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## THE RESOLVENT FOR A CONVOLUTION KERNEL SATISFYING THE DOMINATION PRINCIPLE

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ABSTRACT. — Let  $N$  be a convolution kernel on a locally compact abelian group. It is shown that if  $N$  satisfies the domination principle and is non-singular, then there exists a splitting  $N = N_0 + N'$  of  $N$  in which  $N_0$  is a resolvent kernel and  $N'$  is  $N$ -invariant. Furthermore, the singular part  $N'$  of  $N$  is either  $N_0$ -invariant or a  $N_0$ -potential of a  $N$ -invariant measure. These results simplify Theorems of M. Irô.

RÉSUMÉ. — Soit  $N$  un noyau de convolution dans un groupe abélien localement compact. Pour  $N$  satisfaisant au principe de domination et étant non singulier, on démontre qu'il existe une partition  $N = N_0 + N'$  de  $N$ , où  $N_0$  est un noyau à résolvante et  $N'$  est  $N$ -invariante. De plus, la partie singulière  $N'$  de  $N$  est ou bien  $N_0$ -invariante ou bien un  $N_0$ -potentiel d'une mesure  $N$ -invariante. Ces résultats simplifient des théorèmes de M. Irô.

### Introduction

Let  $G$  be a locally compact abelian group and  $N$  a convolution kernel on  $G$  satisfying the domination principle. In [2], Irô introduced a family  $(N_p)_{p>0}$  of convolution kernels, which in later papers ([3], [5]) turned out to be the resolvent family for the regular part  $N_0$  of  $N$ . Some of the proofs in these papers are complicated, so it is of interest to give a simple and unified treatment of the resolvent and the regular part of  $N$  based entirely on the Riesz decomposition theorem, and this is the aim of the present paper.

A complete proof of the Riesz decomposition theorem was given in [7], which will be a prerequisite for the present paper. Less general versions of the Riesz decomposition Theorem appeared in [3] and [4], and the treatment in [3] assumes knowledge of the resolvent and the regular part.

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The idea behind our treatment is as follows:

For each  $p > 0$ , we have  $pN + \varepsilon_0 < N$ , so let  $N = (pN + \varepsilon_0) \star N_p + \eta_p$  be the Riesz decomposition of  $N$  with respect to  $pN + \varepsilon_0$  as sum of a  $(pN + \varepsilon_0)$ -potential, generated by a measure  $N_p$ , and a  $(pN + \varepsilon_0)$ -invariant measure  $\eta_p$ . The measures  $N_p$  and  $\eta_p$  are uniquely determined, and this leads to the resolvent equation for  $(N_p)_{p>0}$ . For  $p$  tending to zero,  $N_p$  increases to the regular part of  $N$ .

### Preliminaries

In the following,  $G$  denotes an arbitrary locally compact abelian group, and  $N$  a convolution kernel on  $G$  satisfying the domination principle.

A positive measure  $\xi$  on  $G$  is called  $N$ -excessive, if  $N$  satisfies the relative domination principle with respect to  $\xi$  ( $N < \xi$ ) (cf. [4], [7]). The set of  $N$ -excessive measures is a vaguely closed convex cone  $E(N)$ , which is infimum-stable, and every  $\xi \in E(N)$  is the vague limit of an increasing net of  $N$ -potentials. For an open subset  $\Omega \subseteq G$  and a measure  $\xi \in E(N)$ , the reduced measure  ${}_N R_\xi^\Omega$  of  $\xi$  over  $\Omega$  (with respect to  $N$ ) is defined (cf. [7]) as

$${}_N R_\xi^\Omega = \inf \{ \tau \in E(N); \tau \geq \xi \text{ in } \Omega \}.$$

We write  $R_\xi^\Omega$  instead of  ${}_N R_\xi^\Omega$  when  $N$  is clear from the context.

Let  $\mathcal{V}$  denote the set of compact neighbourhoods of 0 in  $G$ . A measure  $\xi \in E(N)$  is called  $N$ -invariant if  $R_\xi^{V'} = \xi$  for all  $V \in \mathcal{V}$ . The set of  $N$ -invariant measures is a convex cone  $I(N)$ , closed under increasing limits.

*Definition.* — The singular part  $N'$  of  $N$  is the limit  $N' = \lim_{V \uparrow G} R_N^{V'}$  of the decreasing net  $(R_N^{V'})_{V \in \mathcal{V}}$ , when  $V \in \mathcal{V}$  increases to  $G$ .

The regular part  $N_0$  of  $N$  is  $N_0 = N - N'$ . Note that  $N_0 \geq 0$ .

The convolution kernel  $N$  is called *singular* (resp. *non-singular*) if  $N_0 = 0$  (resp.  $N_0 \neq 0$ ).

The following Riesz decomposition theorem will be essential later (cf. [7]).

