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SOME REMARKS ABOUT ALGEBRAIC INDEPENDENCE MEASURES IN HIGH DIMENSION

BY

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RÉSUMÉ. — Pour $\omega \in \mathbb{C}^n$ et $k \in \mathbb{N}$, $1 \leq k \leq n$, il est possible de définir deux mesures. La première est la *mesure d'indépendance algébrique*, qui donne une minoration de la grandeur en ω d'un idéal I de polynômes de codimension k. La deuxième est la *mesure d'approximation*, qui donne une minoration d'approximabilité des coordonnées de ω par des nombres algébriques sur une extension transcendante pure de $\mathbb{Q}$ de dimension $n-k$.

En une variable, ces mesures sont équivalentes entre elles. En plusieurs variables les relations qui les lient ne semblent pas optimales. Dans ce texte nous étudions des mesures d'indépendance algébrique pour "idéaux lisses" afin d'établir des relations plus précises avec les mesures d'approximation.

ABSTRACT. — For $\omega \in \mathbb{C}^n$ and $k \in \mathbb{N}$, $1 \leq k \leq n$, it is possible to define two different measures. Firstly, an *algebraic independence measure*, which gives a lower bound for the "smallness" at ω of a k-rank ideal I of polynomials; secondly an *approximation measure*, which provides a lower bound for the approximability of ω with n-tuples of complex numbers, algebraic over a pure transcendental extension of dimension $n-k$ over the rationals. If $n=1$ these measures are equivalent, but in higher dimensions the relations between them do not seem to be optimal. In this paper algebraic independence measures "for smooth ideal" are analized and a stronger relation between them and approximation measures is found.

0. Introduction

In a recent paper ([P3]) PHILIPPON considers some relations between algebraic independence measures and approximation measures.

Let us recall some definitions. For any proper homogeneous unmixed ideal $I \subset \mathbb{Z}[x_0, \ldots, x_n]$, with $I \cap \mathbb{Z} = \{0\}$, we define $t(I)$ as the size of the Chow form F associated with I. The norm $\|I\|_\omega$ of I at $\omega \in \mathbb{C}^{n+1} - \{0\}$ is defined as $H(F(S^1\omega, \ldots, S^{n-k+1}\omega))|\omega|^{-(n+1-k)\delta}$ where $k = \operatorname{rank} I$, δ is the degree of F in the first group of variables and $S^1, \ldots, S^{n-k+1}$

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are skew-symmetric matrices in the new variables $s_{k\ell}^{(j)}$, $0 \leq k < \ell < n$, $j = 1, \ldots, n-k+1$. Here $t(I)$ and $\|I\|_\omega$ play a role analogous to that of the size $t(Q)$ and the modulus $|Q(\omega)|$ for a polynomial $Q \in \mathbb{Z}[x]$.

Given $\omega \in \mathbb{C}^n$, we denote by ω also $(1, \omega) \in \mathcal{C}^{n+1} - \{0\}$. An algebraic independence measure in codimension k $(0 \leq k \leq n+1)$ for ω is a function $\varphi_k : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality

$$\|{}^h I\|_\omega \geq \exp\bigl(-\varphi_k(t(I))\bigr)$$

holds for any proper unmixed ideal $I \subset \mathbb{Z}[x_1, \ldots, x_n]$ with $I \cap \mathbb{Z} = \{0\}$ and $\operatorname{rank} I = k$.

