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SYMMETRY OF MANIFOLDS AND THEIR LOWER HOMOTOPY GROUPS

BY

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Introduction

In this paper we study the relation between lower homotopy groups, specifically π_1 and π_2 of a manifold and its symmetry properties. Such relation on the level of fundamental group has been gradually understood, (*see* [Y], [A – B], [B – S], [F – M], [K – K], [S – Y], and in [B – H] has been treated in its most useful form. In this paper we establish, first, a precise relationship between the degree of symmetry and the canonical homomorphism $H_*(M; \mathbb{Q}) \to H_*(\pi_1(M); \mathbb{Q})$ (Theorem A). Next, we consider the more general homomorphism $H_*(M; \mathbb{Q}) \to H_*(M(r); \mathbb{Q})$ induced by taking the canonical Postnikov map $M \to M(r)$, $r \geqslant 1$. Remarkably, there is a partial generalization of Theorem A, namely Theorem B below which provides us with such a relationship between the degree of symmetry and the homomorphism $H_*(M; \mathbb{Q}) \to H_*(M(2); \mathbb{Q})$, in particular, $H_*(M; \mathbb{Q}) \to H_*(K(\pi_2(M), 2); \mathbb{Q})$ for simply-connected M. Apparently, this phenomenon cannot be expected for $r \geqslant 3$, simply because $\pi_3(G) \otimes \mathbb{Q} \neq 0$ for a general compact Lie group G.

As an application, we show that the changes in the differentiable structure of $M^n = S^2 \times T^{n-2}$ affects its semi-simple degree of symmetry.

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That is, $S_s(M^n \# \Sigma^n) = 0$, where $\Sigma^n \in \theta_n$ is a non-standard homotopy sphere. This may be viewed as a generalization of an earlier result of ours that $S(T^n \# \Sigma^n) = 0$.

In some sense this paper might be viewed as an application to [B-H], where the authors have introduced the concept of an i-fold stratified space and its associated stratified r-stage Postnikov system (*see* II.3 and II.5 below), as well as a continuation of [B-S].

I. BERNSTEIN [Be] provided elementary proofs for some particular cases of Theorem A and B which are however, general enough to cover the cases studied by SCHOEN-YAU in [S-Y].

We would like to thank William BROWDER for an inspiring discussion and making available to us the manuscript [B-H] prior to its publication.

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Section 1

Following W. Y. HSIANG, the differentiable degree of symmetry $S(M)$ of a smooth manifold M^n is defined to be the supremum of $\dim G$ where G is a compact Lie group acting smoothly and effectively on M^n. The semi-simple degree of symmetry $S_s(M)$ is defined similarly, where we consider only actions of semi-simple compact Lie groups G on M^n. *Cf.* BURGHELEA-SCHULTZ [B-S].

Following P. CONNER and F. RAYMOND $[\text{C-R}]_2$ an injective total action is a smooth action $\mu : T^k \times M \to M$, (where T^k denotes the k-dimensional torus) so that for any $x \in M$, the map $f^x : T^k \to M$ given by $f^x(t) = \mu(t, x)$ induces a monomorphism $f^x_* : \pi_1(T^k, 1) \to \pi_1(M, x)$. Let $\rho(M)$ be the supremum of k, so that there exists injective total actions $\mu : T^k \times M \to M$.

In a study of smooth actions of simple classical groups on manifolds with vanishing first rational Pontrijagin class, W. C. HSIANG and W. Y. HSIANG have shown that under some dimension restrictions, all isotropy groups are also simple. More generally, we may consider the symmetry properties of manifolds of the following type.

I.1. DEFINITION. — An effective action $\mu : G \times M^n \to M^n$ is called *isotropically semi-simple* if G and all isotropy groups of the action are semi-simple. The *isotropic semi-simple degree of symmetry* of M, denoted by $S_{is}(M)$, is the supremum of $\dim G$ for all isotropically semi-simple actions $\mu : G \times M \to M$.

We are primarily interested in smooth actions and differentiable degree of symmetry. However, the results are equally valid for a wider class of actions which include locally smooth action. Such generalizations can be deduced from examination of the proofs of the theorems below.

