k)(k')}^* \mu$. The exterior differential d and the exterior product \wedge extend in a natural way to the direct limits giving the graded linear space Ω the structure of both a cochain complex and a commutative graded algebra. We write $\mu \in \Omega_{(k)}^q$ for an element μ of Ω^q if μ has a representative in $\Omega_{(k)}^q$. This notation could be justified by identifying $\Omega_{(k)}^q$ with the image of the canonical injection $\Omega_{(k)}^q \rightarrow \Omega^q$. A collection $a = \{a^q\}$ of linear mappings $a^q : \Omega^q \rightarrow \Omega^{q+r} : \mu \rightarrow a^q \mu$ is called a graded linear mapping of degree r . We write a instead of a^q if this can be done without causing any confusion. The exterior differential d is a graded linear mapping of degree 1.

DEFINITION 2.1. — A graded linear mapping $a = \{a^q\}$ of degree r is called a *derivation* of Ω of degree r if

$$a(\mu \wedge \nu) = a\mu \wedge \nu + (-1)^{qr} \mu \wedge a\nu, \quad \text{where } q = \text{degree } \mu.$$

The exterior differential d is a derivation of Ω of degree 1. If a and b are derivations of Ω of degrees r and s respectively, then

$$[a, b] = \{a^{q+s} b^q - (-1)^{rs} b^{q+r} a^q\}$$

is a derivation of Ω of degree $r+s$ called the commutator of a and b .

It follows from the general theory of derivations [2] that derivations of Ω are completely characterized by their action on Ω^0 and Ω^1 . In fact, a derivation is completely determined by its action on equivalence classes of f_n and df_n for each function f on M and each $n \in \mathbb{N}^p$. Following FRÖLICHER and NIJENHUIS [2], we call a derivation a a derivation of type i_* if it acts trivially on Ω^0 . We call a a derivation of type d_* if $[a, d] = 0$.

For each $m \in \mathbb{N}^p$, each $k \in \mathbb{N}$ and each $q > 0$ there is a linear mapping

$$i_{F_m} : \Omega_{(k)}^q \rightarrow \Omega_{(k)}^q : \mu \mapsto i_{F_m} \mu,$$

defined by

$$\begin{aligned} &\langle w_1 \wedge \dots \wedge w_q, i_{F_m} \mu \rangle \\ &= \langle F_m(w_1) \wedge w_2 \wedge \dots \wedge w_q, \mu \rangle \\ &+ \langle w_1 \wedge F_m(w_2) \wedge \dots \wedge w_q, \mu \rangle + \dots + \langle w_1 \wedge w_2 \wedge \dots \wedge F_m(w_q), \mu \rangle, \end{aligned}$$

where w_1, \dots, w_q are vectors in $TT^{(k)} M$ such that $\tau_{T^{(k)}M}(w_1) = \dots = \tau_{T^{(k)}M}(w_q)$ and $\mathbf{F}_m : TT^{(k)} M \rightarrow TT^{(k)} M$ is the mapping defined in Section 1. Due to commutativity of diagrams

$$\begin{array}{ccc} \Omega_{(k')}^q & \xrightarrow{i_{\mathbf{F}_m}} & \Omega_{(k')}^q \\ \rho_{(k') (k)}^* \downarrow & & \downarrow \rho_{(k') (k)}^* \\ \Omega_{(k)}^q & \xrightarrow{i_{\mathbf{F}_m}} & \Omega_{(k)}^q \end{array}$$

the mappings $i_{\mathbf{F}_m}$ extend to a derivation $i_{\mathbf{F}_m}$ of Ω of type i_* and degree 0. If $\mu \in \Omega_{(k)}^q$, then $i_{\mathbf{F}_m} \mu \in \Omega_{(k)}^q$ and $i_{\mathbf{F}_m} \mu = 0$ if $\mu \in \Omega_{(k)}^q$ and $|m| > k$.

