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# ON THE BOREL CLASS OF THE DERIVED SET OPERATOR, II

BY

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RÉSUMÉ. — Soit $X$ un espace non-énumérable topologique métrisable compact, $2^X$ l'espace topologique des compacts de $X$ avec la topologie de Hausdorff et soit $D$ la dérivation de Cantor. KURATOWSKI a démontré que $D$ est borélienne et précisément de la deuxième classe, et a posé le problème de trouver la classe précise des dérivés successifs $D^\alpha$. Nous démontrons que si $n$ est fini, alors $D^n$ est précisément de la classe $2n$ et si $\lambda$ est un ordinal de seconde espèce et $n$ fini, alors $D^{\lambda+n}$ est précisément $d$ la classe $\lambda+2n+1$.

ABSTRACT. — KURATOWSKI showed that the derived set operator $D$, acting on the space of closed subsets of the Cantor space $2^N$, is a Borel map of class exactly two and posed the problem of determining the precise classes of the higher order derivatives $D^\alpha$. In part I of our work [*Bull. Soc. Math. France*, 110, 4, 1982, p. 357-380], we obtained upper and lower bounds for the Borel class of $D^\alpha$ and in particular showed that for limit ordinals $\lambda$, $D^\lambda$ is exactly of class $\lambda+1$. The first author recently showed, using different methods (*cf.* [1]) that for finite $n$, $D^n$ is exactly of Borel class $2n$. We now complete the solution of KURATOWSKI's problem by showing that for any limit ordinal $\lambda$ and any finite $n$, the operator $D^{\lambda+n}$ is of Borel class exactly $\lambda+2n+1$.

In this paper, we determine the exact Borel classes of the iterated derived set operators $D^\alpha$, acting on the space $\mathscr{H}$ of closed subsets of the Cantor space $2^N$ with the usual Vietoris topology. This completes the solution of the problem of KURATOWSKI [3] which was begun in part I of our work [2].

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The results in the present paper depend strongly on those of its predecessor. We begin with some basic definitions and results from [2].

The derived set operator $D$ maps $\mathscr{K}$ into $\mathscr{K}$ and is defined by :

$$D(F) = F' = \{ x : x \in \mathrm{Cl}(F - \{ x \}) \}.$$

The $\alpha$'th iterate $D^\alpha$ of the derived set operator map be defined for all ordinals $\alpha$ by letting $D^0(F) = F$, $D^{\alpha+1}(F) = D(D^\alpha(F))$ for all $\alpha$ and $D^\lambda(F) = \bigcap \{ D^\alpha(F) : \alpha < \lambda \}$ for limit ordinals $\lambda$. The set $F$ is said to be scattered if $D^{\alpha+1}(F) = \emptyset$ for some $\alpha$; the derived set order $o(F)$ of $F$ is the least such ordinal $\alpha$.

The countable subset $S$ of $2^N$ is defined to be $\{ x : (\exists m)(\forall n > m), x(n) = 0 \}$. If $2^N$ is identified with the family $\mathscr{P}(N)$ of subsets of $N$, then $S$ corresponds to the family of finite sets. Let $\bar{0} = (0, 0, 0, \ldots)$. The stitching operator $\Phi$ mapping $\mathscr{K}^N$ into $\mathscr{K}$ is defined as follows:

$$\Phi(F_0, F_1, F_2, \ldots) = \{ \bar{0} \} \cup \{ (\overbrace{0, 0, \ldots, 0}^{n}, 1, x(0), x(1), \ldots) : x \in F_n \}.$$

Note that $\Phi$ preserves both finite intersections and unions, that is:

$$\Phi(F_0 \cup G_0, F_1 \cup G_1, \ldots) = \Phi(F_0, F_1, \ldots) \cup \Phi(G_0, G_1, \ldots)$$

and similarly for intersections. This also implies that $\Phi$ is monotone, that is, whenever $F_i \subset H_i$ for all $i$, then $\Phi(F_1, F_1, \ldots) \subset \Phi(H_0, H_1, \ldots)$. The two fundamental results on the stitching operator, Lemmas 3.7 and 3.8 of [2] concern the derived set order of the stitched set and the continuity of the stitching map. We actually need an extension of the former lemma to infinite ordinals; the proof goes through without difficulty.