Let $\theta \in \mathbb{C}$ be algebraic over a purely transcendental extension $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_D)$. Let

$$f(y) = \sum_{h=0}^{\delta} P_h(\alpha_1, \ldots, \alpha_D) y^h, \qquad P_h \in \mathbb{Z}[x_1, \ldots, x_D],$$

be the irreducible polynomial of $\mathbb{Z}[x_1, \ldots, x_D, y]$ such that $f(\alpha_1, \ldots, \alpha_D, \theta) = 0$. The size of θ with respect to K is defined as

$$t_K(\theta) = \max_{0 \leq h \leq \delta} \bigl(t(P_h), \delta\bigr).$$

An approximation measure in dimension D $(0 \leq D \leq n+1)$ for ω is a function $\gamma_D : \mathbb{R}_+ \to \mathbb{R}_+$ such that the inequality

$$|\theta - \omega| \geq \exp\Bigl(-\gamma_D\bigl(\max_{1 \leq i \leq n} (t_K(\theta_i))\bigr)\Bigr)$$

holds for any $\theta \in \mathbb{C}^n$ with algebraic coordinates over a pure transcendental extension K of $\mathbb{Q}$ with $\operatorname{tr}\deg_Q K = D$.

PHILIPPON finds the following results :

THEOREM 0.1. — *Let $\omega \in \mathbb{C}^n$. There exist positive constants α_1, α_2, α_3, α_4 depending only on n and $|\omega|$, with the following properties. If φ_k is an algebraic independence measure in codimension k for ω, then the function*

$$\gamma_{n-k}(T) = \varphi_k(a_1 T^n) - a_2 T^n$$

is an approximation measure in dimension $n-k$ for ω. Conversely, let γ_D be an approximation measure in dimension D for ω. Then the function

$$\varphi_{n-D}(T) = T\bigl[\gamma_D(a_3 T) - a_4\bigr]$$

is an algebraic independence measure in codimension $n-d$ for ω.

If the approximation measures which can be derived from algebraic independence measures seem to be the best possible ones, in the opposite direction the result seems to be weaker. For example, if $n=1$, it is well-known (see [W, page 133] for instance) that $\varphi_1(T) = a_5 \max(\gamma_0(T), T^2)$ is an algebraic independence measure in codimension 1 for $\omega \in \mathbb{C}$ if γ_0 is any approximation measure in dimension 0 and a_5 is some constant depending only on $|\omega|$. The reason for this gap between dimension one and higher dimensions is in the "inversion theorem" (see [P2, lemma 2.7]) which is the main tool for the proof of the second part of THEOREM 0.1 :

THEOREM 0.2 ("Inversion theorem"). — *Let $\omega \in \mathbb{C}^n$. There exists a positive constant a_6 depending only on n and $|\omega|$ with the following properties. Let P be a proper prime ideal of $\mathbb{Z}[x_1, \dots, x_n]$ with $P \cap \mathbb{Z} = \{0\}$. Then there exists $\theta \in \mathbb{V}_{\mathbb{C}^n}(P) = \{x \in \mathbb{C}^n \mid f(x) = 0 \ \forall f \in P\}$ such that*

$$|\omega - \theta| \le a_6 \, \|P\|_\omega^{1/t(P)}.$$

We observe that a stronger result is available in dimension one :

THEOREM 0.3. — *Let $\omega \in \mathbb{C}$. There exists a positive constant a_7 depending only on $|\omega|$ with the following properties. Let $Q \in \mathbb{Z}[x]$ be a prime polynomial. Then there exists a root α of Q such that*

$$|\omega - \alpha| \le |Q(\omega)| \exp\bigl(a_7 t(Q)^2\bigr).$$

In this paper we extend THEOREM 0.3 to higher dimensions, under suitable "smoothness" conditions for the variety of zeros of P, obtaining a stronger version of THEOREM 0.2. These results yields the following improvements of THEOREM 0.1 :

THEOREM 0.4. — *Let $\omega \in \mathbb{C}^n$. There exists a positive constant a_8 depending only on n and $|\omega|$ with the following properties. Let φ_D be an approximation measure in dimension D for ω. Suppose that $\varphi_D(x)/x$ is an increasing function. Then for any proper unmixed ideal $I \in \mathbb{Z}[x_1, \dots, x_n]$ with $I \cap \mathbb{Z} = \{0\}$ and $\operatorname{rank} I = n - D$ the following inequality*

$$\|{}^h I\|_\omega > \exp\Bigl(\max\Bigl(a_8 t(I)^{D+2}, \varphi_D\bigl(a_8 t(I)\bigr)\Bigr)\Bigr)$$

holds, provided that $\mathbb{V}_{\mathbb{P}(\mathbb{C}^n)}({}^h I)$ is a smooth variety. In other terms

$$\tilde{\psi}_{n-D}(T) = \max\bigl(a_8 T^{D+2}, \varphi_D(a_8 T)\bigr)$$

is an algebraic independence measure "for smooth ideals" in codimension $n - D$ *for* ω.