**PROPOSITION 1.** — *Suppose  $N$  is non-singular. Every  $\xi \in E(N)$  has a decomposition*

$$\xi = N \star \mu + \eta, \quad \text{where } \eta \in I(N).$$

*The invariant part  $\eta$  is uniquely determined, and the measure  $\mu$  is uniquely determined if (and only if)  $N$  satisfies the principle of unicity of mass.*

We shall use some alternative characterizations of  $N$ -excessive and  $N$ -invariant measures.

LEMMA 2 (ITÔ [4], LAUB [7]). — Suppose  $N$  is non-singular. A positive measure  $\xi$  is  $N$ -invariant if (and only if) there exists a net  $(\lambda_\alpha)_{\alpha \in A}$  of positive measures with compact support such that

$$N \star \lambda_\alpha \uparrow \xi, \quad \lambda_\alpha \rightarrow 0.$$

In Corollary 6 below the conclusion of Lemma 2 is shown to be valid also for singular kernels.

The following result is well-known and not difficult to establish.

LEMMA 3. — Let  $N = 1/a \sum_{n=0}^{\infty} \sigma^n$  be an elementary kernel ( $a > 0$ ). Then

- (i)  $\xi \in E(N) \Leftrightarrow \sigma \star \xi \leq \xi,$
- (ii)  $\eta \in I(N) \Leftrightarrow \sigma \star \eta = \eta.$

For  $c > 0$ , the convolution kernel  $N + c \varepsilon_0$  satisfies the domination principle and the principle of unicity of mass, where  $\varepsilon_0$  denotes the Dirac measure at 0. Moreover, it is easily seen that if  $N < \xi$  then  $N + c \varepsilon_0 < \xi$ , i. e.  $E(N) \subseteq E(N + c \varepsilon_0)$ . For  $V \in \mathcal{V}$ , we consequently have

$${}_N R_N^{cV} \geq_{(N+c\varepsilon_0)} R_{N+c\varepsilon_0}^{cV},$$

so that  $N + c \varepsilon_0$  is non-singular.

The following Lemma is an extension of [4] (Corollaire 1, p. 340); the hypothesis of  $N$  being non-singular is removed.

LEMMA 4. — For  $c > 0$ , we have

$$I(N) = I(N + c \varepsilon_0).$$

*Proof.* — Suppose first that  $\eta \in I(N + c \varepsilon_0)$ . By Lemma 2, there exists a net  $(\lambda_\alpha)_{\alpha \in A}$  of positive measures such that  $(N + c \varepsilon_0) \star \lambda_\alpha \uparrow \eta$  and  $\lambda_\alpha \rightarrow 0$ . Therefore, we have  $\eta = \lim N \star \lambda_\alpha$  and then  $\eta \in E(N)$ . For  $V \in \mathcal{V}$ , we find

$$\eta = {}_{(N+c\varepsilon_0)} R_\eta^{cV} \leq {}_N R_\eta^{cV} \leq \eta.$$

hence  $\eta \in I(N)$ .

Suppose next that  $\eta \in I(N)$ . Then  $\eta \in E(N) \subseteq E(N + c \varepsilon_0)$ , and since  $N + c \varepsilon_0$  is non-singular,  $\eta$  has a Riesz decomposition

$$\eta = (N + c \varepsilon_0) \star \nu_c + \eta_c,$$

where  $\eta_c \in I(N + c \varepsilon_0) \subseteq I(N)$ . We shall prove that  $\nu_c = 0$ .

Let  $V \in \mathcal{V}$ , and choose a net of positive measures  $(\mu_\alpha)_{\alpha \in A}$  with compact support in  $\overline{\mathbb{C}}V$  such that  $N \star \mu_\alpha \uparrow R_N^{\mathbb{C}V}$ . Since  $N < \eta$  the net  $(\eta \star \mu_\alpha)_{\alpha \in A}$  is increasing and

$$\lim_A \eta \star \mu_\alpha \leq \eta.$$

Now choose  $(\lambda_\beta)_{\beta \in B}$  such that  $N \star \lambda_\beta \uparrow \eta$ , and if  $N$  is non-singular with the additional property that  $\lambda_\beta \rightarrow 0$ . We then claim that

$$\lim_B (N - R_N^{\mathbb{C}V}) \star \lambda_\beta = 0.$$

This is true, because if  $N$  is singular then  $N = R_N^{\mathbb{C}V}$ , and if  $N$  is non-singular then  $N - R_N^{\mathbb{C}V}$  has compact support and  $\lambda_\beta \rightarrow 0$ . We then have

$$\begin{aligned} \eta &= \lim_B N \star \lambda_\beta = \lim_B R_N^{\mathbb{C}V} \star \lambda_\beta \\ &= \lim_B (\lim_A N \star \mu_\alpha \star \lambda_\beta) \leq \lim_A \eta \star \mu_\alpha, \end{aligned}$$

hence  $\eta = \lim_A \eta \star \mu_\alpha$ .

