BULLETIN DE LA S. M. F.

ZBIGNIEW PIOTROWSKI

Continuity points in $\{x\} \times Y$

Bulletin de la S. M. F., tome 108 (1980), p. 113-115
<http://www.numdam.org/item?id=BSMF_1980__108__113_0>

Bull. Soc. math. France,
108, 1980, p. 113-115.

CONTINUITY POINTS IN $\{x\} \times Y$

BY

ZBIGNIEW PIOTROWSKI (*)

RÉSUMÉ. — Le résultat principal de cet article est un peu plus fort que le théorème suivant : soit X un espace à base dénombrable en tout point soit Y un espace de Baire et soit Z un espace métrique. Si une fonction $f: X \times Y \to Z$ est séparément continue, l'ensemble des points de continuité de f est un dense G_δ dans $\{x\} \times Y$, pour chaque $x \in X$.

ABSTRACT. — The main result of this paper is somewhat stronger than the following: let X be a first countable space, let Y be a Baire one and let Z be a metric space. If a function $f: X \times Y \to Z$ is separately continuous, then the set of points of continuity of f is a dense G_δ subset in $\{x\} \times Y$, for all $x \in X$.

There are many papers which deal with the classical problem of determination of points of continuity of a separately continuous function, for some references, *see* [1].

The general problem is: find conditions on topological spaces X, Y and Z so that each separately continuous function $f: X \times Y \to Z$ (i.e., function continuous in each variable while the other is fixed) is jointly continuous at points of a "substantial" (in some topological sense) subset of $X \times Y$, *cf.* [1], p. 515.

We will answer this problem showing how the set of points of continuity looks like in the sets of form $\{x\} \times Y$, for each x, while X is assumed to be first countable, Y is Baire, Z is metric and f is somewhat weaker than separately continuous. As a useful tool we make use of quasi-continuous functions. Namely:

A function $f: X \to Y$ is called *quasi-continuous* if for every point $x \in X$ and every neighborhoods U of x and V of $f(x)$, there exists an open, non-empty set G, $G \subset U$, such that $f(G) \subset V$.

(*) Texte reçu le 12 février 1979, révisé le 22 octobre 1979.

Z. PIOTROWSKI, Institute of Mathematics, University, pl. Grunwaldzki 2/4, 50-384, Wrocław, Poland.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE - 0037-9484/1980/113/$ 5.00

Recall that S. Marcus proved that there exists a quasi-continuous function which is not Lebesgue measurable. Of course, every continuous function is quasi-continuous.

A function $f : X \times Y \to Z (X, Y, Z$, arbitrary topological spaces) is said to be *quasi-continuous with respect to the variable* x, if for every point (p, q) of $X \times Y$ and for every neighborhood N of $f(p, q)$ and for every neighborhood $U \times V$ of (p, q) there exists a neighborhood U' of p, with $U' \subset U$ and a non-empty open set $V' \subset V$ such that for all $(x, y) \in U' \times V'$ we have $f(x, y) \in N$. Analogously, one may define functions which are quasi-continuous with respect the variable y. If f is quasi-continuous with respect to the variable x and quasi-continuous with respect to the variable y, then we say that f is *symmetrically quasi-continuous*.

One can easily construct symmetrically quasi-continuous functions which are not separately continuous. From [2], Theorem 1 it follows:

LEMMA. — *Let X be first countable, Y be Baire and Z be metric. If f : $X \times Y \to Z$ is a function such that all its x-sections f_x are quasi-continuous and all its y-sections f_y are continuous, then f is quasi-continuous with respect to x.*

Now, under the same assumptions as in Lemma, let us fix an arbitrary element x from X. Consider the function $y \to \omega(x, y)$. Observe, that the open set $\{ y \mid \omega(x, y) < 1/n \}$ is dense in Y! Hence, standard arguments let us state the following:

THEOREM. — *Let X be first countable, Y be Baire and Z be metric. If a function $f : X \times Y \to Z$ has all its x-sections f_x quasi-continuous and all its y-sections f_y continuous, then for all $x \in X$, the set of points of continuity of f is a dense, G_δ subset in $\{ x \} \times Y$.*

COROLLARY. — *Let X and Y be first countable, Baire spaces and Z be metric. If a function $f : X \times Y \to Z$ is separately continuous, then the set of points of continuity of f is dense, G_δ in the sets of form $X \times \{y\}$ and $\{ x \} \times Y$, for all $x \in X$ and all $y \in Y$.*

The following Question remains open:

Question. — For which "nice" topological (neither metric nor satisfying any sort of countability conditions, *see* [1], p. 515_{20}) spaces X and Y, our Lemma holds?

Good answers to this Question will let to extend our Theorem.

REFERENCES

[1] NAMIOKA (I.). — *Separate continuity and joint continuity, Pacific J. Math.*, vol. 51, 1974, p. 515-531.

[2] PIOTROWSKI (Z.). — *Quasi-continuity and product spaces, Proc. Conf. Geom. Topology*, Warsaw, 1978.