For each $\alpha = 1, \dots, p$, each $k \in \mathbb{N}$, and each $q \in \mathbb{N}$, there is a linear mapping

$$i_{\mathbf{T}^\alpha} : \Omega_{(k)}^{q+1} \rightarrow \Omega_{(k+1)}^q : \mu \mapsto i_{\mathbf{T}^\alpha} \mu,$$

defined by

$$\langle w_1 \wedge \dots \wedge w_q, i_{\mathbf{T}^\alpha} \mu \rangle = \langle x \wedge u_1 \wedge \dots \wedge u_q, \mu \rangle,$$

where

$$\begin{aligned} x &= \mathbf{T}^\alpha(v), & v &= \tau_{T^{(k+1)}M}(w_1) = \dots = \tau_{T^{(k+1)}M}(w_q), \\ u_1 &= T\rho_{(k+1), (h)}(w_1), & \dots, & & u_q &= T\rho_{(k+1), (h)}(w_q), \end{aligned}$$

and $\mathbf{T}^\alpha : T^{(k+1)} M \rightarrow TT^{(k)} M$ is the mapping defined in Section 1. Due to commutativity of diagrams

$$\begin{array}{ccc} \Omega_{(k')}^{q+1} & \xrightarrow{i_{\mathbf{T}^\alpha}} & \Omega_{(k'+1)}^q \\ \rho_{(k') (k)}^* \downarrow & & \downarrow \rho_{(k'+1) (k+1)}^* \\ \Omega_{(k)}^{q+1} & \xrightarrow{i_{\mathbf{T}^\alpha}} & \Omega_{(k+1)}^q \end{array}$$

the mappings $i_{\mathbf{T}^\alpha}$ extend to a derivation $i_{\mathbf{T}^\alpha}$ of Ω of type i_* and degree -1 . A derivation $d_{\mathbf{T}^\alpha}$ of Ω of type d_* and degree 0 is defined by $d_{\mathbf{T}^\alpha} = [i_{\mathbf{T}^\alpha}, d]$. If $\mu \in \Omega_{(k)}^{q+1}$, then $i_{\mathbf{T}^\alpha} \mu \in \Omega_{(k+1)}^q$, and $d_{\mathbf{T}^\alpha} \mu \in \Omega_{(k+1)}^{q+1}$.

For each $\alpha = 1, \dots, p$ let e^α denote the element $(e_1^\alpha, \dots, e_p^\alpha)$ of \mathbb{N}^p defined by $e_\beta^\alpha = 1$ if $\alpha = \beta$, and $e_\beta^\alpha = 0$ if $\alpha \neq \beta$. Let \geq denote the partial ordering relation in \mathbb{N}^p defined by $(n_1, \dots, n_p) \geq (n'_1, \dots, n'_p)$ if

$$n_1 \geq n'_1, \dots, n_{p-1} \geq n'_{p-1} \quad \text{and} \quad n_p \geq n'_p.$$

For each $m \in \mathbb{N}^p$, let $m!$ denote $m_1! \dots m_p!$.

PROPOSITION 2.1. — *If $m \geq e^\alpha$ then*

$$[i_{F_m}, d_{T^\alpha}] = \frac{m!}{(m - e^\alpha)!} i_{F_{m - e^\alpha}}, \quad \text{and} \quad [i_{F_m}, d_{T^\alpha}] = 0$$

in all cases other than $m \geq e^\alpha$.

Proof. — The commutator $[i_{F_m}, d_{T^\alpha}]$ is a derivation and it is of type i_* since it acts trivially on Ω . It can be easily shown for each $n \in \mathbb{N}^p$ and each function f on M that $i_{F_m} df_n = (n!/(n - m)!) df_{n - m}$ if $n \geq m$, and $i_{F_m} df_n = 0$ in all other cases. Also $d_{T^\alpha} f_n = f_{n + e^\alpha}$. It follows that

$$[i_{F_m}, d_{T^\alpha}] df_n = \frac{m!}{(m - e^\alpha)!} i_{F_{m - e^\alpha}} df_n \quad \text{if } m \geq e^\alpha,$$

and $[i_{F_m}, d_{T^\alpha}] df_n = 0$ in all cases other than $m \geq e^\alpha$. This completes the proof since a derivation of type i_* is completely determined by its action on equivalence classes of df_n for each f and each $n \in \mathbb{N}^p$.

