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LIMIT POINTS IN THE LAGRANGE SPECTRUM OF A QUADRATIC FIELD

BY

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ABSTRACT. — We show that the Lagrange spectrum $L(\mathbb{Q}[\sqrt{D}])$ of a real quadratic field has limit points. In the process we prove a curious relationship between the periods of the continued fractions of certain quadratic numbers.

RÉSUMÉ. — Nous démontrons que le spectre de Lagrange, $L(\mathbb{Q}[\sqrt{D}])$, d'un corps quadratique réel a des points d'accumulation. En passant, nous obtenons un rapport entre les périodes des fractions continues de certains nombres quadratiques.

Let D be a non-square positive integer. Cusick and Mendès-France [1] have asked whether the Lagrange spectrum of $\mathbb{Q}[\sqrt{D}]$ contains limit points. Woods [2] (in examining the equivalent problem for the Markov spectrum) has shown that the answer is yes in the case $D=5$. The main purpose of this paper is to show that the general answer is yes. To do this we establish an interesting Theorem on continued fractions which, more or less, generalizes the result used by Woods. I should like to acknowlege the helpful conversations that I have had with Mendès-France.

We give a short résumé of the facts we need.

If $\alpha \in \mathbb{R}$ then its Lagrange number, $L(\alpha)$, is defined to be $\limsup (q \| \alpha q \|)^{-1}$. Here $\| , \|$ denotes "distance from the nearest integer". $L(\alpha)$ is a measure of the approximability of α by rational numbers.

If $\alpha \in \mathbb{Q}[\sqrt{D}]$ then $L(\alpha)=2 \lim \sup_{n \to \infty} \operatorname{Irr}(\alpha_n)$ where the α_n are the successive remainders in the development of α as a continued fraction and $\operatorname{Irr} \alpha =$ "irrational part of α". Hence, if

$$\alpha=[n_1, \ldots, n_t, \overline{m_1, \ldots, m_s}],$$
$$L(\alpha)=2 \operatorname{Max}_r \operatorname{Irr}[\overline{m_r \ldots m_s m_1 \ldots m_{r-1}}]$$

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and if, moreover, $m_1 - m_r \geqslant 2$ for $r \neq 1$ we know that the maximum is achieved with $r=1$.

We write (n) for the matrix $\begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix}$ and observe that, for $\alpha>0$, $\alpha=\overline{[m_1, \ldots, m_s]}$ if and only if $V(\alpha)=\alpha$ where $V=(m_1)\ldots(m_s)$.

A matrix of the form $(m_1)\ldots(m_s)$, $m_i \in \mathbb{N}^*$, we call *positive* and we know that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ is positive if and only if $a \geqslant c \geqslant d \geqslant 0$.

If $\alpha \in \mathbb{Q}[\sqrt{D}]$, $\bar{\alpha}$ denotes its conjugate and $[\alpha]$ the largest integer less than or equal to α.

For the Lemma and for Theorem 1 we make the following definitions. D is a non-square integer greater than 2; $\lambda=2[1/2(D+\sqrt{D})]-D$ (so that $\lambda=[\sqrt{D}]$ or $[\sqrt{D}]-1$ according as $[\sqrt{D}]-D$ is even or odd); x and y are positive integers such that $x^2-Dy^2=4\varepsilon$ ($\varepsilon=\pm 1$) and u, v is the least such pair; $Y=Y_x$ denotes the matrix

$$\begin{pmatrix} \dfrac{x+\lambda y}{2} & \dfrac{D-\lambda^2}{4}y \\ y & \dfrac{x-\lambda y}{2} \end{pmatrix}$$

$\theta=1/2(u+v\sqrt{D})$; $u_n=\theta^n+\theta^n$; $v_n=\theta^n-\bar{\theta}^n/\sqrt{D}$ (so $x=u_n$ for some $n>0$).

LEMMA. — *Let $\alpha=1/2(\lambda+\sqrt{D})$. The continued fraction for α is purely periodic. If $\alpha=\overline{[m_1, \ldots, m_s]}$ is a minimal expression and $V=(m_1)\ldots(m_s)$ then $V^n=Y_{u_n}$.* (m_1 is in fact λ).

