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TWO REMARKS IN PRIME NUMBER THEORY

BY

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RÉSUMÉ. — Nous considérons deux problèmes de théorie additive des nombres premiers. Le premier concerne une inégalité dont un cas particulier est le suivant : $\min |p - q \alpha^a|$ (quand α est un nombre réel fixe plus grand que 1 ; p et q sont des nombres premiers, et le minimum est pris sur l'ensemble des entiers positifs inférieurs ou égaux à $4/\varepsilon$, où ε est un nombre fixe avec $0 < \varepsilon < 1/2$ est inférieur à p^ε pour une infinité de couples (p, q)). Le second résultat montre que si $\theta > 7/72$, l'intervalle $X, X + X^\theta$ contient au moins $\gg X^\theta$ nombres distincts qui sont sommes de deux nombres premiers impairs.

SUMMARY. — Two questions in additive prime number theory are considered. First is an inequality of which a special case is this $\min |p - q \alpha^a|$ (where α is a fixed real number exceeding 1 ; p, q are primes, and the minimum is over all positive integers a not exceeding $4/\varepsilon$ [$0 < \varepsilon < 1/2, \varepsilon$ fixed]) is less than p^ε for an infinity of prime pairs (p, q) . The second result is that if $\theta > 7/72$ then the interval $X, X + X^\theta$ containing at least $\gg X^\theta$ distinct numbers which are expressible as a sum of two odd primes.

1. Introduction

In this note, we consider two questions of an additive nature on prime numbers. Our results are as follows.

THEOREM 1. — *Let ε be a positive constant less than 1, and let N be any natural number exceeding $2 \varepsilon^{-1}$. Let $\alpha_1, \dots, \alpha_N$ be any given positive real numbers no two of which are equal. Then there exist two of the numbers α_j say β and γ such that the inequality*

$$|\beta p - \gamma q| < p^\varepsilon,$$

where p and q are required to be prime numbers has infinity of solutions in p, q .

By choosing $\alpha_1, \dots, \alpha_N$ appropriately, we get the following corollary.

COROLLARY. — *Let α be a real number exceeding 1 and ε as before. Then there exists a natural number a satisfying $1 \leq a < 1 + [2 \varepsilon^{-1}]$, such that the inequality*

$$|p - q \alpha^a| < p^\varepsilon,$$

where p and q are required to be prime numbers has infinity of solutions in p, q .

In other words, the inequality

$$\min_{a=1, 2, \dots, [2\varepsilon^{-1}]} \left| \alpha - \left(\frac{p}{q} \right)^{1/a} \right| < p^{-1+\varepsilon}$$

has infinity of solutions in prime pairs p, q .

THEOREM 2. — Let θ be a constant exceeding $7/72$, $X > X_0(\theta)$ and $h = X^\theta$. Let G denote the number of Goldbach numbers (a natural number n is said to be Golbach if there exist two odd prime numbers whose sum is n) in the interval $[X, X+h]$. Then G exceeds ch where c is an absolute positive constant.

The weaker version $G \geq 1$ of theorem 2 is due to H. L. MONTGOMERY and R. C. VAUGHAN and was communicated to me by Professor H. L. MONTGOMERY in a letter to me a few years back, and is now published [2]. I am very much indebted to Professor H. L. MONTGOMERY both for his letter and for his preprint.

2. Proof of theorem 1

Let $H = X^\varepsilon$ and for $X \leq x \leq 2X$ and any positive constant α , put

$$f(\alpha, x) = \vartheta\left(\frac{x + [\alpha H]}{\alpha}\right) - \vartheta\left(\frac{x}{\alpha}\right),$$

where, for positive real u , we have written $\vartheta(u) = \sum_{p \leq u} \log p$. A simple application of the prime number theorem shows that

$$\sum_{X \leq n \leq 2X} f(\alpha, n) = HX(1+o(1))$$

and so

$$\sum_{X \leq n \leq 2X} \sum_{j=1}^N f(\alpha, n) = NXH(1+o(1)).$$

From this equality it follows that, for some integer n satisfying $X \leq n \leq 2X$,

$$\sum_{j=1}^N f(\alpha_j, n) \geq NH(1+o(1))$$

(it may be remarked that here actually equality holds for some n). In view of the inequality (note that $N > 2\varepsilon^{-1}$),

$$\pi(x) - \pi(x-y) \leq \frac{2y}{\log y} \left(1 + \frac{8}{\log y} \right)$$

(this is a consequence of Selberg sieve ; the first result in this direction is due to G. H. HARDY and J. E. LITTLEWOOD who obtained a bigger constant in place of 2 by the use of the sieve method of V. BRUN) valid for all x, y satisfying $1 < y \leq x$ (see page 107 of [1]), it follows that there exist k_1, k_2 ($k_1 \neq k_2$) for which $f(\alpha_{k_1}, n) \neq 0$ and $f(\alpha_{k_2}, n) \neq 0$. From these, it follows that there exist primes p, q satisfying

$$\frac{n}{\alpha_{k_1}} \leq p \leq \frac{n + [\alpha_{k_1} H]}{\alpha_{k_1}}, \quad \frac{n}{\alpha_{k_2}} \leq q \leq \frac{n + [\alpha_{k_2} H]}{\alpha_{k_2}}$$

(note that $X \leq n \leq 2X$) and so

$$|p \alpha_{k_1} - q \alpha_{k_2}| \leq (\alpha_{k_1} + \alpha_{k_2}) H.$$

This is true for all X and, in particular, for $X = 2^M$ ($M = 1, 2, 3, \dots$). The pair (k_1, k_2) depends on M , but there are only finitely many ($\leq N^2$) pairs and so there exists some pair say $(1, 2)$ for simplicity, which is the same for an infinite subsequence of integers M . This proves Theorem 1.