I.2. DEFINITION. — Let r and s be positive integers, and let $\mathbb{F}$ be a field. A topological space X has the property $P_{r,s}(\mathbb{F})$, if the canonical map into the r-th Postnikov term $f^{(r)}: X \to X(r)$ induces a non-trivial homomorphism:

$$f_*^{(r)}: H_s(X; \mathbb{F}) \to H_s(X(r); \mathbb{F}).$$

We will be interested in spaces with property $P_{r,s}(\mathbb{F})$, where $r=1$ or 2 and $\mathbb{F}=\mathbb{Q}$.

I.3. THEOREM A. — *Suppose M^n is a compact smooth (or topological) manifold which has property $P_{1,k}(\mathbb{Q})$. Then:*

$$S_s(M) \leqslant \frac{(n-k)(n-k+1)}{2} \quad \textit{and} \quad S(M^n) \leqslant \frac{(n-k)(n-k+1)}{2} + \rho(M^n).$$

I.4. PROPOSITION (*well-known*). — (1) *$\rho(M) \leqslant n$ and $\rho(M)=n$ iff M is diffeomorphic to T^n.*

(2) *If M^n is orientable and $\neq T^n$ then $\rho(M^n) \leqslant n-2$ and if $\rho(M^n)=r$ and center $\pi_1(M, x) \to H_1(M; Z)$ is injective, there exists a finite cover of M diffeomorphic to $N^{n-r} \times T^r$.*

(3) *$\rho(M^n) \leqslant$ rank center $(\pi_1(M, x))$.*

(4) *If $\chi(M) \neq 0$ or at least one rational Pontriagin number is $\neq 0$ then $\rho(M)=0$.*

I.5. *Remark.* — (1) To see that the estimates given I.3(a) are sharp, consider $M^n = S^{n-k} \times T^k$, where the rank of center $\pi_1(M)=k$:

$$S_s(M^n) = \frac{(n-k)(n-k+1)}{2} \quad \text{and} \quad S(M^n) = \frac{(n-k)(n-k+1)}{2} + k.$$

(2) It is easy to see that many manifolds satisfy $P_{1,k}(\mathbb{F})$. For instance, if a compact manifold N^k satisfies $P_{1,k}(\mathbb{F})$, then $(N^k \times Q^{n-k}) \# L^n$ also satisfies $P_{1,k}(\mathbb{F})$ if Q and L are closed manifolds. Thus, if $M^{n_1}, M^{n_2}, \ldots, M^{n_r}$ are closed oriented aspherical manifolds and $n = n_1 + n_2 + \ldots + n_r$, then:

$\{(((M^{n_1} \# L^{n_1}) \times M^{n_2}) \# L^{n_1+n_2}) \times M^{n_3} \ldots\} \# L^n$ has property $P_{1,n}(\mathbb{F})$.

(3) Special cases of Theorem A appear in [C-R]$_1$, [S-Y], [B-H] and [K-K] and H. Donnelly-R. Schultz; compact groups actions and maps into aspherical manifolds, Topology 21 (1982), pp. 443-455.

(4) A "gap theory" for manifolds with property $P_{1,k}(\mathbb{Q})$ in the sense of [K-K] can be developed in a rather straightforward manner.

I.6. THEOREM B. — *If M^n is a smooth manifold which satisfies $P_{2,k}(\mathbb{Q})$, then:*

$$S_{is}(M^n) \leqslant \frac{(n-k)(n-k+1)}{2}.$$

I.7. COROLLARY. — *Suppose M^n satisfies either $P_{2,n}(\mathbb{Q})$ or $P_{2,n-1}(\mathbb{Q})$, then any effective smooth action of $SU(2)$ or $SO(3)$ has one-dimensional isotropy groups.*

I.8. *Remarks.* — (1) To construct examples of manifolds which satisfy $P_{2,k}(\mathbb{Q})$, let $n=2n_1+2n_2+\ldots+2n_r$ and consider $L_1=\mathbb{C}P^{n_1}\# K_1^{2n_1}$,

$$L_2=(L_1^{2n_1}\times\mathbb{C}P^{n_2})\# K_2^{2(n_1+n_2)}, \ldots, L_r=(L_{r-1}\times\mathbb{C}P^{n_r})\# K_r^{2(n_1+\ldots+n_r)},$$

where $K_i^{2(n_1+\ldots n_i)}$ are arbitrary closed manifolds. Then L_i satisfies $P_{2,k_i}(\mathbb{Q})$, $k_i=\sum_{j=1}^{i} 2n_j$. Similarly one constructs examples of manifolds M^n which satisfy $P_{2,k}(\mathbb{Q})$ for $k<n$.

