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RIEMANN-ROCH FOR GENERAL ALGEBRAIC VARIETIES

BY

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ABSTRACT. — Grothendieck proved the Riemann-Roch theorem for a morphism between smooth projective varieties, and Baum-Fulton-MacPherson extended it to proper morphisms between quasi-projective, possibly singular, varieties. The Riemann-Roch theorem presented here is valid in the category of proper morphisms between arbitrary (possibly singular, not quasi-projective) algebraic schemes. The construction of the Riemann-Roch transformation uses Chow's lemma and a little higher K -theory; it is computed, for varieties imbeddable in smooth varieties, using a relative version of Riemann-Roch for local complete intersection morphisms.

RÉSUMÉ. — Grothendieck a démontré le théorème de Riemann-Roch pour un morphisme entre variétés projectives et lisses, et Baum-Fulton-MacPherson l'ont étendu aux morphismes propres entre variétés quasi projectives, éventuellement singulières. Le théorème de Riemann-Roch présenté ici vaut dans la catégorie des morphismes propres entre variétés algébriques arbitraires (qu'on ne suppose ni lisses, ni quasi projectives). Dans la construction de la transformation de Riemann-Roch, on utilise le lemme de Chow et un peu de K -théorie algébrique; on explicite cette transformation pour les variétés plongeables dans les variétés lisses, à l'aide d'une version relative de Riemann-Roch pour les morphismes localement intersection complète.

0. Introduction

In this note we show how to free the Riemann-Roch theorems of [4], [1], and [2] from all projective assumptions, thereby completing a program begun in [5] and [9].

The proof, which is remarkably simple, uses ideas most of which have been available for several years. Not surprisingly, the Riemann-Roch theorem for general proper morphisms is deduced from the projective case *via* Chow's

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lemma; however to do this we must introduce the notion of an "envelope" of a variety and use a little higher K -theory (as far as K_1). It is important to point out that even to extend the Grothendieck-Riemann-Roch theorem of [4] to arbitrary proper morphisms between smooth varieties over a field of characteristic $p > 0$, we need to use, in the absence of resolution of singularities, the singular Riemann-Roch theorem of [1] together with a simple extension of the Verdier-Riemann-Roch theorem of [13].

Fix an arbitrary ground field K . All schemes X will be algebraic K -schemes (i. e., X is of finite type and separated over $\text{Spec}(K)$). Denote by $K_0 X$ (resp. $K^0 X$) the Grothendieck group of coherent algebraic sheaves (resp. algebraic vector bundles) on X . Let $A_* X_{\mathbb{Q}}$ be the group of algebraic cycles modulo rational equivalence on X , with rational coefficients, and let $A^* X_{\mathbb{Q}}$ denote the corresponding cohomology ring, as constructed in [7], paragraph 9.

RIEMANN-ROCH THEOREM. — *For every algebraic scheme X there is a homomorphism*

$$\tau_X : K_0 X \rightarrow A_* X_{\mathbb{Q}}.$$

These homomorphisms satisfy the following properties:

(1) (Covariance) *If $f : X \rightarrow Y$ is proper, and $\alpha \in K_0 X$, then:*

$$f_* \tau_X(\alpha) = \tau_Y f_*(\alpha).$$

(2) (Module) *If $\alpha \in K_0 X$, $\beta \in K^0 X$, then:*

$$\tau_X(\beta \otimes \alpha) = \text{ch}(\beta) \cap \tau_X(\alpha).$$

(3) (Formula) *If $i : X \rightarrow M$ is a closed imbedding in a scheme M which is smooth over $\text{Spec}(K)$, \mathcal{F} a coherent sheaf on X , E a resolution of \mathcal{F} by a bounded complex of locally free sheaves on M , then:*

$$\tau_X(\mathcal{F}) = \text{td}(i^* T_M) \cap \text{ch}_X^M(E).$$

Here T_M is the tangent bundle of M , td is the Todd class, and $\text{ch}_X^M(E) \in A_ X_{\mathbb{Q}}$ is the localized Chern character of E . ([1], § II. 1).*

(4) (Local complete intersections) *Let $f : X \rightarrow Y$ be a l.c.i. morphism. Assume that there are closed imbeddings $X \subset M$, $Y \subset P$, with M*

and P smooth. Then for all $\alpha \in K_0 Y$:

$$\tau_X f^*(\alpha) = \text{td}(T_f) \cdot f^* \tau_Y(\alpha).$$

Here T_f is the virtual tangent bundle to f .