PROPOSITION 2.2. — *For each $\alpha, \beta = 1, \dots, p, [d_{T^\alpha}, d_{T^\beta}] = 0$.*

Proof. — Obvious.

3 The Lagrange complex (Λ, δ) ⁽²⁾

Let $\tau = \{ \tau^q \}$ be the graded linear mapping of Ω into Ω of degree 0 defined by $\tau^0 = 1$ and

$$\tau^q \mu = \frac{1}{q} \sum_{|m| \leq k} (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m} \mu,$$

where $q > 0, \mu \in \Omega_{(k)}^p$, and $d_T^m = (d_{T^1})^{m_1} \dots (d_{T^p})^{m_p}$. The sum in the above definition contains all nonzero terms $(-1)^{|m|} (m!)^{-1} d_T^m i_{F_m} \mu$ since $i_{F_m} \mu = 0$ unless $|m| \leq k$. We write

$$\tau^q = \frac{1}{q} \sum_m (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m}$$

without explicitly restricting the summation range which is understood to be wide enough to include in the sum all nonzero terms when τ^q is applied to an element of Ω^q .

PROPOSITION 3.1. — *If $q > 0$, then $\tau^q d_{T^\alpha} = 0$ for each $\alpha = 1, \dots, p$.*

⁽²⁾ For definitions of algebraic topology terms used in this and the following sections, see reference [5].

Proof :

$$\begin{aligned} \tau^q d_{T^\alpha} &= \frac{1}{q} \sum_m (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m} d_{T^\alpha} \\ &= \frac{1}{q} \sum_m (-1)^{|m|} (m!)^{-1} (d_T^{m+e^\alpha} i_{F_m} + d_T^m [i_{F_m}, d_{T^\alpha}]) \\ &= \frac{1}{q} \sum_m (-1)^{|m|} (m!)^{-1} d_T^{m+e^\alpha} i_{F_m} \\ &\quad + \frac{1}{q} \sum_{m \geq e^\alpha} (-1)^{|m|} ((m-e^\alpha)!)^{-1} d_T^m i_{F_{m-e^\alpha}} = 0. \end{aligned}$$

It follows from proposition 3.1, that $\tau\tau = \tau$ and $\tau d\tau = \tau d$.

PROPOSITION 3.2. — *The graded linear mapping $\tau d = \{\tau^{q+1} d^q\}$ is a differential of degree 1.*

Proof. — $\tau d\tau d = \tau dd = 0$ and $\text{degree}(\tau d) = \text{degree } \tau + \text{degree } d = 1$.

We introduce the graded linear space $\Lambda = \{\Lambda^q\}$, where $\Lambda^q = \text{im } \tau^q$. The differential τd can be restricted to Λ due to $\tau d\tau = \tau d$.

The restriction of τd to Λ is a differential of degree 1 denoted by δ .

DEFINITION 3.1. — The differential $\delta = \{\delta^q\}$ is called the *Lagrange differential*, and the cochain complex $\{\Lambda^q, \delta^q\}$ is called the *Lagrange complex*.

THEOREM 3.1 (*δ -Poincaré lemma*). — *If the manifold M is contractible then the Lagrange complex $\{\Lambda^q, \delta^q\}$ is acyclic for $q > 0$.*

Let \mathbf{R} denote the subspace of $\Lambda^0 = \Omega^0$ consisting of equivalence classes of constant functions and let $\gamma : G \rightarrow \Lambda^0$ be the canonical injection of the subspace $G = \mathbf{R} \oplus (d_{T^1}(\Omega^0) + \dots + d_{T^p}(\Omega^0))$.

THEOREM 3.2. — *The mapping $\gamma : G \rightarrow \Lambda^0$ is an augmentation of the Lagrange complex and the sequence*

$$0 \rightarrow G \xrightarrow{\gamma} \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \dots \xrightarrow{\delta^{q-1}} \Lambda^q \xrightarrow{\delta^q} \dots$$

is a resolution of G .