(2) The main interest of Theorem B is primarily in its applications to simply-connected manifolds.

(3) According to HSIANG and HSIANG [H-H] if the first rational Pontrijagin class of M^n vanishes, then smooth actions of $SO(m)$, $SU(m)$ or $Sp(m)$ on M^n must have isotropy groups of the form $SO(k)$, $SU(k)$ or $Sp(k)$, respectively, under some dimension restrictions on m and m. Thus the actions of relatively large dimensional classical groups on manifolds with vanishing first Pontrijagin class are necessarily isotropically (semi-) simple.

Section II

Let G be a compact connected Lie group, and let $\mu: G\times M^n \to M^n$ be a continuous action on the connected manifold M^n. Let $m\in M$, and $e\in G$ be the identity element. Consider the restriction of the action $\mu: G\times\{m\}\to M$. This induces a homomorphism $\mu_*:\pi_1(G, e)\to\pi_1(M, m)$. It is easy to see that $\mu_*\pi_1(G, e)$ lies in the center $(\pi_1(M, m))$: hence it is a normal subgroup of $\pi_1(M, m)$ (*cf.* [C-R]). Let $\Gamma=\pi_1(M, m)/\mu_*\pi_1(G, e)$, which is seen to be idependent of choice of $m\in M$ up to isomorphism. Let $\mu: M\to M/G$ be the canonical projection onto the orbit space. Also, $f^{(1)}: M\to M(1)=K(\pi_1(M), 1)$ denotes the first Postnikov term, and $H_*(\Gamma; \mathbb{F})$ be the homology of the group Γ with coefficients in the field $\mathbb{F}$. The following theorem is due to BROWDER-HSIANG [B-H].

II.1. Theorem A′. — *Let G be a compact connected Lie group, $\mu : G\times M \to M$ be a smooth action $\Gamma=\pi_1(M)/\mu_*\pi_1(G)$, as a defined above, and $q:\pi_1(M)\to\Gamma$ the quotient map. Then:*

(1) *There exists a homomorphism $\theta : H_*(M/G;\mathbb{Q})\to H_*(\Gamma;\mathbb{Q})$ such that the following diagram commutes:*

$$\begin{tikzcd} H_*(M;\mathbb{Q}) \arrow[r, "f^{(1)}_*"] \arrow[d, "\pi_*"'] & H_*(\pi_1(M);\mathbb{Q}) \arrow[d, "q_*"] \\ H_*(M/G;\mathbb{Q}) \arrow[r, "\theta"] & H_*(\Gamma;\mathbb{Q}) \end{tikzcd}$$

The following theorem shows how the second Postnikov term of a manifold M imposes restrictions on the existence of effective actions on M. The case of simply-connected manifolds is particularly noteworthy.

II.2. Theorem B′. — *Let $\mu : G\times M\to M$ be a smooth isotropically semi-simple action, and $f^{(2)} : M\to M(2)$ be the canonical map into the second Postnikov term of M. Then there exists a homomorphism $\theta : H_*(M/G;\mathbb{Q})\to H_*(M(2);\mathbb{Q})$ such that the following diagram commutes:*

$$\begin{tikzcd} H_*(M:\mathbb{Q}) \arrow[r, "f^{(2)}_*"] \arrow[d, "\pi_*"'] & H_*(M(2);\mathbb{Q}) \\ H_*(M/G;\mathbb{Q}) \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

The proofs of Theorems A, B, and their corollaries follow from Theorems A′ and B′:

Proof of Theorem A. — Let G be a compact connected Lie group acting quasi-effectively on M (i.e. the ineffective kernel of the action is a finite subgroup of G). By passing to a finite cover, if necessary, one may assume that $G=T^r\times T^l\times G'$, where G' is compact semi-simple, so that $\mu_*:\pi_1(T^r\times 1)\to\pi_1(M)$ is injective, and $\mu_*:\pi_1(1\times T^l\times G'_s)\to\pi_1(M,m)$ has a finite group as its image. Note that $r\leqslant\rho(M)$ because $\mu_*\mid\pi_1(T^r\times 1)$ injects into (the center of) $\pi_1(M)$. It suffices to show that:

$$\dim(T^l\times G_s)\leqslant\frac{(n-k)(n-k+1)}{2},$$

or equivalently, that for any action $\mu : G\times M\to M$ with $\mu_*(\pi_1(G))$ finite:

$$\dim G\leqslant\frac{(n-k)(n-k+1)}{2}.$$

Assume that M satisfies $P_{1,k}(\mathbb{Q})$, and consider the following commutative diagram from Theorem A′.

$$\begin{tikzcd} H_*(M;\mathbb{Q}) \arrow[r] \arrow[d] & H_*(\pi_1(M);\mathbb{Q}) \arrow[d] \\ H_*(M/G;\mathbb{Q}) \arrow[r] & H_*(\Gamma;\mathbb{Q}) \end{tikzcd}$$

where the right-hand vertical map is an isomorphism, since $\overline{H}_*(\mu_*(\pi_1(G));\mathbb{Q})=0$.

Clearly, the property $P_{1,k}$ implies that $\dim M/G \geqslant k$. As a result, the dimension of the principal orbit is less than or equal to $n-k$. Since G acts effectively on each principal orbit, by a classical theorem of differential geometry (Frobenius-Birkhoff Theorem):

$$\dim G \leqslant \frac{(n-k)(n-k+1)}{2}. \quad \blacksquare$$

Observation to the proof of Theorem A. — The essence of the above proof is the verification of the following statement: in the presence of the property $P_{1,k}$ for any action $\mu: G\times M\to M$ with G compact and $\mu_*^x(\pi_1(G))$ finite (in particular if G is semisimple) the dimension of the principal orbit is $\leqslant n-k$.

Proof of Proposition I.4. — (1) is immediate since the hypothesis implies that for each x f^x is an immersion. (2) follows from Theorem 4.2 of [C-R]$_2$. (3) follows by observing that for each x, $f^x(\pi_1(T,1))\subset \text{Center } \pi_1(M,x)$. To check (4) we recall that if $\chi(M^n)\neq 0$ or at least one rational characteristic number is nonzero, then any circle action on M must have a fixed point. Consequently $\rho(M^n)=0$.

Proof of Theorem B. — Suppose G has an isotropically semi-simple action on M^n. Then the hypothesis implies that $\dim M/G\geqslant k$. Hence the dimension of the principal orbit is at most n-k, which, in turn, shows that:

$$\dim G \leqslant \frac{(n-k)(n-k+1)}{2}. \quad \blacksquare$$

Proof of Corollary I.7. — The hypotheses implies that M^n does not admit an isotropically semi-simple action, and the conclusion follows immediately from consideration of subgroups of $SO(3)$ or $SU(2)$. ■

Before proving Theorem B′, we recall the concept of the stratified N-stage Postnikov system, due to Browder-Hsiang. Let $\mathscr{C}$ be a category (of topological spaces) and let $i\geqslant 0$ be an integer. An *i-fold $\mathscr{C}$-stratified space X* is defined inductively by taking elements of $\mathscr{C}$ for $i=0$, and for $i \geqslant 1$ *via* a push out diagram

$$\begin{array}{ccc} Y_{i-1} & \xrightarrow{f_{i-1}} & X_{i-1} \\ {\scriptstyle h}\downarrow & & \downarrow \\ Z & \rightarrow & X_i \end{array}$$

where Y_{i-1} and X_{i-1} are $(i-1)$-fold spaces, and $Z \in \mathscr{C}$, and f_{i-1} is a morphism of $(i-1)$-fold $\mathscr{C}$-stratified spaces. Morphisms are maps of such diagrams. Let us denote the category of i-fold $\mathscr{C}$-stratified spaces by $\mathscr{C}_i$.