Since  $\eta_c \in I(N)$ , we similarly find  $\lim_A \eta_c \star \mu_\alpha = \eta_c$ . If  $\mu_{\mathbb{C}V}$  denotes a vague accumulation point of  $(\mu_\alpha)_{\alpha \in A}$ , we may assume that  $\mu_\alpha \rightarrow \mu_{\mathbb{C}V}$ , and since  $N \star \mu_\alpha \leq N$  and  $N \star v_c$  exists, Deny's convergence Lemma ([1], Lemma 5.2) shows that

$$\lim_A \mu_\alpha \star v_c = \mu_{\mathbb{C}V} \star v_c.$$

If we convolve all terms in the Riesz decomposition of  $\eta$  with  $\mu_\alpha$  and go to the limit, we obtain

$$\eta = R_N^{\mathbb{C}V} \star v_c + c \mu_{\mathbb{C}V} \star v_c + \eta_c.$$

Finally, letting  $V$  increase to  $G$ , Deny's convergence Lemma shows that  $\mu_{\mathbb{C}V} \star v_c \rightarrow 0$  because  $\text{supp}(\mu_{\mathbb{C}V}) \subseteq \overline{\mathbb{C}V}$ , and hence

$$\eta = N' \star v_c + \eta_c,$$

which compared to the original decomposition gives  $v_c = 0$ , so we have

$$\eta = \eta_c \in I(N + c\varepsilon_0).$$

As an application of Lemma 4, we prove the following result which will not be used in the sequel, but it might be of independent interest.

**PROPOSITION 5.** — *The following conditions about  $N$  are equivalent:*

- (i)  $N$  is singular.
- (ii)  $I(N) = E(N)$ .
- (iii) *There exists a net  $(\lambda_\alpha)_{\alpha \in A}$  of positive measures with compact support such that  $N \star \lambda_\alpha \uparrow N$  and  $\lambda_\alpha \rightarrow 0$ .*

*Proof.*

(i)  $\Rightarrow$  (ii): Let  $\mu$  be a positive measure such that  $N \star \mu$  exists. By Lemma 1.8 in [7], the net  $R_N^{\mathfrak{C}V} \star \mu$  decreases to  $N \star \mu$  as  $V$  increases to  $G$ , hence  $R_N^{\mathfrak{C}V} \star \mu = N \star \mu$  for all  $V \in \mathcal{V}$  so that  $N \star \mu \in I(N)$ . Since every measure  $\xi \in E(N)$  is an increasing limit of potentials  $N \star \mu$ , we get  $E(N) \subseteq I(N)$ .

(ii)  $\Rightarrow$  (iii): By Lemma 4, we have  $N \in I(N + \varepsilon_0)$ , so by Lemma 2 there exists a net  $(\lambda_\alpha)_{\alpha \in A}$  such that  $(N + \varepsilon_0) \star \lambda_\alpha \uparrow N$  and  $\lambda_\alpha \rightarrow 0$ . Therefore,  $N \star \lambda_\alpha \rightarrow N$ , and  $(N \star \lambda_\alpha)_{\alpha \in A}$  is increasing because  $N + \varepsilon_0 < N$ .

(iii)  $\Rightarrow$  (i): Let  $V \in \mathcal{V}$ , and suppose that  $N \star \lambda_\alpha \uparrow N$  and  $\lambda_\alpha \rightarrow 0$ . Writing  $\lambda_\alpha$  as sum of its restrictions  $\lambda_\alpha|_W$  and  $\lambda_\alpha|\mathfrak{C}W$  to  $W$  and  $\mathfrak{C}W$ , where  $W \in \mathcal{V}$  is a compact neighbourhood of  $V$ , we have  $N \star (\lambda_\alpha|\mathfrak{C}W) \rightarrow N$ . By the domination principle for measures, we find  $R_N^{\mathfrak{C}V} \geq N \star (\lambda_\alpha|\mathfrak{C}W)$ , so taking limits for  $\alpha \in A$  we get  $N \leq R_N^{\mathfrak{C}V}$ , which proves (i).

**COROLLARY 6.** — *The conclusion of Lemma 2 is valid also for singular convolution kernels satisfying the domination principle.*

*Proof.* — In order to show that a  $N$ -invariant measure  $\xi$  is the limit of an increasing net  $(N \star \lambda_\alpha)$  for which  $\lambda_\alpha \rightarrow 0$  one proceeds like in (ii)  $\Rightarrow$  (iii) above. In order to prove the converse one proceeds like in (iii)  $\Rightarrow$  (i).

A family  $(N_p)_{p>0}$  of convolution kernels is called a *resolvent* if

$$N_p = N_q + (q - p) N_p \star N_q \quad \text{for } p, q > 0.$$

A convolution kernel  $N$  is called a *resolvent kernel* if there exists a resolvent  $(N_p)_{p>0}$  such that  $N = \lim_{p \rightarrow 0} N_p$ .