3. Proof of Theorem 2

For integral h, x , and Y with $h \leq x^\epsilon$ and $x^\epsilon \leq Y \leq x/3$, consider the sum

$$\begin{aligned} S &= \sum_{y=Y}^{2Y} (\vartheta(x+h-y) - \vartheta(x-y)) (\vartheta(y+h) - \vartheta(y)), \\ &= \sum_{y=Y}^{2Y} (\vartheta(x+h-y) - \vartheta(x-y)) ((\vartheta(y+h) - \vartheta(y) - h) + h), \\ &= h \sum_{y=Y}^{2Y} (\vartheta(x+h-y) - \vartheta(x-y)), \\ &\quad + O(\max_{Y \leq y \leq 2Y} (\vartheta(x+h-y) - \vartheta(x-y)) Y^{1/2} \\ &\quad \times (\sum_{Y \leq y \leq 2Y} (\vartheta(y+h) - \vartheta(y) - h)^2)^{1/2}). \end{aligned}$$

The O -term is easily proved to be $O(h^2 Y \exp(-(\log x) 1/6))$ provided $h \geq Y^{(1/6)+\epsilon}$ (these results are due to A. SELBERG and M. N. HUXLEY, see [3]). The main term is easily seen to be

$$h \sum_{x-2Y \leq n \leq x-2Y+h-1} (\vartheta(n+Y) - \vartheta(n)),$$

which is asymptotic to $h^2 Y$ provided that $Y \geq x(7/12) + \epsilon$ (these results are due to A. E. INGHAM and M. N. HUXLEY, see [3]). Thus we have following result.

LEMMA 1. — *If h, x, Y are integers with $Y \geq h \geq Y^{1/6+\varepsilon}$ $Y \geq x^{7/12+\varepsilon}$ and $Y \leq x/3$, then*

$$\sum_{y=Y}^{2Y} (\vartheta(x+h-y) - \vartheta(x-y)) (\vartheta(y+h) - \vartheta(y)) = h^2 Y(1 + o(1)).$$

Next we record the following lemma.

LEMMA 2. — *There exists, under the conditions of lemma 1, an integer y_0 satisfying $Y \leq y_0 \leq 2Y$ such that*

$$(\vartheta(x+h-y_0) - \vartheta(x-y_0)) (\vartheta(y_0+h) - \vartheta(y_0)) \geq h^2 (1 + o(1))$$

(actually equality may be secured for a suitable y_0).

Proof. — Follows from lemma 1.

LEMMA 3. — *Let $r(n)$ denote the number of solutions of the equation $n = p_1 + p_2$ where p_1 and p_2 are prime numbers satisfying $x - y_0 \leq p_1 \leq x + h - y_0$ and $y_0 \leq p_2 \leq y_0 + h$. Then*

$$(\log x)^2 \sum_{n=x}^{x+2h} r(n) \geq h^2 (1 + o(1)).$$

Proof. — Follows from lemma 2.

LEMMA 4. — *We have*

$$r(n) \leq 16 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \times \prod_{2 < p | n} \left(1 + \frac{1}{p-2}\right) \frac{h}{(\log h)^2} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right)$$

where the constant implied by the O -symbol is absolute.

Proof. — This is corollary 5.8.3 on page 179 of [1].

LEMMA 5. — *We have*

$$\sum_{n=x}^{x+2h} \prod_{2 < p | n} \left(1 + \frac{1}{p-2}\right)^2 \leq 2h(1 + o(1)) \prod_{p>2} \left(1 + \frac{2p-3}{p(p-2)^2}\right).$$

Proof. — In the product on the left, the contributions from the primes $p > \log x$ are negligible, and so we may restrict to those $p \leq \log x$.

Accordingly the left side is

$$< 2h + 1 + \sum_r \sum_{x \leq p_1 \dots p_2 \leq x+2h} \prod_{j=1}^r (2(p_j-2)^{-1} + (p_j-2)^{-2}),$$

with $2 < p_1 < p_2 < \dots < p_r \leq \log x$,

$$< 2h + \sum_r \left(\frac{2h}{p_1 \dots p_r} + 2 \right) \prod_{j=1}^r (2(p_j-2)^{-1} + (p_j-2)^2)$$

with $2 < p_1 < p_2 < \dots < p_r \leq \log x$, and this proves lemma 5.

We now fix $h = [Y^{(1/6)+\varepsilon}]$, $Y = [x^{(7/12)+\varepsilon}]$ and apply Hölder's inequality to the inequality of lemma 3 and use lemmas 4 and 5. We see that theorem 2 is proved with any positive constant c satisfying

$$1/2 > 16 \cdot \left(\frac{72}{5} \right)^2 c^{1/2} \prod_{p>2} \left((1-(p-1)^{-2}) \left(1 + \frac{2p-3}{p(p-2)^2} \right) \right).$$

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