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## A NOTE ON ELLIPTIC CURVES OVER FINITE FIELDS

BY

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RÉSUMÉ. — Nous déterminons tous les groupes que l'on peut obtenir comme groupe des points rationnels d'une courbe elliptique sur un corps fini donné.

ABSTRACT. — We determine all groups that can occur as the group of rational points of an elliptic curve over a given finite field.

Let  $\mathbb{F}_q$  denote the finite field of  $q$  elements. Given  $t$  an integer,  $|t| \leq 2q^{1/2}$  then WATERHOUSE [3] proved that there exists an elliptic curve over  $\mathbb{F}_q$  with  $q + 1 - t$  rational points if and only if, writing  $q = p^h$ ,  $p$  prime, one of the following conditions is satisfied :

- (i)  $(t, q) = 1$ ,
- (ii)  $t = 0$ ,  $h$  odd or  $p \neq 1(4)$ ,
- (iii)  $t = \pm q^{1/2}$ ,  $h$  even or  $p \neq 1(3)$ ,
- (iv)  $t = \pm 2q^{1/2}$ ,  $h$  even,
- (v)  $t = \pm \sqrt{2q}$ ,  $h$  odd and  $p = 2$ ,
- (vi)  $t = \pm \sqrt{3q}$ ,  $h$  odd and  $p = 3$ .

SCHOOF then proved [2] that the possible structures for the group in cases (ii)–(vi) are :

- (ii)  $\mathbb{Z}/2 \oplus \mathbb{Z}/(q + 1)/2$  or cyclic if  $q = 3(4)$ , cyclic otherwise,
- (iii) Cyclic,
- (iv)  $(\mathbb{Z}/(q^{1/2} \pm 1))^2$ ,
- (v) Cyclic,
- (vi) Cyclic.

The purpose of this paper is to give the list of possibilities for the groups occurring as elliptic curves over  $\mathbb{F}_q$  in case (i). Let, for a prime  $\ell$ ,  $v_\ell(n)$  be the largest integer with  $\ell^{v_\ell(n)} \mid n$ .

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THEOREM. — *If  $t$  is an integer with  $|t| \leq 2q^{1/2}$  and  $(t, q) = 1$ , the possible groups that an elliptic curve over  $\mathbb{F}_q$  with  $N = q + 1 - t$  can be are*

$$(*) \quad \mathbb{Z}/p^{v_p(N)} \oplus \bigoplus_{t \neq p} \mathbb{Z}/\ell^{r_\ell} \oplus \mathbb{Z}/\ell^{s_\ell}$$

with  $r_\ell + s_\ell = v_\ell(N)$  and  $\min(r_\ell, s_\ell) \leq v_\ell(q - 1)$ .

*Proof.* — Let  $E[n]$  stand for the group of  $n$ -torsion points of an elliptic curve  $E$  over the algebraic closure of  $\mathbb{F}_q$ . It is well known that  $E[p] = \{0\}$  or  $\mathbb{Z}/p$  and that  $E[\ell] = (\mathbb{Z}/\ell)^2$ ,  $\ell$  prime,  $\ell \neq p$  (see, e.g. [1, Theorem 8.1]). So, clearly the group of points of an elliptic curve over  $\mathbb{F}_q$  is of the form  $(*)$  with  $r_\ell + s_\ell = v_\ell(N)$ . To see that also  $\min(r_\ell, s_\ell) \leq v_\ell(q - 1)$ , we notice that, if  $r_\ell \leq s_\ell$ , then all points of  $E[\ell^{r_\ell}]$  are defined over  $\mathbb{F}_q$ , hence  $\ell^{r_\ell} | q - 1$  by [2, Proposition 3.8]. It then follows that the conditions of the theorem are necessary. We now prove that they are sufficient. For this we need two lemmas.

LEMMA 1. — *Given  $N \not\equiv 1 \pmod{p}$  such that there exists an elliptic curve with  $N$  points over  $\mathbb{F}_q$  then there exists at least one such elliptic curve with its group of rational points being cyclic.*

*Proof.* — Let  $\ell_1, \dots, \ell_r$  be the primes such that  $\ell_i^2 | N$  and  $\ell_i | q - 1$ . If there is no such prime then by the preceding discussion any elliptic curve over  $\mathbb{F}_q$  with  $N$  points will do. So we assume that  $r \geq 1$ .

In [2, Theorem 4.9 (i)], SCHOOF proves that given an integer  $n$ , the number of isomorphism classes of elliptic curves with  $N = q + 1 - t$  points over  $\mathbb{F}_q$  with all points of  $E[n]$  defined over  $\mathbb{F}_q$ , when  $p \nmid t$  and  $n^2 | N$ ,  $n | q - 1$ , is  $H((t^2 - 4q)/n^2)$  where  $H(\Delta)$  is the class number of binary quadratic forms of discriminant  $\Delta$ . (note that although Theorem 4.9 of [2] its stated only for  $n$  odd the proof of item (i) is valid for all  $n$ ). Hence the number  $M$ , say, of elliptic curves satisfying the conclusion of the lemma is clearly:

$$M = H(t^2 - 4q) - \sum_{i=1}^r H((t^2 - 4q)/\ell_i^2) + \sum_{1 \leq i < j \leq t} H((t^2 - 4q)/\ell_i^2 \ell_j^2) + \dots + (-1)^r H((t^2 - 4q)/\ell_1^2 \dots \ell_r^2)$$

$$H(\Delta) = \sum_{\mathcal{O}(\Delta) \subseteq \mathcal{O} \subseteq \mathcal{O}_{\max}} h(\mathcal{O}),$$

where  $\mathcal{O}(\Delta)$  is the quadratic order of discriminant  $\Delta$ ,  $h(\mathcal{O})$  is the class number of  $\mathcal{O}$  and  $\mathcal{O}$  runs through the orders of  $\mathcal{O}(\Delta) \otimes \mathbb{Q}$ . It follows that  $M \geq h(\mathcal{O}(t^2 - 4q)) \geq 1$ . The lemma is thus proved.

*Definition.* — We shall call two elliptic curves  $\ell^\infty$ -isogenous, for a prime  $\ell$ , if there exists an isogeny between them of degree a power of  $\ell$ .

LEMMA 2. — *If  $E$  is an elliptic curve defined over  $\mathbb{F}_q$  and  $\ell \neq p$  is a prime such that  $E$  has a cyclic subgroup of order  $\ell^n$ , then for any  $r \leq s$  with  $r + s = n$  and  $\ell^r | q - 1$ , there exists an elliptic curve defined over  $\mathbb{F}_q$ ,  $\ell^\infty$ -isogenous to  $E$  and containing a subgroup isomorphic to  $\mathbb{Z}/\ell^r \oplus \mathbb{Z}/\ell^s$ .*

*Proof.* — Let  $P \in E$  be a point of order  $\ell^n$  in  $E$  and let  $\Gamma$  be the group generated by  $\ell^s P$ . Let  $E' = E/\Gamma$  and  $\lambda : E \rightarrow E'$  the natural isogeny [1, Lemma 8.5].  $\lambda$  has degree  $\ell^r$ , hence is an  $\ell^\infty$  isogeny. We shall prove that  $E'$  satisfies the conclusions of the lemma. Let  $\hat{\lambda}$  be the dual isogeny [1, pg. 216] and  $M = \ker \hat{\lambda}$ , the points of  $M$  are defined over  $\mathbb{F}_q$  by [1, Lemma 8.4]. Let  $N$  be the group generated by  $\lambda(P)$ , then  $N$  is cyclic of order  $\ell^s$  and as  $\hat{\lambda} \circ \lambda$  is multiplication by  $\ell^r$  [1, 8.7], it follows that  $\hat{\lambda}$  is injective on  $N$ . So  $M \cap N = \{0\}$  and as  $\#M = \deg \hat{\lambda} = \ell^r$  [1, 8.8] it follows that  $M \oplus N \simeq \mathbb{Z}/\ell^r \oplus \mathbb{Z}/\ell^s$ , as desired.

We now complete the proof of the theorem. Take  $N \not\equiv 1 \pmod{p}$  and  $E$  the elliptic curve given by LEMMA 1, so  $E(\mathbb{F}_q)$  is cyclic of order  $N$ . Let  $\ell_1, \dots, \ell_r$  be the primes such that  $\ell_i^2 | N$  and  $\ell_i | q - 1$ . (If there is no such prime there is nothing to prove). Let  $s_1, \dots, s_r$  be integers with  $s_i \leq v_{\ell_i}(N)$  and  $v_{\ell_i}(N) - s_i \leq v_{\ell_i}(q - 1)$ ,  $i = 1, \dots, r$ . Construct successively by LEMMA 2, elliptic curves  $E_1, \dots, E_r$ , with  $E_1$  being  $\ell_1^\infty$ -isogenous to  $E$  and containing a subgroup isomorphic to  $\mathbb{Z}/\ell_1^{s_1} \oplus \mathbb{Z}/\ell_1^{v_{\ell_1}(N) - s_1}, \dots, E_r$ ,  $\ell_r^\infty$ -isogenous to  $E_{r-1}$  and containing a subgroup isomorphic to  $\mathbb{Z}/\ell_r^{v_{\ell_r}(N) - s_r}$ . Notice that an  $\ell^\infty$ -isogeny induces an isomorphism between the subgroups of order prime to  $\ell$ , so the construction is justified since, for  $i < r$ ,  $E_i$  has a cyclic subgroup of order  $\ell_{i+1}^{v_{\ell_{i+1}}(N)}$ . Then

$$E_r \simeq \mathbb{Z}/p^{v_p(N)} \oplus \bigoplus_{\ell \neq p, \ell_i} \mathbb{Z}/\ell^{v_\ell(N)} \oplus \bigoplus_{i=1}^r \mathbb{Z}/\ell_i^{s_i} \oplus \mathbb{Z}/\ell_i^{v_{\ell_i}(N) - s_i}.$$

As the  $s_i$  were arbitrary satisfying  $s_i \leq v_{\ell_i}(N)$  and  $v_{\ell_i}(N) - s_i \leq v_{\ell_i}(q - 1)$ , the proof of the theorem is complete.

*Added in proof.* — After this paper was submitted, there appeared in print an article by H. G. RUCH (*Math. of Comp.*, t. 49, 1987, p. 301–304), proving the same result but with a different proof.

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