(5) (Cartesian products) For all $\alpha \in K_0 X$, $\beta \in K_0 Y$:

$$\tau_{X \times Y}(\alpha \times \beta) = \tau_X(\alpha) \times \tau_Y(\beta).$$

(6) (Top term) If X is an n -dimensional variety (reduced and irreducible), then:

$$\tau_X(\mathcal{O}_X) = [X] + \text{terms of dimension} < n.$$

In addition, these homomorphisms τ_X are uniquely determined by properties (1), (4) (for open imbeddings of quasi-projective varieties), and (6) (for $X = \mathbb{P}^n$).

Granting this theorem, define the Todd class $\text{Td}(X) \in A_* X$, for any algebraic scheme X by setting:

$$\text{Td}(X) = \tau_X(\mathcal{O}_X),$$

\mathcal{O}_X the structure sheaf of X .

COROLLARY 1. — (i) If X is a smooth scheme (or a l. c. i. scheme which is imbeddable in a smooth scheme) then:

$$\text{Td}(X) = \text{td}(T_X) \cap [X].$$

Here T_X is the tangent bundle (or virtual tangent bundle) of X .

(ii) Let $f: X \rightarrow Y$ be a proper morphism, and let $\beta \in K^0 X$. Assume there is an element $f_*(\beta)$ in $K^0 Y$ such that:

$$f_*(\beta \otimes \mathcal{O}_X) = f_*(\beta) \otimes \mathcal{O}_Y,$$

in $K_0 Y$. Then:

$$f_*(\text{ch}(\beta) \cap \text{Td}(X)) = \text{ch}(f_* \beta) \cap \text{Td}(Y).$$

Note that there is a canonical such element $f_*(\beta)$ whenever f is a l. c. i. morphism, or, more generally, a perfect morphism (cf. [3]).

COROLLARY 2 (Grothendieck-Riemann-Roch). — Let $f: X \rightarrow Y$ be a proper morphism of smooth schemes, $\beta \in K^0 X$. Then:

$$f_*(\text{ch}(\beta) \cdot \text{td}(T_X)) = \text{ch}(f_* \beta) \cdot \text{td}(T_Y).$$

The same formula holds for X, Y l. c. i. schemes which can be imbedded in smooth schemes.

COROLLARY 3 (Hirzebruch-Riemann-Roch). — Assume X is proper over $\text{Spec}(K)$, and E is a vector bundle on X . Then:

$$\chi(X, E) = \int_X \text{ch}(E) \cap \text{Td}(X).$$

In particular $\chi(X, \mathcal{O}_X) = \int_X \text{Td}(X)$. From (5) of the theorem, one has also:

$$\text{Td}(X \times Y) = \text{Td}(X) \times \text{Td}(Y).$$

COROLLARY 4. — For all X , τ_X induces an isomorphism:

$$K_0 X \otimes \mathbb{Q} \rightarrow A_* X_{\mathbb{Q}}.$$

Corollary 1 (i) follows from (3) in the non-singular case, and (4) (for $f: X \rightarrow Y$, Y smooth, $\alpha = \mathcal{O}_Y$) if X is a l. c. i.; (ii) then follows from (1) and (2). Corollary 2 is a special case of Corollary 1. Corollary 3 follows from (1) (for $X \rightarrow \text{Spec}(K)$) and (2). Corollary 4 follows from (1) and (6), as in [1], § III (cf. Step 7 below).

For quasi-projective schemes, τ was constructed in [1]. The proof that τ satisfies (1)-(6), in the category of quasi-projective schemes, was given in [1] and [13]. In paragraph 1 we use Chow's lemma and an exact sequence involving a (first) higher K -group, to extend τ to all algebraic schemes. Paragraph 1 also contains the definition and a discussion of the properties of an envelope. In paragraph 2 we show that for schemes which are imbeddable in smooth schemes, the formula (3) is independent of the imbedding; this fact is deduced from a relative version of (4), for which VERDIÉ's proof in [13] suffices. The proof is then easily completed (§ 3).