THEOREME 0.5. — *Let* $\omega \in \mathbb{C}^n$. *There exists a positive constant* a_9 *depending only on* n *and* $|\omega|$ *with the following properties. Let* φ_{D-1}, φ_D *be approximation measures in dimension* $D-1$, D *for* ω. *Suppose that* $\varphi_h(x)/x$ *is an increasing function for* $h = D-1$, D. *Then*

$$\psi_{n-D}(T) = \max\Big(a_9 T^{D+2}, \varphi_D(a_9 T),\, T^2 \varphi_{D-1}(a_9 T^2)\Big)$$

is an algebraic independence measure in codimension $n - D$ *for* ω.

1. Inversion of ideals

The following lemmas will be useful :

LEMMA 1.1. — *Let* P *be a prime homogeneous ideal of* $\mathbb{Z}[x_0, \dots, x_n]$ *with* $P \cap \mathbb{Z} = \{0\}$, $\operatorname{rank} P = k$. *Let* $f_1, \dots, f_m$ *be homogeneous polynomials of* $Z[x_0, \dots, x_n]$ *of total degree at most* δ. *Let* $r = \operatorname{rank}(P, f_1, \dots, f_m)$. *Then there exists homogeneous polynomials* $\lambda_{ij} \in \mathbb{Z}[x_0, \dots, x_n]$ *with* $(1 \le i \le r-k,\ 1 \le j \le m)$, $\deg \lambda_{ij} = \delta - \deg f_j$ *and* $H(\lambda_{ij}) \le \delta^{r-k} \deg P$ *such that the polynomials* $Q_1, \dots, Q_{r-k}$ *defined by*

$$Q_i = \lambda_{i1} f_1 + \cdots + \lambda_{im} f_m, \quad 1 \le i \le r-k,$$

are a "semi"-regular sequence in $\mathbb{Z}[x_0, \dots, x_n]/P$, *i.e. if* $I_0 = P$ *and we define for* $i = 1, \dots, r-k$ *the ideal* I_i *as the intersection of the isolated components of* (I_{i-1}, Q_i), *we have* $I_i : Q_{i+1} = I_i$ *for* $i = 1, \dots, r-k$.

The proof of this lemma is standard. See for instance [P2, lemma 1.9] or [M–W, lemma 2], although these authors take $P = 0$.

LEMMA 1.2. — *Let* P *be a homogeneous prime ideal of* $\mathbb{C}[x_0, \dots, x_n]$ *of rank* $n-d+1$. *Let* $F \in \mathbb{C}[u_{ij}]_{i=1,\dots,d,\, j=0,\dots,n}$ *be a Chow form associated to* P. *Let us denote by* F_j, $j = 0, \dots, n$ *the derivation of* F *with respect to* u_{dj}. *For* $\omega \in \mathbb{C}^{n+1} - \{0\}$ *let us consider the homomorphism* $\sigma_\omega : \mathbb{C}[u_{ij}] \to \mathbb{C}[s^i_{k\ell}]_{0 \le k < \ell \le n,\, i=1,\dots,d}$ *given on each* u_i *by* $u_i \mapsto S^i \omega$ *where the* S^i *are skew-symmetric matrices in the variable* $s^i_{k\ell}$. *Let us assume*

$$\sigma_\omega F \equiv \sigma_\omega F_0 \equiv \cdots \equiv \sigma_\omega F_n \equiv 0.$$

Then ω *is a singular point of* $\mathbb{V} = \mathbb{V}_{\mathbf{P}(\mathbb{C}^n)}(P)$.