We give proofs of the two theorems in the following section after having constructed a resolution of the graded linear space $\Lambda' = \{\Lambda^q\}_{q>0}$.

4. A resolution of Λ'

Let K be the simplicial complex with vertices $1, \dots, p$, and let $\Delta_r(K)$ denote the free abelian group generated by the ordered r -simplexes of K [5].

We introduce a bigraded linear space $\Phi = \{ \Phi_r^q \}$, where $\Phi_r^q = \Delta_{r-1}(K) \otimes \Omega^q$ for $r > 0$, $\Phi_0^q = \Omega^q$, and $\Phi_r^q = 0$ for $r < 0$. Elements of Φ_r^q are said to be of bidegree (q, r) . The exterior differential in Ω is extended to a bigraded linear mapping $d = \{ d_r^q \}$ of bidegree $(1, 0)$ by the formula

$$d_r^q((\alpha_1, \dots, \alpha_r) \otimes \mu) = (\alpha_1, \dots, \alpha_r) \otimes d\mu,$$

where $(\alpha_1, \dots, \alpha_r)$ is an ordered $r+1$ -simplex and $\mu \in \Omega^q$. A bigraded linear mapping $\partial = \{ \partial_r^q \}$ of bidegree $(0, -1)$ is defined by

$$\partial_r^q((\alpha_1, \dots, \alpha_r) \otimes \mu) = \sum_{1 \leq i \leq r} (-1)^{i-1} (\alpha_1, \dots, \alpha_i, \dots, \alpha_r) \otimes d_{T\alpha_i} \mu.$$

For each fixed r , $\{ \Phi_r^q, d_r^q \}$ is a cochain complex, and for each fixed q , $\{ \Phi_r^q, \partial_r^q \}$ is a chain complex. Since $\partial_r^{q+1} d_r^q = d_{r-1}^q \partial_r^q$, for each fixed r the collection $\{ \partial_r^q : \Phi_r^q \rightarrow \Phi_{r-1}^q \}$ is a cochain mapping, and for each fixed q the collection $\{ d_r^q : \Phi_r^q \rightarrow \Phi_r^{q+1} \}$ is a chain mapping.

PROPOSITION 4.1. — For each fixed $q > 0$ the chain complex $\{ \Phi_r^q, \partial_r^q \}$ is acyclic for $r > 0$.

Proof. — For each $\alpha = 1, \dots, p$, let a graded linear mapping

$$\sigma_\alpha = \{ \sigma_\alpha^q : \Omega^q \rightarrow \Omega^q \}$$

be defined by $\sigma_\alpha^0 = 0$ and

$$\sigma_\alpha^q = -\frac{1}{q} \sum_{m \in I_\alpha} (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m}, \quad \text{where } q > 0,$$

$I_\alpha = \{ m \in \mathbb{N}^p; m_\alpha > 0, m_\beta = 0 \text{ for } \beta > \alpha \}$ and the summation range is governed by a convention similar to the one used in the definition of τ in Section 3. From Proposition 2.1, it follows easily for $q > 0$ that $\sigma_\alpha^q d_{T\beta} = 0$ if $\beta < \alpha$, $\sigma_\alpha^q d_{T\alpha} = 1 - \sum_{\gamma < \alpha} d_{T\gamma} \sigma_\gamma^q$, and $\sigma_\alpha^q d_{T\beta} = d_{T\beta} \sigma_\alpha^q$ if $\beta > \alpha$. A bigraded linear mapping $D = \{ D_r^q \}$ is defined by $D_0^q \mu = \sum_\beta (\beta) \otimes \sigma_\beta^q \mu$ and

$$D_r^q((\alpha_1, \dots, \alpha_r) \otimes \mu) = \sum_{\beta < \alpha_1} (\beta, \alpha_1, \dots, \alpha_r) \otimes \sigma_\beta^q \mu,$$