When $K = \mathbb{C}$, the same construction determines, for all complex algebraic schemes X , a homomorphism:

$$\alpha_X : K_0 X \rightarrow K_0^{\text{top}}(X),$$

where $K_0^{\text{top}}(X)$ is the homology topological K -theory of X , satisfying analogues of (1)-(6). This generalizes [3] and [7] PRR precisely as the above Riemann-Roch theorem generalizes [1] and [13]. In particular, every

complex algebraic variety has a canonical orientation $\{X\}$ in $K_0^{top}(X)$, namely $\{X\} = \alpha_X(\mathcal{O}_X)$. This is compatible with the previous theorem: there is a commutative diagram:

$$\begin{array}{ccc}
 K_0 X & \xrightarrow{\alpha_X} & K_0^{top}(X) \\
 \tau_X \downarrow & & \downarrow \text{ch}_* \\
 A_* X_{\mathbb{Q}} & \longrightarrow & H_*(X; \mathbb{Q})
 \end{array}$$

where ch_* is the homology Chern character, and the lower horizontal map is the cycle map.

In § 5 of the second author's paper [9] it was implicitly assumed that if X is a singular scheme quasi-projective over two different base schemes S and T then the two, *a priori* different, Riemann-Roch transformations $\tau^S, \tau^T : K_0 \rightarrow A_{\mathbb{Q}}^*$ on the categories $\mathcal{C}_S, \mathcal{C}_T$ of schemes quasi-projective over S and T respectively, coincide on X . The equality $\tau_X^S = \tau_X^T$ follows from Proposition 2 of the present paper.

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1. The construction of τ .

PROPOSITION 1. — Consider a fibre square:

$$\begin{array}{ccc}
 Y' & \xrightarrow{j} & X' \\
 q \downarrow & & \downarrow p \\
 Y & \xrightarrow{i} & X
 \end{array}$$

with i a closed imbedding and p projective. Assume that p maps $X' - Y'$ isomorphically onto $X - Y$. Then the sequence:

$$K_0 Y' \xrightarrow{a} K_0 Y \oplus K_0 X' \xrightarrow{b} K_0 X \rightarrow 0$$

is exact, where $a(\alpha) = (q_* \alpha, -j_* \alpha)$, and $b(\alpha, \beta) = i_* \alpha + p_* \beta$.

Proof. — If $f : Z \rightarrow W$ is any proper morphism, let $\mathbf{F}(Z, f)$ be the full subcategory of the exact category $\mathbf{M}(Z)$ of coherent sheaves on Z consisting

of those \mathcal{F} for which $R^i f_* \mathcal{F} = 0$ for $i > 0$. Writing $U' = X' - Y'$, $U = X - Y$, $u : U' \rightarrow X'$ and $v : U \rightarrow X$ for the natural inclusions, and $r : U' \rightarrow U$ for the natural isomorphism, we have a commutative diagram of exact functors:

$$\begin{array}{ccccc}
 \mathbf{F}(Y', q) & \xrightarrow{j_*} & \mathbf{F}(X', p) & \xrightarrow{u_*} & \mathbf{M}(U') \\
 \downarrow q_* & & \downarrow p_* & & \downarrow r_* \\
 \mathbf{M}(Y) & \xrightarrow{i_*} & \mathbf{M}(X) & \xrightarrow{r_*} & \mathbf{M}(U).
 \end{array}$$

By QUILLEN [12], paragraph 7, 2.7 we know that if p is projective then $K_i(\mathbf{F}(X', p)) \simeq K_i(X') (= K_i(\mathbf{M}(X'))$ by definition) and $K_i(\mathbf{F}(Y', q)) \simeq K_i(Y')$. Hence by [12], paragraph 7, Proposition 3.2, applying BQ to this diagram gives a map of fibration sequences and hence a map of long exact sequences:

$$\begin{array}{ccccccccc}
 \longrightarrow & K_1 U' & \longrightarrow & K_0 Y' & \xrightarrow{j_*} & K_0 X' & \longrightarrow & K_0 U' & \longrightarrow & 0 \\
 & \cong \downarrow r_* & & \downarrow q_* & & \downarrow p_* & & \cong \downarrow r_* & & \\
 \longrightarrow & K_1 U & \longrightarrow & K_0 Y & \xrightarrow{i_*} & K_0 X & \longrightarrow & K_0 U & \longrightarrow & 0
 \end{array}$$

The proposition then follows by a simple diagram chase.