Proof. — Let us denote by δ the degree of F with respect to the variables u_i. Let us assume ω be a regular point of $\mathbb{V}$. We show that

$$\sigma_\omega F \equiv \sigma_\omega F_h \implies \omega_h = 0.$$

We can find $d-1$ skew-symmetric matrices $T^1,\dots,T^{d-1}$ such that ω is a regular point of

$$\mathbb{V}' = \mathbb{V} \cap \{xT^1\omega = 0\} \cap \cdots \cap \{xT^{d-1}\omega = 0\}$$

and $\dim \mathbb{V}' = 0$. Let us consider the homomorphism $\tau_\omega : \mathbb{C}[u_{ij}] \to \mathbb{C}[u_{dj}]$ given on each u_i, $1 \le i < d$, by $u_i \mapsto T^i\omega$. Then $\tau_\omega F$ is a Chow form of $\mathbb{V}'$:

$$\tau_\omega F = a \prod_{\ell=1}^{\delta} \theta_\ell u_d \quad a \in \mathbb{C}^*, \quad \theta_\ell \in \mathbb{V}'.$$

We have :

$$\tau_\omega F_h = \frac{\partial}{\partial u_{dh}} \tau_\omega F = a \sum_{\ell=1}^{m=\delta} \theta_{\ell h} \prod_{m \ne \ell} \theta_m u_d$$

$\tau_\omega F|_{u_d = S^d\omega} \equiv 0$, thus $\omega = \theta_1$, say. Hence

$$0 \equiv \tau_\omega(F_h)|_{u_d = S^d\omega} = a\omega_h \prod_{m \ne 1} \theta_m S^d\omega$$

ω is a regular point of $\mathbb{V}'$, thus $\prod_{m\ne 1} \theta_m S^d\omega \not\equiv 0$. Hence $\omega_h = 0$. □

THEOREM 1.1. — *Let $\omega \in \mathbb{C}^n$. There exists a positive constant c_1 depending only on n and $|\omega|$ with the following properties. Let P be a homogeneous prime ideal of $\mathbb{Z}[x_0,\dots,x_n]$ with $P \cap \mathbb{Z} = \{0\}$. If $\mathbb{V} = \mathbb{V}_{\mathbb{P}(\mathbb{C}^n)}(P)$ is a smooth variety of dimension D then there exists $\theta \in \mathbb{V}$ with $\theta_0 = 1$ such that*

$$|\theta - \omega| \le \|P\|_\omega \exp\big(2c_1 t(P)^{D+2}\big).$$

Proof. — We use the same notation as in LEMMA 1.2. We can assume $\|P\|_\omega < \exp(-c_1 t(P)^{D+2})$. We denote by $c_2, c_3, \dots$ positive constants depending only on n and $|\omega|$. Let $d = D+1$. We can find a finite subset Λ of monomials in the variables $s^i_{k\ell}$ and homogeneous polynomials $f_{\mathcal{M}h} \in \mathbb{Z}[x_0,\dots,x_n]$, $\mathcal{M} \in \Lambda$, $0 \le h \le n$ of total degree $d\delta - 1$ such that

$$\sigma_x F_h = \sum_{\mathcal{M} \in \Lambda} f_{\mathcal{M}h}(x)\mathcal{M}.$$

By LEMMA 1.2 the ideal $(P, f_{\mathcal{M}h})_{\mathcal{M}\in\Lambda,\, 0\le h\le n}$ has rank $n+1$. Using LEMMA 1.1 we can find homogeneous polynomials $Q_1,\dots,Q_d$ with $t(Q_i) \le c_2 t(P)$ and $\max_i |Q_i(\omega)| \le c_2 \max_{\mathcal{M}} |f_{\mathcal{M}}(\omega)|$ such that $(Q_1,\dots,Q_d)$ is a "semi"-regular sequence in $\mathbb{Z}[x_0, \ldots, x_N]/P$. By induction, we define the following sequence of Chow forms :

$$\begin{cases} F_0 = F\,; \\ F_k = \mathrm{Res}\big(F_{k-1}, Q_k\big), \quad 1 \le k \le d. \end{cases}$$