A resolvent kernel  $N$  satisfies the domination principle, and for every  $V \in \mathcal{V}$  there exists a balayaged measure  $\varepsilon'_{\mathfrak{C}V}$  of  $\varepsilon_0$  on  $\mathfrak{C}V$  with respect to  $N$  such that  $R_N^{\mathfrak{C}V} = N \star \varepsilon'_{\mathfrak{C}V}$  (cf. [4], § 3). Since  $\text{supp } \varepsilon'_{\mathfrak{C}V} \subseteq \overline{\mathfrak{C}V}$ , we have  $\lim_{V \uparrow G} \varepsilon'_{\mathfrak{C}V} = 0$ , and therefore  $N' = \lim_{V \uparrow G} R_N^{\mathfrak{C}V} = 0$  because of the dominated convergence property of a resolvent kernel (cf. [4] or [6]).

Suppose now that  $N$  is a non-zero resolvent kernel. KISHI showed in [6] that  $\lim_{p \rightarrow \infty} p N_p$  exists and is the normalized Haar measure  $\omega_K$  of a compact subgroup  $K$  of  $G$ . The group  $K$  is the periodicity group for  $N$ , i. e.  $K = \{x \in G; N \star \varepsilon_x = N\}$ .

If  $\mu$  is a positive measure such that  $N \star \mu = N$ , it follows that  $N_p \star \mu = N_p$  for all  $p$ , hence by the convergence Lemma of DENY that  $\mu \star \omega_K = \omega_K$ .

This shows that  $\mu$  is a probability measure supported by  $K$ . In particular, every pseudo-period of  $N$  (i. e. a point  $x \in G$  such that  $N \star \varepsilon_x$  is proportional to  $N$ ) is a period for  $N$ .

Denoting by  $\mathcal{V}_K$  the set of compact neighbourhoods of  $K$  we consequently have  $N \star \varepsilon'_V \neq N$  for any  $V \in \mathcal{V}_K$ . This implies that the series  $\sum_{n=0}^{\infty} (\varepsilon'_V)^n$  converges and the following formula holds

$$N = (N - N \star \varepsilon'_V) \star \sum_{n=0}^{\infty} (\varepsilon'_V)^n, \quad V \in \mathcal{V}_K.$$

Using this notation, the sets  $E(N)$  and  $I(N)$  can be characterized in the following way:

**PROPOSITION 7.** — *Let  $N$  be a non-zero resolvent kernel with resolvent  $(N_p)_{p>0}$ . Then*

- (i)  $\xi \in E(N) \Leftrightarrow \forall p > 0 : p N_p \star \xi \leq \xi$ .
- (ii)  $\eta \in I(N) \Leftrightarrow \forall p > 0 (\exists p > 0) : p N_p \star \eta = \eta$ .
- (iii)  $\xi \in E(N) \Leftrightarrow \forall V \in \mathcal{V}_K : \varepsilon'_V \star \xi \leq \xi$  and  $\omega_K \star \xi = \xi$ .
- (iv)  $\eta \in I(N) \Leftrightarrow \forall V \in \mathcal{V}_K : \varepsilon'_V \star \eta = \eta$  and  $\omega_K \star \eta = \eta$ .

The invariant part of  $\xi \in E(N)$  is given as  $\lim_{p \rightarrow 0} p N_p \star \xi$ .

*Proof:*

(ii): If  $N$  is a resolvent kernel then  $N + 1/p \varepsilon_0 = 1/p \sum_{n=0}^{\infty} (p N_p)^n$  is an elementary kernel for every  $p > 0$  and hence Lemma 3 and 4 show that

$$\eta \in I(N) \Leftrightarrow \eta \in I\left(N + \frac{1}{p} \varepsilon_0\right) \Leftrightarrow p N_p \star \eta = \eta.$$

“(i)  $\Rightarrow$  ”: For  $\xi \in E(N)$ , Proposition 1 shows that  $\xi = N \star \mu + \eta$ , where  $\eta \in I(N)$ , and from this Riesz decomposition we obtain

$$p N_p \star \xi = p N_p \star N \star \mu + \eta \leq \xi.$$

“(i)  $\Leftarrow$  ”: From  $p N_p \star \xi \leq \xi$  follows by Lemma 3 that  $N + (1/p) \varepsilon_0 < \xi$ . Letting  $p$  tend to infinity we find that  $N < \xi$ .

“(iii)  $\Rightarrow$  ”: The statement holds for  $N$ -potentials and hence for every  $N$ -excessive measure.