It can be shown more generally that this proposition is true if p is proper rather than projective.

Let us define an *envelope* of a scheme X to be a proper morphism $p : X' \rightarrow X$ such that for every closed subvariety V of X there is a closed subvariety V' of X' such that p maps V' birationally onto V . We call p a *Chow envelope* if, in addition, X' is quasi-projective over $\text{Spec}(K)$.

LEMMA. — (1) If $p : X' \rightarrow X$ and $q : X'' \rightarrow X'$ are envelopes, then $qp : X'' \rightarrow X$ is an envelope.

(2) If $p : X' \rightarrow X$ is an envelope, and $f : Y \rightarrow X$ is an arbitrary morphism, then the fibre product $X' \times_X Y \rightarrow X'$ is an envelope.

(3) For any scheme X there is a closed subscheme $Y \subset X$ with $X - Y$ dense in X , and a Chow envelope $p : X' \rightarrow X$ such that p maps $X' - p^{-1}(Y)$ isomorphically onto $X - Y$.

(4) If $p_1 : X'_1 \rightarrow X$, $p_2 : X'_2 \rightarrow X$ are envelopes, then there is a Chow envelope $p : X' \rightarrow X$, with morphisms $q_i : X' \rightarrow X'_i$ such that $p_i q_i = p$ for $i = 1, 2$.

(5) For any morphism $f: Y \rightarrow X$, and any Chow envelope $p: X' \rightarrow X$, there is a Chow envelope $q: Y' \rightarrow Y$ and a morphism $f': Y' \rightarrow X'$ such that $pf' = fq$. If f is proper, f' may also be taken to be proper.

(6) If $p: X' \rightarrow X$ is an envelope, then the induced morphism $p_*: K_0 X' \rightarrow K_0 X$ is surjective.

Proof. — (1) and (2) are straightforward. For (3), by Chow's lemma [10], paragraph 5.6, there is a $Y \subset X$ with $X - Y$ dense in X , and a proper morphism $p'_1: X'_1 \rightarrow X$, with X'_1 quasi-projective, such that p'_1 restricts to an isomorphism over $Y - X$. By noetherian induction there is a Chow envelope $p'_2: X'_2 \rightarrow Y$. Then the disjoint union of X'_1 and X'_2 , with its canonical map to X , is a Chow envelope of X . (4) and (5) follow from (1), (2) and (3). (6) follows from the fact that $K_0 X$ is generated by classes of structure sheaves of closed subvarieties of X (cf. [5]).

In [1], within the category of quasi-projective schemes over $\text{Spec}(K)$, homomorphisms:

$$\tau_X: K_0 X \rightarrow A_* X_{\mathbb{Q}}$$

were defined, satisfying properties (1)-(3), (5), (6) of the Riemann-Roch Theorem stated in the introduction. Property (4) was proved in [13]. Some additional argument was needed in [1], paragraph II.1.2 to handle the case where K is not algebraically closed, but subsequent improvements in intersection theory have taken care of this (see [6]).

In the rest of this section we show how to extend the construction of τ_X to all algebraic schemes X , so that properties (1), (2), (5) and (6) of the Riemann-Roch Theorem hold. There are several steps in the argument.

Step 1. — Let us say that a homomorphism $\tau: K_0 X \rightarrow A_* X_{\mathbb{Q}}$ is compatible with a Chow envelope $p: X' \rightarrow X$ if the diagram:

$$\begin{array}{ccc} K_0 X' & \xrightarrow{\tau_{X'}} & A_* X'_{\mathbb{Q}} \\ p_* \downarrow & & \downarrow p_* \\ K_0 X & \xrightarrow{\tau} & A_* X_{\mathbb{Q}} \end{array}$$

commutes. Here $\tau_{X'}$ is the homomorphism previously constructed for the quasi-projective scheme X' . From Lemma (6) it follows that there can be at most one τ compatible with p .