II.3. *Example.* — Let G be a compact Lie group, and let $\mathscr{M}^{is}$ be the category of G-spaces X which have the G-homotopy type of a isotropically semi-simple G-space Y with only one type of orbit. The $\mathscr{M}^{is}_i$ is called the category of *i-fold monotropic is* G-*spaces* (*).

II.4. Proposition (Browder-Hsiang). — *A smooth isotropically semi-simple compact G-manifold M can be given the structure of an i-fold monotropic is G-space, for some i.*

The proof is the same as [B-H].

II.5. *Example.* — Let N be a fixed positive integer and let $\mathscr{P}(N)$ be the category of spaces X such that for each connected component X_α of X, $\pi_i(X_\alpha)=0$ for $i>N$. Then $\mathscr{P}(N)_i$ will be called the category of *i-fold N-stage Postnikov systems.*

The functor $X \to X(N)$ which assigns to a space X its N-th Postnikov term induces a functor $\Pi : \mathscr{C}_i \to \mathscr{P}(N)_i$ in the following fashion. For $i=0$, $\Pi(X_0)=X_0(N)$. Supoose $\Pi : \mathscr{C}_{i-1} \to \mathscr{P}(N)_{i-1}$ is such that there is a natural map $X_{i-1} \to \Pi(X_{i-1})$ which induces isomorphism on the first N homotopy groups. Thus $X_{i-1} \to \Pi(X_{i-1})$ factors the natural map $X_{i-1} \to X_{i-1}(N)$, where $X_{i-1}(N)$ *is the* N-th Postnikov term of X_{i-1}. Suppose $X_i \in \mathscr{C}_i$ is given, so that we have the following push out diagram:

$$\begin{array}{ccc} Y_{i-1} & \rightarrow & X_{i-1} \\ \downarrow & & \downarrow \\ Z & \rightarrow & X_i. \end{array}$$

(*) Is G-space stands for isotropically semi-simple G-space.

Then $\Pi(X_i)$ is defined by the push-out diagram

$$\begin{tikzcd}
\Pi(Y_{i-1}) \arrow[r] \arrow[d] & \Pi(X_{i-1}) \arrow[d, dashed] \\
Z(N) \arrow[r, dashed] & \Pi(X_i)
\end{tikzcd}$$

and the natural map $X_i \to \Pi(X_i)$ is given by the functoriality of push-outs. Note that $\Pi(Y_{i-1}) \to Z(N)$ is given by the composition:

$$\Pi(Y_{i-1}) \to \Pi(Y_{i-1})(N) = Y_{i-1}(N) \to Z(N).$$

A Mayer-Vietoris sequence arguments shows that $X_i \to \Pi(X_i)$ induces isomorphism on π_i for $i \leqslant N$, and this finishes the inductive definition of Π (*cf.* [B-H] for the details).

Proof of Theorem B'. — Let $\mu : G \times M \to M$ be a smooth isotropically semi-simple action. Then M is an i-fold monotropic is G-space by II.4, where $1+i$ is the number of orbit-types. This induces i-fold structures on X/G, $E_G \times X$, and $E_G \times_G X$, where $E_G \to BG$ is the universal principal G-bundle and $E_G \times_G X$ is the orbit space of the diagonal G-action on $E_G \times X$. Apply the functor $\Pi : \mathscr{C}_i \to \mathscr{P}(2)_i$, the i-fold second stage Postnikov functor, to the diagram of i-fold spaces and maps:

$$\begin{tikzcd}
& E_G \times M \arrow[dl, "p_2"'] \arrow[ddr, "\omega"] & \\
M \arrow[ddr, "p"'] & & \\
& & E_G \times_G M \arrow[dl, "q"] \\
& M/G &
\end{tikzcd}$$