LEMMA 1. — *For any sequence* $(F_0, F_1, \ldots)$ of sets from $\mathscr{K} \cap \mathscr{P}(S)$ and any ordinal $\alpha$:

$$D^\alpha(\Phi(F_0, F_1, \ldots)) = \begin{cases} \Phi(D^\alpha(F_0), D^\alpha(F_1), \ldots), \\ \quad \text{if } (\forall \beta < \alpha) \{ n : D^\beta(F_n) \neq \emptyset \} \text{ is infinite}, \\ \Phi(D^\alpha(F_0), D^\alpha(F_1), \ldots) - \{ \bar{0} \}, \text{ otherwise.} \end{cases} \quad \square$$

LEMMA 2. — Let $(H_0, H_1, \ldots)$ be a sequence of continuous functions mapping a topological space $X$ into the space $\mathscr{K}$ of closed subsets of $2^N$

such that each $H_n(x) \subset S$. Then the function $H$, defined by $H(x) = \Phi(H_0(x), H_1(x), \ldots)$ is also continuous. □

Calculation of the exact Borel classes of the iterated derived set operators begins with Theorem 1.3 of [2].

THEOREM 3. — For any finite $n$ and any limit ordinal $\lambda$:

($a$) $D^n$ is of Borel class $2n$;

($b$) $D^{\lambda+n}$ is of Borel class $\lambda+2n+1$. □

Proofs that the Borel classes cited in Theorem 3 are exact proceed as follows. First we note that $\{\emptyset\}$ is both a closed and an open subset of $\mathscr{H}$. Thus if $D^n$ were of class $2n-1$, then $T_n = (D^n)^{-1}(\{\emptyset\})$ would have to be a Borel subset of $\mathscr{H}$ of both additive and multiplicative class $2n-1$; similarly, if $D^{\lambda+n}$ were of class $\lambda+2n$, then $T_{\lambda+n} = (D^{\lambda+n})^{-1}(\{\emptyset\})$ would be of both additive and multiplicative class $\lambda+2n$. To show that $T_n$ is not of multiplicative class $2n-1$, we prove that $T_n$ is actually universal for Borel sets of additive class $2n-1$; a similar result is given for $T_{\lambda+n}$. Both results will be proved by induction on $n$. We need two more propositions from [2]; the first is Proposition 4.1:

THEOREM 4. — For any $F_\sigma$ subset $B$ of $N^N$, there is a continuous function $H$ mapping $N^N$ into $\mathscr{H} \cap \mathscr{P}(S) \cap T_2$ such that, for all $x$, $x \in B$ if and only if $H(x) \in T_1$. □

We actually need the following improvement of Theorem 6.2 of [2].

THEOREM 5. — For any countable limit ordinal $\lambda$ and any Borel subset $B$ of $N^N$ of additive class $\lambda$, there is a continuous function $H$ mapping $N^N$ into $\mathscr{H} \cap \mathscr{P}(S) \cap T_{\lambda+1}$ such that, for all $x$, $x \in B$ if and only if $H(x) \in T_\lambda$.

*Proof.* — Let $B$ be a Borel subset of $N^N$ of additive class $\lambda$. By Theorem 6.2 of [2], there is a continuous function $G$ from $N^N$ into $\mathscr{H} \cap \mathscr{P}(S) \cap T_{\lambda+2}$ such that, for all $x$, $x \in B$ if and only if $G(x) \in T_\lambda$; furthermore, $G(x)$ is also normal, as defined in 5.1 of [2]. Now let $C = C_\lambda$ be some canonical normal set with $o(C) = \lambda$ (*see* 5.10 of [2]). Define the function $H$ by:

$$H(x) = G(x) \cap C_\lambda.$$

Recall from Lemma 5.2 of [2] that, for two normal sets $F$ and $G$:

$$o(F \cap G) = \min(o(F), o(G)).$$

It follows that:

$$o(H(x)) = \min(o(G(x)), \lambda).$$

This implies that $H$ maps into $T_{\lambda+1}$ and that, for any $x$, $x \in B$ if and only if $H(x) \in T_\lambda$. Recall from Lemma 5.12 of [2] that the intersection map is continuous for normal sets. Of course the constant map $F(x) = C_\lambda$ is continuous. It follows that $H$ is also continuous. □