Step 2. — Suppose $\tau : K_0 X \rightarrow A_* X_0$ is a homomorphism compatible with some Chow envelope $p : X' \rightarrow X$. Then for any proper morphism $f : Y \rightarrow X$, with Y quasi-projective, the diagram:

$$\begin{array}{ccc} K_0 Y & \xrightarrow{\tau_Y} & A_* Y_0 \\ \downarrow f_* & & \downarrow f_* \\ K_0 X & \xrightarrow{\tau} & A_* X_0 \end{array}$$

commutes. In particular, τ is compatible with every Chow envelope of X . To see this, choose $q : Y' \rightarrow Y, f' : Y' \rightarrow X'$ as in Lemma (5). Given $\alpha \in K_0 Y$, choose $\alpha' \in K_0 Y'$ with $\alpha = q_* \alpha'$ (Lemma (6)). Then:

$$f_* \tau_Y(\alpha) = f_* \tau_Y(q_* \alpha') = f_* q_* \tau_{Y'}(\alpha') = p_* f'_* \tau_{Y'}(\alpha') = p_* \tau_{X'}(f'_*(\alpha')),$$

by the covariance property for quasi-projective schemes. By the compatibility of τ with p :

$$p_* \tau_{X'}(f'_*(\alpha')) = \tau p_*(f'_* \alpha') = \tau f_*(\alpha).$$

as required.

Step 3. — We construct a homomorphism $\tau_X : K_0 X \rightarrow A_* X_0$, compatible with some (and hence any) Chow envelope of X , by induction on the dimension, the cases of dimensions -1 and 0 being trivial. Given X , choose a Chow envelope $p : X' \rightarrow X$, with $Y \subset X'$ as in Lemma (3). Form the fibre square:

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ \downarrow q & & \downarrow p \\ Y & \xrightarrow{i} & X \end{array}$$

Since j is a closed imbedding, Y' is quasi-projective. By induction, there is a homomorphism $\tau_Y : K_0 Y \rightarrow A_* Y_0$ compatible with the Chow envelope q (cf. Lemma (2)). Consider the diagram:

$$\begin{array}{ccccc} K_0 Y' & \xrightarrow{a} & K_0 Y \oplus K_0 X' & \xrightarrow{b} & K_0 X \rightarrow 0 \\ \downarrow \tau_Y & & \downarrow \tau_Y \oplus \tau_{X'} & & \\ A_* Y'_0 & \xrightarrow{a} & A_* Y_0 \oplus A_* X'_0 & \xrightarrow{b} & A_* X_0 \end{array}$$

where a and b are defined as in Proposition 1. The square commutes by the covariance of τ for the inclusion j of quasi-projective schemes, and by the induction hypothesis for q . By Proposition 1 the top row is exact. The composite of the two homomorphisms in the lower row is clearly zero (in fact, this row is also exact). There is therefore a unique homomorphism τ_X from $K_0 X$ to $A_* X_0$ making the right square commute. In particular, τ_X is compatible with p .

Step 4. — Proof of covariance (1). Given $f: Y \rightarrow X$ proper, choose Chow envelopes $p: X' \rightarrow X$, $q: Y' \rightarrow Y$, and a proper $f': Y' \rightarrow Y$ as in Lemma (5). The proof that $f_* \tau_Y = \tau_X f_*$ is exactly the same as in Step 2.