“(iii)  $\Leftarrow$  ”: Lemma 3 proves that  $\xi$  is excessive with respect to the elementary kernel  $N_V = \sum_{n=0}^{\infty} (\varepsilon'_V)^n$ , so there exists a net  $(\lambda_\alpha)_{\alpha \in A}$  of positive measures such that  $N_V \star \lambda_\alpha \uparrow \xi$ . Using  $N = (N - N \star \varepsilon'_V) \star N_V$ , we get

$$N \star \lambda_\alpha \uparrow \xi \star (N - N \star \varepsilon'_V),$$

so that  $\xi \star (N - N \star \varepsilon'_{\mathfrak{t}V}) \in E(N)$ . Defining  $a_V = (N - N \star \varepsilon'_{\mathfrak{t}V})(G)$ , we have

$$\frac{1}{a_V} (N - N \star \varepsilon'_{\mathfrak{t}V}) \rightarrow \omega_K \text{ as } V \downarrow K,$$

which implies that

$$\xi = \xi \star \omega_K \in E(N).$$

“(iv)  $\Rightarrow$ ”: By Lemma 2 there exists a net  $(\lambda_\alpha)_{\alpha \in A}$  of positive measures such that  $N \star \lambda_\alpha \uparrow \eta$ ,  $\lambda_\alpha \rightarrow 0$ . Since  $N - N \star \varepsilon'_{\mathfrak{t}V}$  has compact support, we get

$$\eta - \eta \star \varepsilon'_{\mathfrak{t}V} = \lim_A (N - N \star \varepsilon'_{\mathfrak{t}V}) \star \lambda_\alpha = 0.$$

“(iv)  $\Leftarrow$ ”: By (iii)  $\eta \in E(N)$  and hence  $\eta = N \star \mu + \zeta$ , where  $\zeta \in I(N)$ , but since  $\eta = \eta \star \varepsilon'_{\mathfrak{t}V} = (N \star \varepsilon'_{\mathfrak{t}V}) \star \mu + \zeta$ , we get  $\mu = 0$  and then  $\eta \in I(N)$ .

If  $\xi \in E(N)$  has the Riesz decomposition  $\xi = N \star \mu + \eta$ , where  $\eta \in I(N)$ , we find  $p N_p \star \xi = (N - N_p) \star \mu + \eta$ , hence

$$\eta = \lim_{p \rightarrow 0} p N_p \star \xi.$$

**Main result**

**THEOREM 8.** — *Let  $N$  be a non-singular convolution kernel satisfying the domination principle. There exist a non-zero resolvent  $(N_p)_{p>0}$  and a positive measure  $\nu$  such that*

$$(1) \quad N = N_p \star (pN + \varepsilon_0 + \nu) \quad \text{for } p > 0.$$

The resolvent kernel  $\tilde{N} = \lim_{p \rightarrow 0} N_p$  exists, and denoting by  $K$  the compact periodicity group of  $\tilde{N}$ , the measure  $\nu$  can be chosen such that  $\nu \star \varepsilon_x = \nu$  for all  $x \in K$  and  $\nu \in I(N)$ .

*Proof.* — Let  $p > 0$  be fixed. Then  $pN + \varepsilon_0$  is a non-singular convolution kernel satisfying the domination principle and  $N \in E(pN + \varepsilon_0)$ . By Proposition 1, there exist positive measures  $N_p$  and  $\eta_p$  such that

$$N = (pN + \varepsilon_0) \star N_p + \eta_p,$$

where  $\eta_p \in I(pN + \varepsilon_0) = I(N)$ . Furthermore,  $N_p$  and  $\eta_p$  are uniquely determined,  $N_p$  because  $pN + \varepsilon_0$  satisfies the principle of unicity of mass.

For  $q > p > 0$ , we have

$$\begin{aligned} N &= (qN + \varepsilon_0) \star N_q + \eta_q = (q-p)N \star N_q + (pN + \varepsilon_0) \star N_q + \eta_q, \\ &= (q-p)((pN + \varepsilon_0) \star N_p + \eta_p) \star N_q + (pN + \varepsilon_0) \star N_q + \eta_q, \\ &= (pN + \varepsilon_0) \star (N_q + (q-p)N_p \star N_q) + \eta_q + (q-p)\eta_p \star N_q. \end{aligned}$$



The measure  $\eta_q + (q-p)\eta_p \star N_q$  is  $N$ -invariant because both  $\eta_p$  and  $\eta_q$  are so (cf. [7]). By the unicity of the Riesz decomposition with respect to  $pN + \varepsilon_0$ , we conclude

$$(2) \quad \begin{aligned} N_p &= N_q + (q-p)N_p \star N_q & \text{for } 0 < p < q, \\ \eta_p &= \eta_q + (q-p)\eta_p \star N_q & \text{for } 0 < p < q. \end{aligned}$$

