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Continuity points in \( \{x\} \times Y \)


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CONTINUITY POINTS IN \( \{ x \} \times Y \)

BY

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RESUME. — Le resultat principal de cet article est un peu plus fort que le theorème suivant : soit \( X \) un espace à base dénombrable en tout point soit \( Y \) un espace de Baire et soit \( Z \) un espace métrique. Si une fonction \( f : X \times Y \to Z \) est séparément continue, l'ensemble des points de continuité de \( f \) est un dense \( G_\delta \) dans \( \{ x \} \times Y \), pour chaque \( x \in X \).

ABSTRACT. — The main result of this paper is somewhat stronger than the following: let \( X \) be a first countable space, let \( Y \) be a Baire one and let \( Z \) be a metric space. If a function \( f : X \times Y \to Z \) is separately continuous, then the set of points of continuity of \( f \) is a dense \( G_\delta \) subset in \( \{ x \} \times Y \), for all \( x \in X \).

There are many papers which deal with the classical problem of determination of points of continuity of a separately continuous function, for some references, see [1].

The general problem is: find conditions on topological spaces \( X, Y \) and \( Z \) so that each separately continuous function \( f : X \times Y \to Z \) (i.e., function continuous in each variable while the other is fixed) is jointly continuous at points of a “substantial” (in some topological sense) subset of \( X \times Y \), cf. [1], p. 515.

We will answer this problem showing how the set of points of continuity looks like in the sets of form \( \{ x \} \times Y \), for each \( x \), while \( X \) is assumed to be first countable, \( Y \) is Baire, \( Z \) is metric and \( f \) is somewhat weaker than separately continuous. As a useful tool we make use of quasi-continuous functions. Namely:

A function \( f : X \to Y \) is called quasi-continuous if for every point \( x \in X \) and every neighborhoods \( U \) of \( x \) and \( V \) of \( f( x ) \), there exists an open, non-empty set \( G, G \subset U \), such that \( f(G) \subset V \).

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Recall that S. Marcus proved that there exists a quasi-continuous function which is not Lebesgue measurable. Of course, every continuous function is quasi-continuous.

A function \( f : X \times Y \to Z \) (\( X, Y, Z \), arbitrary topological spaces) is said to be **quasi-continuous with respect to the variable** \( x \), if for every point \((p, q)\) of \( X \times Y \) and for every neighborhood \( N \) of \( f(p, q) \) and for every neighborhood \( U \times V \) of \((p, q)\) there exists a neighborhood \( U' \) of \( p \), with \( U' \subseteq U \) and a non-empty open set \( V' \subseteq V \) such that for all \((x, y) \in U' \times V'\) we have \( f(x, y) \in N \). Analogously, one may define functions which are quasi-continuous with respect the variable \( y \). If \( f \) is quasi-continuous with respect to the variable \( x \) and quasi-continuous with respect to the variable \( y \), then we say that \( f \) is **symmetrically quasi-continuous**.

One can easily construct symmetrically quasi-continuous functions which are not separately continuous. From [2], Theorem 1 it follows:

**Lemma.** — Let \( X \) be first countable, \( Y \) be Baire and \( Z \) be metric. If \( f : X \times Y \to Z \) is a function such that all its \( x \)-sections \( f_x \) are quasi-continuous and all its \( y \)-sections \( f_y \) are continuous, then \( f \) is quasi-continuous with respect to \( x \).

Now, under the same assumptions as in Lemma, let us fix an arbitrary element \( x \) from \( X \). Consider the function \( y \mapsto \omega(x, y) \). Observe, that the open set \( \{ y \mid \omega(x, y) < 1/n \} \) is dense in \( Y \). Hence, standard arguments let us state the following:

**Theorem.** — Let \( X \) be first countable, \( Y \) be Baire and \( Z \) be metric. If a function \( f : X \times Y \to Z \) has all its \( x \)-sections \( f_x \) quasi-continuous and all its \( y \)-sections \( f_y \) continuous, then for all \( x \in X \), the set of points of continuity of \( f \) is a dense, \( G_\delta \) subset in \( \{ x \} \times Y \).

**Corollary.** — Let \( X \) and \( Y \) be first countable, Baire spaces and \( Z \) be metric. If a function \( f : X \times Y \to Z \) is separately continuous, then the set of points of continuity of \( f \) is dense, \( G_\delta \) in the sets of form \( X \times \{ y \} \) and \( \{ x \} \times Y \), for all \( x \in X \) and all \( y \in Y \).

The following Question remains open:

**Question.** — For which “nice” topological (neither metric nor satisfying any sort of countability conditions, see [1], p. 515) spaces \( X \) and \( Y \), our Lemma holds?

Good answers to this Question will let to extend our Theorem.
REFERENCES