to get the commutative diagram

$$\begin{tikzcd}[column sep=small]
& E_G \times M \arrow[rrr] \arrow[dl, "p_2"'] \arrow[ddr, "\omega"] & & & \Pi(E_G \times M) \arrow[rrr] \arrow[dl, "\Pi(p_2)"'] \arrow[ddr, "\Pi(\omega)"] & & & (E_G \times M)(2) \arrow[dl, "\bar{p}_2"'] \arrow[ddr, "\bar{\omega}"] & \\
M \arrow[rrr] \arrow[ddr, "p"'] & & & \Pi(M) \arrow[rrr] \arrow[ddr, "\Pi(q)"'] & & & M(2) \arrow[ddr, "\bar{p}"'] & & \\
& & E_G \times_G M \arrow[rrr] \arrow[dl, "q"] & & & \Pi(E_G \times_G M) \arrow[rrr] \arrow[dl, "\Pi(p)"] & & & (E_G \times_G M)(2) \arrow[dl, "\bar{q}"] \\
& M/G \arrow[rrr] & & & \Pi(M/G) \arrow[rrr] & & & (M/G)(2) &
\end{tikzcd}$$

To obtain the desired factorization, it suffices to show that after applying the functor $H_*(\;\;;\mathbb{Q})$ to the above diagram, the induced maps $\bar{\omega}_*$ and $\Pi(q)_*$ become isomorphisms.

First, consider the map ω, and observe that we have the fibration:

$$E_G \times M \xrightarrow{\omega} E_G \times_G M \xrightarrow{\beta} BG.$$

Since G is a connected semi-simple Lie group, $\pi_j(BG)\otimes\mathbb{Q}=0$ for $j\leqslant 3$. Hence the homotopy-theoretic fibre of $\bar{\omega}$, call it $F_{\bar{\omega}}$, is connected and $\pi_j(F_{\bar{\omega}})$ is finite for $j=1, 2$ and $\pi_j(F_{\bar{\omega}})=0$ for $j>2$. A Serre spectral sequence argument applied to the fibration $\tilde{F}_{\bar{\omega}} \to F_{\bar{\omega}} \to K(\pi_1(F_{\bar{\omega}}), 1)$ shows that $\overline{H}_*(F_{\bar{\omega}};\mathbb{Q})=0$. Consequently:

$$\bar{\omega}_*:\quad H_*((E_G \times M)(2);\mathbb{Q}) \to H_*((E_G \times_G M)(2);\mathbb{Q}),$$

is also an isomorphism by the Serre spectral sequence of:

$$F_{\bar{\omega}} \to (E_G \times M)(2) \to (E_G \times_G M)(2).$$

Next, we prove that $\Pi(q)_*$ is an isomorphism in rational cohomology as well. The proof is by induction on i. If $i=0$, then we have only one type of orbit, say (H). Consequently, we have a fibration $BH \to E_G \times_G M \xrightarrow{q} M/G$ in which $\pi_j(BH)$ are finite for $j\leqslant 3$, since H is semi-simple. Hence the homotopy-theoretic fibre of $\Pi(q)$, say F_q is connected and $\pi_j(F_q)$ are finite for $j=1, 2$ and $\pi_j(F_q)=0$ for $j>2$. An argument similar to the one applied to $F_{\bar{\omega}}$ shows that $\overline{H}_*(F_q;\mathbb{Q})=0$ and hence:

$$\Pi(q)_*:\quad H_*(\Pi(E_G \times_G M);\mathbb{Q}) \to H_*(\Pi(M/G);\mathbb{Q}),$$

is an isomorphism. Now suppose that we have proved the assertion for $i=k-1$, and we would like to prove it for $i=k$. Then we have the following push out diagram:

$$\begin{array}{ccc} Y_{k-1} & \to & M_{k-1} \\ \downarrow & & \downarrow \\ Z & \to & M_k \end{array}$$

where Z has the G-homotopy type of a G-space with only one type of orbit, say (H). We have the push out diagram

$$\begin{tikzcd}
& E_G \times_G Y_{k-1} \arrow[rrr, "q'_{k-1}"] \arrow[ddl] \arrow[dr] & & & Y_{k-1}/G \arrow[ddl] \arrow[dr] & \\
& & E_G \times_G M_{k-1} \arrow[rrr, "q_{k-1}"] \arrow[ddl] & & & M_{k-1}/G \arrow[ddl] \\
E_G \times_G Z \arrow[rrr, "q''"] \arrow[dr] & & & Z/G \arrow[dr] & & \\
& E_G \times_G M_k \arrow[rrr, "q_k"'] & & & M_k/G &
\end{tikzcd}$$