Step 5. — Proof of the module property (2). Choose a Chow envelope $p: X' \rightarrow X$, and $\alpha' \in K_0 X'$ with $p_* \alpha' = \alpha$. Using the projection formula and the known result on X' :

$$\begin{aligned} \tau_X(\beta \otimes \alpha) &= \tau_X p_* (p^* \beta \otimes \alpha) = p_* \tau_{X'} (p^* \beta \otimes \alpha') \\ &= p_* (\text{ch}(p^* \beta) \cap \tau_{X'}(\alpha')) = \text{ch}(\beta) \cap p_* \tau_{X'}(\alpha') = \text{ch}(\beta) \cap \tau_X(\alpha). \end{aligned}$$

Step 6. — Proof of the Cartesian product property (5). Choose Chow envelopes $p: X' \rightarrow X$, $q: Y' \rightarrow Y$, $p_*(\alpha') = \alpha$, $q_*(\beta') = \beta$. As in the previous step, the required equation for $\alpha \times \beta$ follows from the known result for $\alpha' \times \beta'$.

Step 7. — Proof of property (6). Let $F_k K_0 X$ denote the subgroup of $K_0 X$ generated by coherent sheaves whose support has dimension at most k , or, equivalently, by structure sheaves of closed subvarieties of dimension at most k . From the covariance property it follows that τ_X maps $F_k K_0 X$ into $\sum_{i=0}^k A_i X_0$. Choose a quasi-projective variety X' and a proper birational morphism $p: X' \rightarrow X$. The result follows from the known result for X' and the fact that $p_*[\mathcal{O}_{X'}] = [\mathcal{O}_X] + \alpha$, $\alpha \in F_{n-1} K_0 X$.

Step 8. — Proof of uniqueness. This was proved in the category of quasi-projective schemes in [1], paragraph III.2. We saw in Step 3 that the extension to general algebraic schemes was uniquely determined by the covariance property (1).

The remaining properties (3) and (4) will be proved in paragraph 3.

2. Imbeddings in smooth varieties

For the proposition and corollary to be proved in this section, we explicitly ignore the constructions made for non-projective varieties in the preceding section. Instead we make use of a relative version of the theorems of [1] and [13]. Namely, fix a base scheme S which is smooth, but not necessarily quasi-projective over $\text{Spec}(K)$. Consider the category \mathcal{C}_S of schemes which are quasi-projective over S . For X in \mathcal{C}_S , choose a closed imbedding $i : X \rightarrow M$, $M \in \mathcal{C}_S$, M smooth over S . Define:

$$\tau_X : K_0 X \rightarrow A_* X_{\mathbb{Q}},$$

by setting:

$$\tau_X(\mathcal{F}) = \text{td}(i^* T_M) \cap \text{ch}_X^M(E),$$

where E is a resolution of the coherent sheaf $i_*(\mathcal{F})$ by a complex of locally free sheaves, T_M is the tangent bundle to M , and $\text{ch}_X^M(E) \in A_* X_{\mathbb{Q}}$ is the localized Chern character [1] paragraph II. 1.

The proofs of [1] and [13] extend without essential change, to show ⁽¹⁾ that τ_X is independent of the imbedding and the resolution, and that properties (1)-(4) hold with all schemes and morphisms in \mathcal{C}_S . (One small change is needed, to show that every vector bundle E on $X \in \mathcal{C}_S$ is the restriction of a vector bundle on a smooth scheme in \mathcal{C}_S . Choose an imbedding $i : X \rightarrow M$, M smooth in \mathcal{C}_S , and choose a surjection of a vector bundle F on M onto the coherent sheaf $i_* E$. Let $\pi : G \rightarrow M$ be the Grassmann bundle of e -dimensional quotients of F , $e = \text{rank } E$. Then G is smooth and in \mathcal{C}_S , and there is a morphism $s : X \rightarrow G$ such that E is the pull-back of the universal quotient bundle.) For details, see [6], paragraph 18.

As in [1], if $X \subset M$, \mathcal{F}, E are as above we write $\text{ch}_X^M(\mathcal{F})$ in place of $\text{ch}_X^M(E)$; ch_X^M determines a homomorphism from $K_0 X$ to $A_* X$.

PROPOSITION 2. — *Let $i : X \rightarrow M$, $j : X \rightarrow P$ be closed imbeddings of a scheme X in smooth schemes M and P . Let \mathcal{F} be a coherent sheaf on X . Then:*

$$\text{td}(i^* T_M) \cap \text{ch}_X^M(\mathcal{F}) = \text{td}(j^* T_P) \cap \text{ch}_X^P(\mathcal{F}).$$

⁽¹⁾ It will follow from our Riemann-Roch theorem that this τ_X agrees with that constructed in paragraph 1, but this fact cannot be used at this point in the argument.