This shows that  $(N_p)_{p>0}$  is a resolvent family, and since  $N_p \leq N$  for all  $p$  we get that  $\tilde{N} = \lim_{p \rightarrow 0} N_p$  exists. The resolvent kernel  $\tilde{N}$  is non-zero, because  $\tilde{N} = 0$  would imply that  $N \in I(pN + \varepsilon_0) = I(N)$ , hence that  $N$  is singular. Moreover  $N \geq \eta_p \geq \eta_q$  for  $p < q$  so the limit  $\eta_0 = \lim_{p \rightarrow 0} \eta_p$  exists and belongs to  $I(N)$ . From (2), we get

$$(3) \quad \eta_0 = \eta_q + qN_q \star \eta_0 \quad \text{for } q > 0,$$

which by Proposition 7 shows that  $\eta_0$  is excessive with respect to the resolvent kernel  $\tilde{N}$ , but since from (3)  $\lim_{q \rightarrow 0} qN_q \star \eta_0 = 0$ , the  $\tilde{N}$ -invariant part of  $\eta_0$  is 0. There exists consequently a positive measure  $\nu$  such that  $\eta_0 = \tilde{N} \star \nu$ . The measure  $\nu$  need not be uniquely determined. In fact,  $\tilde{N}$  has a compact periodicity group  $K$ , and denoting the normalized Haar measure of  $K$  by  $\omega_K$ , we have as well  $\eta_0 = \tilde{N} \star (\omega_K \star \nu)$ , so by replacing  $\nu$  by  $\omega_K \star \nu$ , we may and will assume that  $\nu$  is periodic with each  $x \in K$  as period. In this case,  $\nu$  is easily seen to be uniquely determined. From (3) follows

$$\eta_q = \tilde{N} \star \nu - qN_q \star \tilde{N} \star \nu = N_q \star \nu,$$

which implies (1).

Using  $pN_p \star \tilde{N} \leq \tilde{N}$  and that  $\tilde{N} \star \nu$  exists, the convergence Lemma of DENY implies that  $\lim_{p \rightarrow \infty} p\eta_p = \omega_K \star \nu = \nu$ , so  $\nu \in E(N)$ . Since  $\eta_0 = \tilde{N} \star \nu \in I(N)$ , it is easy to see that  $\nu \in I(N)$ , (cf. [7], Corollary 2.4).

LEMMA 9. — *Let  $N_1$  and  $N_2$  be non-zero convolution kernels satisfying the domination principle and  $N_1 < N_2$ . Then*

$$I(N_1) \cap E(N_2) \subseteq I(N_2).$$

*Proof.* — The relation  $<$  being transitive (cf. [4]), it follows that  $E(N_2) \subseteq E(N_1)$ . For  $\eta \in I(N_1) \cap E(N_2)$  and  $V \in \mathcal{V}$ , we then have

$$\eta = {}_{N_1}R_\eta^{\mathbf{C}^V} \leq {}_{N_2}R_\eta^{\mathbf{C}^V} \leq \eta,$$

which proves that  $\eta \in I(N_2)$ .

**THEOREM 10.** — *Let  $N$  be a non-singular convolution kernel satisfying the domination principle, and let  $(N_p)_{p>0}$  and  $\nu$  be as in Theorem 8.*

*Then we have  $\lim_{p \rightarrow 0} N_p = N_0$  and  $N_0 \prec N$ , where  $N_0$  is the regular part of  $N$ . The Riesz decomposition of  $N$  with respect to  $N_0$  is*

$$(4) \quad N = N_0 \star (\varepsilon_0 + \nu) + N^i$$

and

$$N^i \in I(N_0) \cap I(N).$$

*The singular part  $N'$  of  $N$  is  $N$ -invariant and given as*

$$(5) \quad N' = N_0 \star \nu + N^i.$$

*Proof.* — From Theorem 8 we know that the resolvent kernel  $\tilde{N} = \lim_{p \rightarrow 0} N_p$  exists, and also that  $p N_p \star N \leq N$ , hence  $\tilde{N} \prec N$ . Furthermore,  $N$  has the  $\tilde{N}$ -invariant part  $N^i = \lim_{p \rightarrow 0} p N_p \star N$ , so by Lemma 9  $N^i$  is also  $N$ -invariant. Letting  $p \rightarrow 0$  in (1), we find

$$(6) \quad N = \tilde{N} \star (\varepsilon_0 + \nu) + N^i.$$

Since  $\eta_0 = \tilde{N} \star \nu \in I(N)$ , we have  $\tilde{N} \star \nu + N^i \in I(N)$ , so for  $V \in \mathcal{V}$ :

$$\tilde{N} \star \nu + N^i = {}_N R_{N \star \nu + N^i}^{\xi_V} \leq {}_N R_N^{\xi_V},$$

which implies

$$(7) \quad \tilde{N} \star \nu + N^i \leq N',$$

where  $N'$  is the singular part of  $N$ .