where $\mu \in \Omega^q$ and $\alpha_1 < \alpha_2 < \dots < \alpha_r$. Relations $\partial_{r+1}^q D_r^q + D_{r-1}^q \partial_r^q = 1$ for $r > 0, q > 0$ are readily verified using the above stated properties of σ_α . It follows that for each fixed $q > 0$ the graded mapping $D^q = \{ D_r^q \}$ defines a chain contraction of $\{ \Phi_r^q, \partial_r^q \}$ for $r > 0$. Hence $\{ \Phi_r^q, \partial_r^q \}$ is acyclic for $r > 0$.

PROPOSITION 4.2. — For each $q > 0$, the mapping $\tau^q : \Phi_0^q \rightarrow \Lambda^q$ is an augmentation of the chain complex $\{ \Phi_r^q, \partial_r^q \}$ and the sequence

$$\dots \rightarrow \Phi_r^q \xrightarrow{\delta_r^q} \Phi_{r-1}^q \xrightarrow{\delta_{r-1}^q} \dots \xrightarrow{\delta_1^q} \Phi_0^q \xrightarrow{\tau^q} \Lambda^q \rightarrow 0$$

is a resolution of Λ^q .

Proof. — The mapping $\tau^q : \Omega^q \rightarrow \Lambda^q$ is an epimorphism, and $\tau^q \partial_1^q = 0$ follows from Proposition 3.1. Further $\tau^q + \partial_1^q D_0^q = 1$, where D_0^q is the mapping defined in the proof of Proposition 4.1. Hence $\tau^q \mu = 0$ implies $\mu = \partial_1^q D_0^q \mu$ for each $\mu \in \Omega^q$. It follows that $\ker \tau^q = \text{im } \partial_1^q$.

Proof of Theorems 3.1 and 3.2. — We define a nonnegative graded linear space $C = \{C_r\}$ by $C_0 = \mathbf{R}$ and $C_r = \Delta_{r-1}(K) \otimes \mathbf{R}$ for $r > 0$, and a collection $\eta = \{\eta_r : C_r \rightarrow \Phi_r^0\}$ by $\eta_r = 1 \otimes \eta_0$, where $\eta_0 : \mathbf{R} \rightarrow \Omega^0$ is the canonical injection of the space $\mathbf{R} \subset \Omega^0$ of equivalence classes of constant functions identified with the field \mathbf{R} of constants. If the manifold M is contractible, then all rows except the bottom row of the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & C_p & \xrightarrow{\eta_p} & \Phi_p^0 & \xrightarrow{a_p} & \Phi_p^1 & \xrightarrow{d_p^1} & \dots & \rightarrow & \Phi_p^q & \xrightarrow{d_p^q} & \dots \\
 & & & & \downarrow \partial_p^0 & & \downarrow \partial_p^1 & & & & \downarrow \partial_p^q & & \\
 & & & & \vdots & & \vdots & & & & \vdots & & \\
 & & & & \downarrow \partial_1^0 & & \downarrow \partial_1^1 & & & & \downarrow \partial_1^q & & \\
 0 & \rightarrow & C_0 & \xrightarrow{\eta_0} & \Phi_0^0 & \xrightarrow{d_0^0} & \Phi_0^1 & \xrightarrow{d_0^1} & \dots & \rightarrow & \Phi_0^q & \xrightarrow{d_0^q} & \dots \\
 & & & & \downarrow \tau^0 & & \downarrow \tau^1 & & & & \downarrow \tau^q & & \\
 0 & \rightarrow & G & \xrightarrow{\gamma} & \Lambda^0 & \xrightarrow{\delta^0} & \Lambda^1 & \xrightarrow{\delta^1} & \dots & \rightarrow & \Lambda^q & \xrightarrow{\delta^q} & \dots \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & & & 0 & & 0 & & & & 0 & &
 \end{array}$$

are known to be exact and all columns for $q > 0$ are exact. For each $q > 0$, the top statement in the sequence