Proof. — Consider the diagram:

$$\begin{array}{ccccc}
 X & \xrightarrow{f} & M \times X & \xrightarrow{k} & M \times P \\
 & & \downarrow q & & \downarrow p \\
 & & X & \xrightarrow{j} & P
 \end{array}$$

where $f=(i, 1_X)$, $k=1_X \times j$, and p and q are the projections. Since q is smooth, f is a regular imbedding, with normal bundle $i^*(T_M)$. Let E . be a resolution of $j_*(\mathcal{F})$ by locally free sheaves on P . Then $p^*(E)$ is a resolution of $q^*(\mathcal{F})$ by locally free sheaves on $M \times P$, so by [1], paragraph II.2.6:

(i)
$$q^* \text{ch}_X^p(\mathcal{F}) = \text{ch}_{M \times P}^{M \times P}(q^* \mathcal{F}).$$

Now let $S=M \times P$, and regard X and $M \times X$ as quasi-projective schemes over S by means of the morphisms kf and k . We apply the formula of property (4) to the morphism f , using (3) to express τ_X and $\tau_{M \times X}$ explicitly. This gives the formula:

(ii)
$$(kf)^*(\text{td}(T_{M \times P})) \cap \text{ch}_X^{M \times P}(f^* \alpha) = \text{td}(i^* T_M)^{-1} \cap f^*(k^*(\text{td}(T_{M \times P})) \cap \text{ch}_{M \times X}^{M \times P}(\alpha)),$$

for any $\alpha \in K_0(M \times X)$. Apply this to $\alpha = q^*[\mathcal{F}] = [q^* \mathcal{F}]$. Then:

$$f^* \alpha = f^* q^* [\mathcal{F}] = (qf)^* [\mathcal{F}] = [\mathcal{F}]$$

and:

$$\text{td}(i^* T_M)^{-1} \cdot (kf)^* \text{td}(T_{M \times P}) = j^* \text{td}(T_P).$$

Using (i) to rewrite $\text{ch}_{M \times X}^{M \times P}(\alpha)$, (ii) becomes:

(iii)
$$(kf)^* \text{td}(T_{M \times P}) \cap \text{ch}_X^{M \times P}(\mathcal{F}) = j^* \text{td}(T_P) \cap f^*(q^* \text{ch}_X^p(\mathcal{F})) = j^* \text{td}(T_P) \cap \text{ch}_X^p(\mathcal{F}).$$

From (iii) it follows that the diagonal imbedding kf of X in $M \times P$ determines the same class as the imbedding j of X in P . By symmetry, the required equality for $X \subset M$ and $X \subset P$ follows.

COROLLARY. — *Let Y be a quasi-projective scheme over $\text{Spec}(K)$, X a scheme, $i : X \rightarrow M$ a closed imbedding in a scheme M which is smooth over*

$\text{Spec}(K)$. Let $f : Y \rightarrow X$ be a proper morphism, $\alpha \in K_0 Y$. Then:

$$\text{td}(i^* T_M) \cap \text{ch}_X^M(f_* \alpha) = f_* \tau_Y(\alpha).$$

Here τ_Y is the Riemann-Roch map constructed for the quasi-projective scheme Y in [1], paragraph II.

Proof. — Choose a closed imbedding $j : Y \rightarrow U$, with U open in some \mathbb{P}^n . Then (f, j) is a closed imbedding of Y in $M \times U$. By Proposition 2 :

$$\tau_Y(\alpha) = (f, j)^*(\text{td}(T_{M \times U})) \cap \text{ch}_Y^{M \times U}(\alpha).$$

Then the required equation amounts to the covariance property for the Riemann-Roch theorem in the category \mathcal{C}_M of schemes quasi-projective over M .

3. Conclusion of the proof

We now return to the proof of the theorem in progress in paragraph 1. Thus τ_X is defined for all schemes, extended by Chow envelopes from the explicitly constructed τ for schemes quasi-projective over $\text{Spec}(K)$.