It should be pointed out that the proof of Theorem 6.2 in [2] required the introduction of a more complex stitching operator acting on the family of normal sets.

L. Piątkiewicz has pointed out that in Proposition 5.8 of [2] $\theta(\hat{F})$ is actually normal if and only if $\gamma = \lim_{n \to \infty} (o(F_n) + 1)$; this does not affect the proof of Theorem 6.2.

The induction step in the proofs that $T_n$ and $T_{\lambda+n}$ are universal depends on Lemmas 1 and 2 and the following well-known result (a version of which can be found in Lusin's classic book [5]).

LEMMA 6. — *Let $X$ be a topological space with a countable basis of clopen sets (such as $2^N$ and $N^N$). Then for any countable ordinal $\alpha$ and any Borel subset $B$ of $X$ of additive class $\alpha$, $B$ can be written as the disjoint countable union of Borel sets $B_m$, each of multiplicative class $<\alpha$.*

THEOREM 7. — (*a*) *For any natural number $k$ and any Borel subset $B$ of $N^N$ of additive class $2k-1$, there is a continuous function $H$ mapping $N^N$ into $\mathscr{K} \cap \mathscr{P}(S) \cap T_{k+1}$ such that, for all $x$, $x \in B$ if and only if $H(x) \in T_k$.* (*b*) *For any countable limit ordinal $\lambda$, any natural number $k$ and any Borel subset $B$ of $N^N$ of additive class $\lambda + 2k$, there is a continuous function $H$ mapping $N^N$ into:*

$$\mathscr{K} \cap \mathscr{P}(S) \cap T_{\lambda+k+1}$$

*such that, for all $x$, $x \in B$ if and only if $H(x) \in T_{\lambda+k}$.*

*Proof.* — The proofs of parts (*a*) and (*b*) proceed from, respectively, Theorems 4 and 5 in a similar manner. We will give the proof of (*b*), which is of course by induction on $k$. Theorem 5 covers the case $k=0$. Suppose therefore that the result is true for $k$ and let $B$ be a Borel subset of $N^N$ of additive class $\lambda+2k+2$. Since $N^N \setminus B$ is of multiplicative class $\lambda+2k+2$, there is a decreasing sequence $\{C_n : n \in N\}$ of sets of additive class $\lambda+2k+1$ such that $N^N \setminus B = \bigcap_n C_n$. Now by Lemma 6, there exists for each $n$ a disjoint sequence $\{C_{n,m} : m \in N\}$ of sets of multiplicative class $\lambda+2k$ such that $C_n = \bigcup_m C_{n,m}$. It is now easy to see that, for all $x$:

(i) $x \in B \leftrightarrow \{(n,m) : x \in C_{n,m}\}$ is finite.

Let $(n_0, m_1), (n_1, m_1), \ldots$ be some one-to-one enumeration of $N \times N$ and let $A_i = N^N \setminus C_{n_i, m_i}$. By the induction hypothesis, there exists a sequence $\{H_i : i \in N\}$ of continuous functions from $N^N$ into:

$$\mathscr{K} \cap \mathscr{P}(S) \cap T_{\lambda+k+1}$$

such that, for all $x$:

(ii) $x \in A_i \leftrightarrow H_i(x) \in T_{\lambda+k}$.

The desired reduction $H$ of $B$ to $T_{\lambda+k}$ is now defined by:

(iii) $H(x) = \Phi(H_0(x), H_1(x), \ldots)$.