Proof. — We first establish that Y is positive. λ and D have the same parity so if $2 \nmid D$ then $x \equiv y \equiv \lambda y \bmod 2$ and if $2 \mid D$ then $x \equiv 0 \equiv \lambda \equiv \lambda y \bmod 2$. In either case $1/2(x \pm \lambda y) \in \mathbb{Z}$. Moreover if $D \equiv 0, 1 \bmod 4$ then $D \equiv \lambda^2 \bmod 4$ and if $D \equiv 2, 3 \bmod 4$ then $D \equiv \lambda^2 \bmod 2$ and $2 \mid y$. In either case $(D-\lambda^2)y/4 \in \mathbb{Z}$. It is clear that $\det Y=\varepsilon$ so $Y \in GL_2(\mathbb{Z})$.

It remains to show that

$$\frac{1}{2}(x+\lambda y) \geqslant y \geqslant \frac{1}{2}(x-\lambda y) \geqslant 0.$$

The first inequality is clear as $\lambda \geqslant 1$. The other two are equivalent to $\lambda+2 \geqslant x/y \geqslant \lambda$ where $x/y=\sqrt{D+4\varepsilon/y^2}$. Here the left hand side holds if $\varepsilon=-1$, $y>1$ or $([\sqrt{D}]+1)^2 \geqslant D+4$ but if $\varepsilon=1$ and $y=1$ then

$D+4=x^2=([\sqrt{D}]+1)^2$. Moreover the right hand side holds if $\varepsilon=1$, $y>1$ or $[\sqrt{D}]^2\leqslant D-4$. But if $\varepsilon=-1$, $y=1$ and $[\sqrt{D}]^2>D-4$ we have $D>[\sqrt{D}]^2>D-4=x^2$. Hence $D=5$, $\lambda=1=x=y=x/y$ and the right hand side is, in any case, satisfied.

We observe that $Y(\alpha)=\alpha$ so the continued fraction for α is purely repeating. Put $V=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Now $V(\alpha)=\alpha$ so $(a-d)/c=\alpha+\bar{\alpha}$ and $b/c=-\alpha\bar{\alpha}$ whence $ad=-\alpha\bar{\alpha}c^2+\det V$. Now $(a+d)^2=(a-d)^2+4ad$ so $(a+d)^2-c^2D=4\det V$. Hence $V=Y_x$ for some x. As s is minimal so are the coefficients of V and so $V=Y_u$. Moreover $Y_{u_n}=V^r$ for some r and as the eigenvalues of Y_{u_n} are θ^n and $\bar{\theta}^n$ we have $n=r$.

THEOREM 1. — *Let*

$$\beta_n=\left(\frac{1+\sqrt{D}}{2}\right)\frac{u_n}{v_n}+\left[\frac{D-\sqrt{D}}{2}\right]$$

and

$$X_n=\begin{pmatrix} 1 & D-1-\lambda \\ 0 & 1 \end{pmatrix}V^n\begin{pmatrix} 1 & 1-\lambda \\ 0 & -1 \end{pmatrix}V^n\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

then $X_n(\beta_n)=\beta_n$.

Moreover $\exists\, r, t$ *such that* $V^r\begin{pmatrix} 1 & 1-\lambda \\ 0 & -1 \end{pmatrix}V^r$ *and* $V^t\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ *are positive and so for* $n\geqslant r+t$ *and for some* $n_1, \ldots, n_x, p_1, \ldots, p_y$ *we have*

$$\varepsilon_n=\overline{[D-1, m_2, \ldots, m_s, \underbrace{m_1, \ldots, m_s}_{n-r-1 \text{ times}},}$$

$$\overline{n_1, \ldots, n_x, \underbrace{m_1, \ldots, m_s}_{n-r-t \text{ times}}, p_1, \ldots, p_y]}.$$

[In fact $t=1$ will do unless $u=v=1$.]

Remark. — There is "experimental evidence" that this result generalizes along the following lines but the exact relationship (along with any necessary restrictions), and hence the proof, eludes me. Let

$$\alpha'=\frac{m}{2}\,\frac{D+\sqrt{D}}{D-1}=[a_1, \ldots, a_r, \overline{m_1', \ldots, m_{s'}'}],$$

$V' = (m_1') \ldots (m_s')$, Φ be the positive eigenvalue of V', $u_n' = \Phi^n + \bar{\Phi}^n$ and $v_n' = (\Phi^n - \bar{\Phi}^n)\sqrt{D}$ then, for sufficiently large n:

$$\beta_n' = \frac{m}{2}\,\frac{1+\sqrt{D}}{D-1}\,\frac{u_n'}{v_n'} = [a_1 \ldots a_r, \overline{\underbrace{m_1', \ldots, m_{s'}',}_{n-i \text{ times}}}$$

$$\overline{b_1, \ldots, b_t, \underbrace{m_1', \ldots, m_{s'}',}_{n-j \text{ times}} c_1, \ldots, c_n}].$$

Proof. — We omit the routine verification that $X_n(\beta_n) = \beta_n$ (The only "explanatory" proof I have constructs X_n from β_n and factorizes it by assuming such a factorization exists. It is unsuitable for inclusion here because of its length and its convoluted logic.)