$$\begin{aligned}
 \ker(\partial_p^{q+p+1} d_p^{q+p}) &= \text{im } d_p^{q+p-1}, \\
 \ker(\partial_{p-1}^{q+p} d_{p-1}^{q+p-1}) &= \text{im } d_{p-1}^{q+p-2} + \text{im } \partial_p^{q+p-1}, \\
 &\dots\dots\dots \\
 \ker(\partial_1^{q+2} d_1^{q+1}) &= \text{im } d_1^q + \text{im } \partial_2^{q+1}, \\
 \ker(\tau^{q+1} d_0^q) &= \text{im } d_0^{q-1} + \text{im } \partial_1^q,
 \end{aligned}$$

is true, and each of the remaining statements follows from the one immediately above. Hence the bottom statement is true. The same holds for $q = 0$ if the bottom statement is replaced by

$$\ker(\tau^1 d_0^0) = \text{im } \eta_0 \otimes \text{im } \partial_1^0.$$

If $q > 0$ and μ is an element of $\Lambda^q \subset \Omega^q$, then $\tau^q \mu = \mu$, and $\delta^q \mu = \tau^{q+1} d_0^q \mu$. If $\delta^q \mu = 0$, then there are elements $\kappa \in \Phi_0^{q-1}$ and $\lambda \in \Phi_1^q$ such that $\mu = d_0^{q-1} \kappa + \partial_1^q \lambda$. It follows that

$$\mu = \tau^q \mu = \tau^q d_0^{q-1} \kappa = \tau^q d_0^{q-1} \tau^{q-1} \kappa = \delta^{q-1} \tau^{q-1} \kappa.$$

Hence $\ker \delta^q = \text{im } \delta^{q-1}$ and the Lagrange complex is acyclic for $q > 0$. We note that $\delta^0 = \tau^1 d_0^0$ and

$$G = \mathbf{R} \otimes (d_{T^1}(\Omega^0) + \dots + d_{T^p}(\Omega^0)) = \text{im } \chi_0 \otimes \text{im } \delta^0.$$

Hence $\ker \delta^0 = G$. It follows that the sequence

$$0 \rightarrow G \xrightarrow{\gamma} \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \dots \rightarrow \Lambda^q \xrightarrow{\delta^q} \dots$$

is exact.

5 . Applications of the δ -Poincaré lemma in the calculus of variations

A smooth mapping $\chi : \mathbf{R}^{p+1} \rightarrow M : (s, t_1, \dots, t_p) \mapsto \chi(s, t_1, \dots, t_p)$ will be called a *homotopy*. For each $s \in \mathbf{R}$, we denote by χ_s the mapping

$$\chi_s : \mathbf{R}^p \rightarrow M : (t_1, \dots, t_p) \mapsto \chi(s, t_1, \dots, t_p).$$

The mapping $\gamma = \chi_0$ will be called the *base* of the homotopy χ . We say that the homotopy χ is *constant* on $A \subset \mathbf{R}^p$ if $\chi(s, t_1, \dots, t_p) = \chi(0, t_1, \dots, t_p)$ for each $s \in \mathbf{R}$ and each $(t_1, \dots, t_p) \in A$. For each mapping

$$\varphi : \mathbf{R}^p \rightarrow M : (t_1, \dots, t_p) \mapsto \varphi(t_1, \dots, t_p),$$

we denote by $\varphi^{(k)}$ the mapping

$$\varphi^{(k)} : \mathbf{R}^p \rightarrow T^{(k)} M : (t_1, \dots, t_p) \mapsto j_{(t_1, \dots, t_p)}^{(k)}(\varphi).$$

For each homotopy χ , we denote by $\chi'^{(k)}$ the mapping

$$\chi'^{(k)} : \mathbf{R}^p \rightarrow TT^{(k)} M : (t_1, \dots, t_p) \mapsto j_{(0, t_1, \dots, t_p)}^{(1, k)}(\chi),$$

where $j_{(0, t_1, \dots, t_p)}^{(1, k)}(\chi)$ is a jet-like object similar to $j_0^{(1, k)}(\chi)$ defined in terms of partial derivatives at $(0, t_1, \dots, t_p)$ instead of $(0, 0, \dots, 0)$ and identified with an element of $TT^{(k)} M$.