For  $V \in \mathcal{V}_K$ , let  $\varepsilon'_{\mathfrak{t}V}$  be a  $\tilde{N}$ -balayaged measure of  $\varepsilon_0$  on  $\mathfrak{t}V$ . Then

$$\xi_V = N \star \varepsilon'_{\mathfrak{t}V} + (\tilde{N} - \tilde{N} \star \varepsilon'_{\mathfrak{t}V}) \star \nu \in E(N),$$

and using (6) and  $\varepsilon'_{\mathfrak{t}V} \star N^i = N^i$ , we find

$$\xi_V = \tilde{N} \star \varepsilon'_{\mathfrak{t}V} + \tilde{N} \star \nu + N^i.$$

In  $\mathfrak{t}V$ , we have  $\tilde{N} \star \varepsilon'_{\mathfrak{t}V} = \tilde{N}$ , hence  $\xi_V = N$  in  $\mathfrak{t}V$ , so by the definition of reduced measure, we get  ${}_N R_N^{\xi_V} \leq \xi_V$ . Letting  $V$  increase to  $G$ , we find using  $\lim_{V \uparrow G} \tilde{N} \star \varepsilon'_{\mathfrak{t}V} = 0$  that

$$N' = \lim_{V \uparrow G} {}_N R_N^{\xi_V} \leq \lim_{V \uparrow G} \xi_V = \tilde{N} \star \nu + N^i,$$

which combined with (7) yields  $N' = \tilde{N} \star \nu + N^i$  and hence  $N_0 = \tilde{N}$ .

With the notation as in Theorem 8 and 10, we further have the following proposition.

PROPOSITION 11. — *If  $N' = N_0 \star v + N^i$  is the Riesz decomposition of the singular part  $N'$  of  $N$  with respect to the regular part  $N_0$  of  $N$ , then either  $v$  or  $N^i$  is zero.*

*Proof.* — Suppose that  $v \neq 0$ . Since  $v \in E(N)$  there exists a net  $(\lambda_\alpha)_{\alpha \in A}$  of positive measures such that  $N \star \lambda_\alpha \uparrow v$ , and since  $N_0 \star v$  exists, this shows that also  $N \star N_0$  exists. Finally, since  $N^i \leq N$  also  $N^i \star N_0$  exists. Using  $N^i \in I(N_0)$ , it follows that

$$N^i = p N_p \star N^i \leq p N_0 \star N^i \quad \text{for all } p > 0,$$

and hence  $N^i = 0$ .

PROPOSITION 12. — *Let  $N$  be a non-singular convolution kernel with regular part  $N_0$ . Then  $N$  and  $N_0$  have the same pseudo-periods. In particular, the group of pseudo-periods for a non-singular convolution kernel is compact.*

*Proof.* — Suppose that  $N_0 \star \varepsilon_x = c N_0$ . Since  $N_0 \prec N$ , it follows that  $N \star \varepsilon_x = c N$ . Conversely, if  $N \star \varepsilon_x = c N$ , then  $N' \star \varepsilon_x = c N'$  because  $N \prec N'$ . Using  $N = N_0 + N'$ , we get  $N_0 \star \varepsilon_x = c N_0$ .

Let  $V \in \mathcal{V}$  be fixed. For every open relatively compact set  $\omega \subseteq G$  such that  $V \subseteq \omega$ , let  $\mu_{\omega \setminus V}$  be a balayaged measure of  $\varepsilon_0$  on  $\omega \setminus V$  with respect to  $N$  such that  ${}_N R_N^{\omega \setminus V} = N \star \mu_{\omega \setminus V}$ . With this notation, we have the following result.

PROPOSITION 13.

(i) *Every accumulation point for the net  $(\mu_{\omega \setminus V})_\omega$  as  $\omega$  increases to  $G$  is a balayaged measure of  $\varepsilon_0$  on  $\complement V$  with respect to  $N_0$ .*

(ii)  ${}_N R_N^{\complement V} = {}_{N_0} R_{N_0}^{\complement V} + N'$ .

(iii) *If  $N$  satisfies the principle of unicity of mass  $\lim_{\omega \uparrow G} \mu_{\omega \setminus V}$  exists and  ${}_{N_0} R_{N_0}^{\complement V} = N_0 \star \lim_{\omega \uparrow G} \mu_{\omega \setminus V}$ .*

*Proof.* — Since  $N \star \mu_{\omega \setminus V} \leq N$ , the net  $(\mu_{\omega \setminus V})_\omega$  is vaguely bounded. Let  $\mu_{\complement V}$  be an accumulation point and assume that  $\mu_{\omega \setminus V} \rightarrow \mu_{\complement V}$  (For notational simplicity we do not write the subnet). From (1) follows

$$(8) \quad N \star \mu_{\omega \setminus V} = p N_p \star N \star \mu_{\omega \setminus V} + N_p \star \mu_{\omega \setminus V} + v \star N_p \star \mu_{\omega \setminus V}.$$

We have  $N \star \mu_{\omega \setminus V} = {}_N R_N^{\omega \setminus V} \uparrow {}_N R_N^{\complement V}$  so the first term on the right-hand side increases to  $p N_p \star {}_N R_N^{\complement V}$ . Since  $N_p \star N$  exists, Deny's convergence Lemma implies that

$$\lim_{\omega} N_p \star \mu_{\omega \setminus V} = N_p \star \mu_{\complement V}.$$