Using the fact that $u_r/v_r \to \sqrt{D}$ we find that

$$\lim_{r\to\infty} \frac{1}{v_r^2} V^r \begin{pmatrix} 1 & (1-\lambda) \\ 0 & -1 \end{pmatrix} V^r = \begin{pmatrix} \dfrac{\sqrt{D}+\lambda}{2} & \dfrac{D-\lambda^2}{4} \\ 1 & \dfrac{\sqrt{D}-\lambda}{2} \end{pmatrix}.$$

Hence, as this matrix satisfies the required inequalities strictly, there must be an r as required in the Theorem. The existence of t is proved similarly.

CORROLLARY 1. — *If* $D = n(n+4)$ *then*

$$\left(\lambda = n,\ \alpha = [n, 1],\ \theta = \frac{n+2+\sqrt{D}}{2}\ \textit{and}\right)$$

for $r \geq 0$,

$$\beta_{r+2} = [D-1, 1, \underbrace{n, 1,}_{r \text{ times}} n+2, \underbrace{n, 1,}_{r \text{ times}} n+1].$$

Proof. — $V = \begin{pmatrix} n+1 & n \\ 1 & 1 \end{pmatrix}$. Now

$$V\begin{pmatrix} 1 & 1-\lambda \\ 0 & -1 \end{pmatrix} V = \begin{pmatrix} n+2 & 1 \\ 1 & 0 \end{pmatrix} = (n+2)$$

$$\text{and } V\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = (n+1),$$

COROLLARY 2. — *If* $D=n^2+4$ *and* $n>1$ *then*

$$\left(\lambda=n,\ \alpha=[n],\ \theta=\frac{n+\sqrt{D}}{2}\ \textit{and}\right)$$

for $r\geqslant 0$,

$$\beta_{r+3}=\overline{[D-1,\ \underset{r+1\text{ times}}{\underset{\uparrow}{n}},\ n-1,\ n+1,\ \underset{r\text{ times}}{\underset{\uparrow}{n}},\ n-1,\ 1]}.$$

Proof. — $V=\begin{pmatrix} n & 1\\ 1 & 0\end{pmatrix}$. Now

$$V^2\begin{pmatrix} 1 & 1-\lambda\\ 0 & -1\end{pmatrix}\ V^2=\begin{pmatrix} n^2 & n-1\\ n+1 & 1\end{pmatrix}=(n-1)(n+1)$$

and

$$V\begin{pmatrix} 1 & 1\\ 0 & -1\end{pmatrix}=\begin{pmatrix} n & n-1\\ 1 & 1\end{pmatrix}=(n-1)(1),$$

whence the result.

THEOREM 2. — *The Lagrange spectrum of* $\mathbb{Q}[\sqrt{D}]$ *is not discrete.*

Proof. — We choose $n>1$ such that $(n+2)^2-Dc^2=4$ and replace D by c^2D. Then $D=n(n+4)$ and as $D-1-(n+2)\geqslant 2$ we have

$$L(\beta_r)=\beta_r-\overline{\beta}_r=\frac{u_r}{v_r}\sqrt{D}$$

which tends to D as $r\to\infty$. (The case $n=1$ works also, *cf.* Woods [2].)

After Cusick and Mendès-France, put

$$A(D)=\left\{r\in L(\mathbb{Q}[\sqrt{D}]),\ \text{s.t.}\frac{r}{n}\notin L(\mathbb{Q}[\sqrt{D}])\text{ for }n>1\right\};$$

it is clear from Theorem 2 that $A(D)$ also has a limit point.

REFERENCES

[1] CUSICK (T. W.) and MENDES-FRANCE (M.). — On the Lagrange Spectrum of a set, *Acta Arithmetica*, 34 (to appear).

[2] WOODS (A.C.). — The Markoff Spectrum of an Algebraic Number Field, *J. Austral. Math. Soc.* (series A), 25, 1978, pp. 486-488.