Each element $L \in \Omega_{(k)}^0$ gives rise to a family of functions

$$\gamma \mapsto \int_V L \circ \gamma^{(k)},$$

defined on the set of smooth mappings of \mathbf{R}^p into M for each domain $V \subset \mathbf{R}^p$.

DEFINITION 5.1. — A mapping $\gamma : \mathbf{R}^p \rightarrow M$ is called an *extremal* of the family of functions

$$\gamma \mapsto \int_V L \circ \gamma^{(k)} \quad \text{if} \quad \left. \frac{d}{ds} \int_V L \circ \chi_s^{(k)} \right|_{s=0} = 0,$$

for each domain $V \subset \mathbf{R}^p$ and each homotopy χ with base γ constant on the boundary ∂V of V .

DEFINITION 5.2. — A form $\lambda \in \Omega_{(k')}^1$ is called an *Euler-Lagrange* form associated with $L \in \Omega_{(k)}^0$ if $i_{\mathbf{F}_m} \lambda = 0$ for each $m > 0$ and if

$$\int_V \langle \chi'^{(k)}, dL \rangle = \int_V \langle \chi'^{(k)}, \lambda \rangle$$

for each domain $V \subset \mathbf{R}^p$ and each homotopy χ constant on ∂V .

It is clear from the definition of \mathbf{F}_m that if $\lambda \in \Omega_{(k')}^1$ satisfies $i_{\mathbf{F}_m} \lambda = 0$ for each $m > 0$, then λ can be interpreted as a mapping $\lambda : T^{(k')} M \rightarrow T^* M$. If λ is an Euler-Lagrange form associated with L then

$$\begin{aligned} \left. \frac{d}{ds} \int_V L \circ \chi_s^{(k)} \right|_{s=0} &= \int_V \langle \chi'^{(k)}, dL \rangle \\ &= \int_V \langle \chi'^{(k)}, \lambda \rangle \\ &= \int_V \langle \chi'^{(0)}, \lambda \circ \gamma^{(k')} \rangle, \end{aligned}$$

for each homotopy χ with base γ constant on ∂V . It follows that $\gamma : \mathbf{R}^p \rightarrow M$ is an extremal of the family

$$\gamma \mapsto \int_V L \circ \gamma^{(k)},$$

if, and only if, γ satisfies the equation $\lambda \circ \gamma^{(k')} = 0$ called the *Euler-Lagrange equation*.

We show that $\lambda = \delta^0 L$ is the unique Euler-Lagrange form associated with $L \in \Omega^0$. We also show that $i_{\mathbf{F}_m} \lambda = 0$ for each $m > 0$ means that $\lambda \in \Omega^1$ is in Λ^1 . These statements imply applications of the δ -Poincaré lemma. A form $\lambda \in \Omega^1$ is an Euler-Lagrange form if, and only if, $\lambda \in \Lambda^1$ and $\delta^1 \lambda = 0$. Euler-Lagrange forms associated with two elements L and L' of Ω^0 are the same if, and only if, $L' - L \in \mathbf{R} \oplus (d_{\mathbf{T}^1}(\Omega^0) + \dots + d_{\mathbf{T}^p}(\Omega^0))$.

PROPOSITION 5.1. — A form $\lambda \in \Omega^1$ belongs to Λ^1 if, and only if, $i_{F_m} \lambda = 0$ for each $m > 0$.