Finally, since  $v \star N_p \in I(N)$ , we have as in the proof of Lemma 4 that  $\lim_{\omega} v \star N_p \star \mu_{\omega \setminus V} = v \star N_p$  so (8) leads to

$$(9) \quad {}_N R_N^{\xi_V} = p N_p \star {}_N R_N^{\xi_V} + N_p \star (\mu_{\xi_V} + v).$$

This shows that  ${}_N R_N^{\xi_V} \in E(N_0)$ , and since  $N' \leqslant {}_N R_N^{\xi_V} \leqslant N$  the  $N_0$ -invariant part of  ${}_N R_N^{\xi_V}$  is equal to  $N^i$  which is the  $N_0$ -invariant part of  $N'$  as well as of  $N$ . Letting  $p \rightarrow 0$  in (9), we get

$${}_N R_N^{\xi_V} = N_0 \star (\mu_{\xi_V} + v) + N^i = N_0 \star \mu_{\xi_V} + N',$$

so it is clear that  $\mu_{\xi_V}$  is a balayaged measure of  $\varepsilon_0$  on  $\mathbb{C}V$  with respect to  $N_0$ .

Let  $\varepsilon'_{\xi_V}$  be a balayaged measure of  $\varepsilon_0$  on  $\mathbb{C}V$  with respect to  $N_0$  such that  ${}_{N_0} R_{N_0}^{\xi_V} = N_0 \star \varepsilon'_{\xi_V}$ . Then  $N_0 \star \varepsilon'_{\xi_V} \leqslant N_0 \star \mu_{\xi_V}$  and with the notation from the proof of Theorem 10, we have

$${}_N R_N^{\xi_V} \leqslant \xi_V = N_0 \star \varepsilon'_{\xi_V} + N',$$

hence

$$(10) \quad {}_N R_N^{\xi_V} = N_0 \star \mu_{\xi_V} + N' \geqslant N_0 \star \varepsilon'_{\xi_V} + N' \geqslant {}_N R_N^{\xi_V}.$$

We shall finally prove (iii). When  $N$  satisfies the principle of unicity of mass,  $N$  and hence also  $N_0$  have no pseudo-periods, so  $N_0$  is a Hunt kernel. Therefore  $\varepsilon'_{\xi_V}$  is uniquely determined by the formula  ${}_{N_0} R_{N_0}^{\xi_V} = N_0 \star \varepsilon'_{\xi_V}$ , and every accumulation point  $\mu_{\xi_V}$  of  $(\mu_{\omega \setminus V})_{\omega}$  is equal to  $\varepsilon'_{\xi_V}$ . Therefore  $\lim_{\omega} \mu_{\omega \setminus V} = \varepsilon'_{\xi_V}$ .

*Remarks*

1° The singular part of  $N + c \varepsilon_0$  is equal to the singular part  $N'$  of  $N$ .

In fact, for  $V \in \mathcal{V}$ , we have observed that

$${}_N R_N^{\xi_V} \geqslant {}_{N+c\varepsilon_0} R_{N+c\varepsilon_0}^{\xi_V},$$

hence  $N' \geqslant (N+c \varepsilon_0)'$ . Since  $N' \in I(N) = I(N+c \varepsilon_0)$ , we also have

$$N' = {}_{N+c\varepsilon_0} R_{N+c\varepsilon_0}^{\xi_V} \leqslant {}_{N+c\varepsilon_0} R_{N+c\varepsilon_0}^{\xi_V},$$

which shows that  $N' \leqslant (N+c \varepsilon_0)'$ .

2° Suppose that  $N' = N_0 \star v$  where  $v \neq 0$ . If  $N$  is shift-bounded (i. e. the set  $\{ N \star \varepsilon_x; x \in G \}$  is vaguely bounded) then  $N_0(G) < \infty$ .

In fact, since  $v \in E(N)$  there exists a non-zero measure  $\lambda \geqslant 0$  such that  $N \star \lambda \leqslant v$  and then

$$N_0 \star N \star \lambda \leqslant N_0 \star v = N' \leqslant N.$$

The shift-boundedness of  $N$  implies that  $N_0 \star \lambda(G) \leqslant 1$ , hence  $N_0(G) < \infty$ .

If  $N' = N_0 \star \nu$  with  $\nu \neq 0$ , and  $N$  is not shift-bounded,  $N_0$  need not be of finite mass as the following example shows:

$$G = \mathbf{R}, N = (1)_{]0, \infty[}(x) + e^x dx.$$

The regular part of  $N$  is the Heaviside kernel  $(1)_{]0, \infty[}$  and  $N' = \nu = e^x$ .

If  $N$  is shift-bounded and  $N_0(G) < \infty$ , then  $N'$  is a  $N_0$ -potential, because  $I(N_0)$  does not contain any shift-bounded non-zero measures.

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