We apply the functor Π to the above diagram, and observe that by the induction hypothesis, $\Pi(q'_{k-1})_*$, $\Pi(q_{k-1})_*$, and $\Pi(q'')_*$ induce isomorphisms in rational homology. Hence the ladder of Mayer-Vietoris sequences associated to the above push-out diagrams, and the Five Lemma shows that $\Pi(q_k)_*$ is also an isomorphism. This finishes the inductive step and establishes the proof that $\Pi(q)_*$:

$$H_*(\Pi(E_G \times_G M); \mathbb{Q}) \to H_*(\Pi(M/G); \mathbb{Q})$$

is an isomorphism. A simple diagram chase yields the desired factorization. ∎

Section III

As an application of Theorem B, we will show that the non-standard differentiable manifold $S^2 \times T^{n-2} \# \Sigma^n$ does not admit a non-trivial smooth action of a semi-simple compact Lie group, whereas:

$$S^2 \times T^{n-2} = (SO(3)/SO(2)) \times T^{n-2},$$

has semi-simple degree of symmetry 3. Thus changes in the smoothness structure at one point leads to lack of semi-simple degree of symmetry.

III.1. THEOREM. – *Let $\Sigma^n \in \theta_n$ be a non-standard homotopy sphere. Then* $S_s^{\text{diff}}(S^2 \times T^{n-2} \# \Sigma^n) = 0$.

Proof. – It suffices to show that if $M^n = S^2 \times T^{n-2} \# \Sigma^n$ admits an effective action of $G = SO(3)$ or $SU(2)$, then $\Sigma^n = 0$ in θ_n, since any semi-simple compact Lie group has a subgroup isomorphic to $SO(3)$ or $SU(2)$. Since M satisfies the condition $P_{2,n}(\mathbb{Q})$ by Corollary 1.7 any nontrivial G-action of M^n must have one dimensional isotropy subgroups hence the principal orbit has dimension $\geqslant 2$. Since M satisfies the condition $P_{1,n-2}(\mathbb{Q})$ by *Observation to the proof of Theorem A* the principal orbit should have dimension $\leqslant 2$, hence $=2$. Consequently there are no finite isotropy groups and any isotropy group H has to be either a maximal torus S^1 or a normalizer $N(S^1)$ of a maximal torus S^1 or G. If $H=G$ the embedding of the principal orbit is homotopic to the constant map and then by [B-S] Proposition 2 for any:

$$\alpha \in H^2(M;\mathbb{Q}),\quad w_1,\dots,w_{n-2}\in H^1(M;Q)\quad \alpha\cup\dots\cup w_{n-2}=0,$$

which is clearly false for $M=S^2\times T^{n-2}\#\Sigma^n$. If some $N(S^1)$ occurs as an isotropy group then $G/N(S^1)\simeq RP^2$ is embedded with trivial normal bundle (any orbit has a trivial normal bundle) in M which is also impossible. Consequently each isotropy group has to be a maximal torus. Thus $M^n\to M/G$ is a fibre bundle with fibre S^2 and structure group $G=SO(3)$. Let W^{n+1} be the associated disc bundle which is homotopy equivalent to M/G. It is not difficult to see that M/G is homotopy equivalent to T^{n-2}; however, we only need $H^*(M/G;\mathbb{Z})\cong H^*(T^{n-2};\mathbb{Z})$ and this follows from the observation that the fibre S^2 is totally non-homologuous to zero in M^n.