Proof. — If $i_{F_m} \lambda = 0$ for each $m > 0$, then

$$\tau^1 \lambda = \sum_m (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m} \lambda = i_{F_0} \lambda = \lambda.$$

Hence $\lambda \in \text{im } \tau^1 = \Lambda^1$. From Proposition 2.1, it follows that

$$i_{F_{e^\alpha}} d_T^m = d_T^m i_{F_{e^\alpha}} + (m! / (m - e^\alpha)!) d_T^{m - e^\alpha} i_{F_0}$$

if $m \geq e^\alpha$ and $i_{F_{e^\alpha}} d_T^m = d_T^m i_{F_{e^\alpha}}$ in all other cases. Since $i_{F_m} i_{F_n} \mu = i_{F_{m+n}} \mu$ for each $\mu \in \Omega^1$, it follows that

$$\begin{aligned} i_{F_{e^\alpha}} \tau^1 &= \sum_m (-1)^{|m|} (m!)^{-1} i_{F_{e^\alpha}} d_T^m i_{F_m} \\ &= \sum_m (-1)^{|m|} (m!)^{-1} d_T^m i_{F_{m+e^\alpha}} \\ &\quad + \sum_{m \geq e^\alpha} (-1)^{|m|} ((m - e^\alpha)!)^{-1} d_T^{m - e^\alpha} i_{F_m} = 0. \end{aligned}$$

Consequently, $i_{F_m} \tau^1 = 0$ for each $m > 0$, and if $\lambda \in \Lambda^1$ then $i_{F_m} \lambda = 0$ for each $m > 0$.

PROPOSITION 5.2. — The space Ω^1 is the direct sum of Λ^1 and

$$d_{T^1}(\Omega^1) + \dots + d_{T^p}(\Omega^1).$$

Proof. — Let μ be an element of Ω^1 . Then $\mu = \lambda + v$, where $\lambda = \tau^1 \mu \in \Lambda^1$, and

$$v = -\sum_{m > 0} (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m} \mu \in d_{T^1}(\Omega^1) + \dots + d_{T^p}(\Omega^1).$$

It follows from $\tau^1 \tau^1 = \tau^1$ and $\tau^1 d_{T^\alpha} = 0$ that this decomposition of μ into elements of Λ^1 and $d_{T^1}(\Omega^1) + \dots + d_{T^p}(\Omega^1)$ is unique.

PROPOSITION 5.3. — Let μ be an element of $\Omega_{(k)}^1$. Then

$$\int_V \langle \chi'^{(k)}, \mu \rangle = 0,$$

for each domain $V \subset \mathbf{R}^p$ and each homotopy $\chi : \mathbf{R}^{p+1} \rightarrow M$ constant on ∂V if, and only if, $\mu \in d_{T^1}(\Omega^1) + \dots + d_{T^p}(\Omega^1)$.

Proof. — If $\mu = \sum_\alpha d_{T^\alpha} \omega^\alpha$ then

$$\int_V \langle \chi'^{(k)}, \mu \rangle = \sum_\alpha \int_V \frac{\partial}{\partial t^\alpha} \langle \chi'^{(k)}, \omega^\alpha \rangle = \sum_\alpha \int_{\partial V} n_\alpha \langle \chi'^{(k)}, \omega^\alpha \rangle,$$

where n_z are the components of the normal vector. If χ is constant on ∂V , then

$$\int_V \langle \chi'^{(k)}, \mu \rangle = 0.$$

Let $\mu = \lambda + v$ be the unique decomposition of $\mu \in \Omega^1$ used in the proof of proposition 5.2. If $\int_V \langle \chi'^{(k)}, \mu \rangle = 0$, then

$$\int_V \langle \chi'^{(k)}, \lambda \rangle = \int_V \langle \chi'^{(0)}, \lambda \circ \gamma^{(k')} \rangle = 0,$$

where γ is the base of χ , and λ is interpreted as a mapping $\lambda : T^{(k)} M \rightarrow T^* M$. It follows that $\lambda = 0$ and $\mu = v$. Hence $\mu \in d_{T^1}(\Omega^1) + \dots + d_{T^p}(\Omega^1)$.

COROLLARY. — *If L is an element of Ω^0 , then $\lambda = \delta^0 L$ is the unique element of Λ^1 such that $dL - \lambda \in d_{T^1}(\Omega^1) + \dots + d_{T^p}(\Omega^1)$. It follows that λ is the unique Euler-Lagrange form associated with L .*

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