Step 9. — Proof of property (3). Given $i : X \rightarrow M$, \mathcal{F}, E , as in (3), choose a Chow envelope $p : Y \rightarrow X$, and choose $\alpha \in K_0 Y$ such that $p_* \alpha = [\mathcal{F}]$. By the corollary in paragraph 2:

$$\text{td}(i^* T_M) \cap \text{ch}_X^M(\mathcal{F}) = f_* \tau_Y(\alpha).$$

By Step 2, $f_* \tau_Y(\alpha) = \tau_X(\mathcal{F})$, which proves (3).

Note. — It now follows from property (3) that the homomorphism τ_X defined in paragraph 2 for a scheme X which is quasi-projective over some smooth base S is the same as the τ_X constructed in paragraph 1.

Step 10. — Proof of property (4). Given $f : X \rightarrow Y$, $X \subset M$, $Y \subset P$, form the commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{h} & M \times Y & \xrightarrow{k} & M \times P \\ & \searrow f & \downarrow q & & \downarrow p \\ & & Y & \xrightarrow{j} & P \end{array}$$

As usual, the truth of (4) for q and h implies (4) for the composite f . For q , (4) follows from the commutativity of localized Chern character with flat pull-back, as in the proof of Proposition 2, equation (ii). For h , (4) follows from the known property (4) in the category of quasi-projective schemes over $M \times P$. In both cases the note preceding this step is used to know that the τ 's constructed relative to $M \times P$ or P agree with the τ 's constructed in paragraph 1. This concludes the proof of the theorem.

Remarks 1. — In characteristic zero, the GRR formula (Corollary 2) for a proper morphism $f: X \rightarrow Y$ of smooth schemes can be deduced easily from the case of quasi-projective morphisms of smooth varieties, using Hironaka's resolution of singularities to construct a Chow envelope $X' \rightarrow X$ with X' smooth. Without resolution of singularities we need the whole singular Riemann-Roch theorem to prove GRR for non-singular varieties.

2. When $K = \mathbb{C}$, the construction of:

$$\alpha_X: K_0 X \rightarrow K_0^{\text{top}} X,$$

proceeds by exactly the same plan, using [2] and [7] PRR in place of [1] and [13]. We leave the details to the reader.

3. For complex *analytic* spaces, if $K_0 X$ denotes the Grothendieck group of coherent analytic sheaves, there should be homomorphisms:

$$K_0 X \rightarrow K_0^{\text{top}} X$$

and therefore $K_0 X \rightarrow H_*(X; \mathbb{Q})$ satisfying the corresponding properties. For complex manifolds the best result so far is the theorem of O'BRIAN, TOLEDO, and TONG [11] which gives a version of GRR, but with values in the cohomology $H^*(X, \Omega_X^*)$.

4. We still do not know how to extend the ideas of this paper to higher K -theory, thus completing the work started in [8]. The essential difference between the K_0 and $K_i (i > 0)$ situations is that instead of the exact sequence of Proposition 1 we have (for $i > 0$):

$$K_i Y \xrightarrow{a} K_i Y \oplus K_i X' \xrightarrow{b} K_i X \xrightarrow{d} K_{i-1} Y' \rightarrow$$

hence even if $\tau_Y, \tau_{Y'}, \tau_X$ are all defined, this does not determine $\tau_{X'}$.

5. The notion of an envelope (see paragraph 1) is distilled from the rather murky concept of a projective decomposition ([9], § 4). One can make the relationship more explicit by showing that envelopes have the

property of “universal homological descent” (this is analogous to the universal cohomological descent of SGA 4 V bis). Projective decompositions are then hypercoverings for the Grothendieck topology constructed using envelopes as covering maps. For example one may prove that if $\pi : \tilde{X} \rightarrow X$ is an envelope then:

$$K_0(\tilde{X} \times_X \tilde{X}) \xrightarrow{d} K_0(\tilde{X}) \xrightarrow{\pi_*} K_0(X) \rightarrow 0$$

(where $d = (p_1)_* - (p_2)_*$ with $p_i : \tilde{X} \times_X \tilde{X} \rightarrow \tilde{X}$ the natural projections) is an exact sequence.

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