Now we are ready to apply the argument of Theorem 1.1 of [A-B] to this situation. First, consider $S^2\times T^{n-2}$ embedded in the interior of D^{n+1}, bounding $D^3\times T^{n-2}$. Taking out the interior of $D^3\times T^{n-2}$ from D^{n+1}, we obtain a cobordism V between $S^2\times T^{n-2}$ and S^n. Then we take the connected sum of V and $\Sigma^n\times[0,1]$ along the "generator" of the cylinder $\Sigma^n\times[0,1]$ to obtain a new cobordism V' between $S^n\#\Sigma^n=\Sigma^n$ and $S^2\times T^{n-2}\#\Sigma^n=M^n$. Since we assumed that $S_s(M^n)\neq 0$, the above argument shows that $M^n\cong\partial W^{n+1}$, with $H_*(W^{n+1})\cong H_*(D^3\times T^{n-2})$. Let $V''=V'\cup_{M^n}W^{n+1}$, and observe that $\Sigma^n=\partial V''$. The comparison of the Mayer-Vietoris sequences of the triads $(V', W; M^n)$ and $(V, D^3\times T^{n-2}; S^2\times T^{n-2})$, and the fact that all cohomology automorphisms of M^n can be obtained by diffeomorphisms show that we may find a diffeomorphism $f: M^n\to M^n$ such that $V''=V'\cup_f W$ becomes acyclic. It is easy to see that by performing surgeries on spheres of dimension one and two in the interior of V'', we can kill the fundamental group without changing the homology of V'', thus obtaining a contractible manifold with boundary Σ^n. This shows that $\Sigma^n=0$ and finishes the proof of the theorem. ■

III.2. As another application, suppose M^n is a spin manifold and admits an isotropically semi-simple action, then the argument of [B-H] § 4 goes through with no difficulty to show that for any cohomology class:

$$x\in H^*(M(2);\mathbb{Q}),\quad (f^{(2)*}(x)\cup \hat{A})[M]=0,$$

where $\hat{A}\in H^{4*}(M;\mathbb{Q})$ is the $\hat{A}$-class of M and:

$$f^{(2)*}:\quad H^*(M(2);\mathbb{Q})\to H^*(M;\mathbb{Q}),$$

is induced by taking second Postnikov term. This is an appropriate substitute for the "higher $\hat{A}$-genus Theorem" of BROWDER-HSIANG when M is simply-connected or it has a finite fundamental group.

REFERENCES

[Be] BERSTEIN (I.). – On Covering Spaces and Lie Group Actions, preprint, June 1981.

[B] BREDON (G.). – Introduction to Compact Transformation Groups, Academic Press, 1972.

[A-B] ASSADI (A.) and BURGHELEA (D.). – Examples of Asymmetric Differentiable Manifolds, *Math. Ann.*, Vol. 255, 1981, p. 423-430.

[B-H] BROWDER (W.) and HSIANG (W. C.). – *G*-actions and the Fundamental Group (to appear in *Inv. Math.*).

[B-S] BURGHELEA (D.) and SCHULTZ (R.). – On the Semisimple Degree of Symmetry, *Bull. Soc. math. France*, T. 103, 1975, pp. 433-440.

[C-R] CONNER (P.) and RAYMOND (F.). – Actions of Compact Lie Groups on Aspherical Manifolds, Topology of Manifolds, CANTRELL and EDWARDS, Ed., Markham, Chicago, 1970, pp. 227-264.

[C-R]$_2$ CONNER (P.) and RAYMOND (F.). – Injective Operations on the Toral Actions, *Topology*, Vol. 10, pp. 280-296.

[F-M] FREEDMAN (M.) and MEEKS (W.). – Une obstruction élémentaire à l'existence d'une action continue de groupe dans une variété, *C.R. Acad. Sc. Paris*, T. 28 , Série A, 1978.

[H-H] HSIANG (W. C.) and HSIANG (W. Y.). – Differentiable Actions of Compact Connected Classical Groups II, *Ann. of Math.*, Vol. 92, 1970, pp. 189-223.

[K-K] KU (H. T.) and KU (M. C.). – Group Actions on A_k-Manifolds, *Trans. A.M.S.*, Vol. 245, 1978, pp. 469-492.

[S-Y] SCHOEN (R.) and YAU (S. T.). – Compact Group Actions and the Topology of Manifolds with Non-positive Curvature, *Topology*, Vol. 18, 1979, pp. 361-380.

[Y] YAU (S. T.). – Remarks on the Group of Isometrics of a Riemannian Manifold, *Topology*, Vol. 16, 1977.