$H$ is continuous by Lemma 2. We must now calculate the possible derived set order of $H(x)$. First of all, from the induction hypothesis $D^{\lambda+k+1}(H_i(x)) = \emptyset$; it follows from Lemma 1 that $D^{\lambda+k+2}(H(x)) = \Phi(\emptyset, \emptyset, \ldots) \setminus \{\bar{0}\} = \emptyset$. Thus $H(x) \in T_{\lambda+k+2}$ for any $x$. Next suppose that $x \in B$. Then by (i) and the definition of the $A_i$, $\{i : x \notin A_i\}$ is finite. It follows from (ii) that:

$$\{i : D^{\lambda+k}(H_i(x)) \neq \emptyset\}.$$

is finite. Then by Lemma 1, $D^{\lambda+k+1}(H(x)) = \emptyset$ as desired. Finally, suppose that $x \notin B$. Then again using (i) and (ii), it follows that:

$$\{i : D^{\lambda+k}(H_i(x)) \neq \emptyset\}$$

is infinite. Applying Lemma 1 and the fact that each $D^{\lambda+k+1}(H_i(x)) = \emptyset$, we obtain:

$$D^{\lambda+k+1}(H(x)) = \Phi(\emptyset, \emptyset, \ldots) = \{\bar{0}\},$$

so that $H(x) \notin T_{\lambda+k+1}$. □

THEOREM 8. — *(a) For any natural number $k$, $T_k$ is a Borel subset of $\mathscr{K}$ of additive class $2k-1$ but not of multiplicative class $2k-1$. (b) For any countable limit ordinal $\lambda$ and any finite $k$, $T_{\lambda+k}$ is a Borel subset of $\mathscr{K}$ of additive class $\lambda+2k$ but not of multiplicative class $\lambda+2k$.*

*Proof.* — The positive direction is proved by induction, as follows. $T_1 = \{F : F \text{ is finite}\}$ is an $F_\sigma$ set by Lemma 1.1 of [2]. For any limit ordinal $\lambda$, $T_\lambda = \bigcup_{\alpha<\lambda} T_\alpha$ and will therefore be of additive class $\lambda$

if the result is assumed for $\alpha<\lambda$. Finally, $T_{\alpha+1}=D^{-1}(T_\alpha)$; since $D$ is a mapping of Borel class 2, the result can always be extended from $\alpha$ to $\alpha+1$. The other direction has similar proofs for parts (*a*) and (*b*); we give the proof of (*b*). Let $B$ be an arbitrary subset of $N^N$ which is of additive class $\lambda+2k$ but not of multiplicative class $\lambda+2k$ (*see* [4], p. 425). By Theorem 7, there is a continuous function $H$ such that $B=H^{-1}(T_{\lambda+k})$. Now if $T_{\lambda+k}$ were of multiplicative class $\lambda+2k$, it would follow that $B$ must also be of multiplicative class $\lambda+2k$, contradicting our choice of $B$.

We can now give the complete solution of the problem of Kuratowski.

THEOREM 9. — (*a*) *For any natural number* $k$, *the iterated derived set operator* $D^k$ *is of Borel class exactly* $2k$. (*b*) *For any countable limit ordinal* $\lambda$ *and any natural number* $k$, $D^{\lambda+k}$ *is of Borel class exactly* $\lambda+2k+1$.

*Proof.* — One direction is given by Theorem 3. The other direction has similar proofs for parts (*a*) and (*b*); we give the proof of (*b*). Recall that $\{\emptyset\}$ is a closed subset of $\mathscr{K}$. Thus if $D^{\lambda+k}$ were of Borel class $\lambda+2k$, then:

$$T_{\lambda+k}=(D^{\lambda+k})^{-1}(\{\emptyset\})$$

would have to be a Borel set of multiplicative class $\lambda+2k$, which would contradict Theorem 8.

The finite cases of Theorems 7, 8 and 9 were previously obtained by the first author in [1] using different methods.

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