Using proposition 3 of [N] it is easy to see

$$t(F_d) \le c_3 t(P)^{d+1}\,;$$
$$\|F_d\|_\omega \le c_3 t(P)^{d+1} \max_{\substack{0 \le h \le n \\ \mathcal{M} \in \Delta}} \big(\|P\|_\omega, |f_{\mathcal{M}h}(\omega)|\,|\omega|^{-d\delta+1}\big).$$

F_d is a Chow form of an ideal of rank $n+1$. Thus, if c_1 is sufficiently large with respect to c_3,

$$\max_{\substack{0 \le h \le n \\ \mathcal{M} \in \Delta}} \big(|f_{\mathcal{M}h}(\omega)|\big) \ge \exp\big(-c_4 t(P)^{d+1}\big).$$

In other terms, for some $\tilde{h}$

$$H(\sigma_\omega F_{\tilde{h}}) \ge \exp\big(-c_4 t(P)^{d+1}\big). \tag{1}$$

By lemma 4 of [N], we can find $a \in \mathbb{Z}[u_1, \ldots, u_{d-1}]$ and $\theta_{\ell j}$, $1 \le \ell \le \delta$, $0 \le j \le n$, algebraic over $Q(u_1, \ldots, u_{d-1})$ such that

$$F = a \prod_{\ell=1}^{\delta} \theta_\ell u_d. \tag{2}$$

Moreover, since $x_{\tilde{h}} \notin P$ (otherwise $F_{\tilde{h}} \equiv 0$), we may assume $\theta_{\ell\tilde{h}} = 1$, $1 \le \ell \le \delta$.

We can specialize the S^i, $1 \le i \le d-1$ to the open set of skew-symmetric matrices $T^i = (t^i_{k\ell})$ with $t^i_{k\ell}$ in the unit polydisk $|t^i_{k\ell}| < 1$ such that under the substitution τ_ω given on each u_i, $1 \le i \le d-1$, by $u_i \mapsto T^i\omega$, we get

$$H\Big(\tau_\omega(F_{\tilde{h}})\big|_{u_d = S^d\omega}\Big) \ge \frac{1}{2} H\big(\sigma_\omega F_{\tilde{h}}\big). \tag{3}$$

We have

$$H\Big(\tau_\omega(F)\big|_{u_d = S^d\omega}\Big) \le (\delta+1)^{c_5} \|P\|_\omega. \tag{4}$$

The derivation of (2) with respect to $u_{d\tilde{h}}$ gives

$$F_{\tilde{h}} = a\sum_{\ell=1}^{\delta}\prod_{m\neq\ell}\theta_\ell u_d \tag{5}$$

by the continuity of $\| \ \|_\omega$ in ω, we can assume $\tau_\omega a \neq 0$. We extend τ_ω to a homomorphism $\tau_\omega : \mathbb{C}[u_i,\theta_\ell]_{i<d} \to \mathbb{C}$ (*see* B, p. 7]). Thus

$$\tau_\omega(F)|_{u_d=S^d\omega} = \tau_\omega a \prod_{\ell=1}^{\delta} \tau_\omega\theta_\ell S^d\omega \tag{6}$$

$$\tau_\omega(F_{\tilde{h}})|_{u_d=S^d\omega} = \tau_\omega a \sum_{\ell=1}^{\delta}\prod_{m\neq\ell} \tau_\omega\theta_\ell S^d\omega \tag{7}$$

Let us suppose

$$H\big(\tau_\omega\theta_1 S^d\omega\big) \leq \cdots \leq H\big(\tau_\omega\theta_\delta.S^d\omega\big). \tag{8}$$

Using GEL'FOND'S inequality, (6), (7) and (8) give

$$H\Big(\tau_\omega(F)|_{u_d=S^d\omega}\Big) \geq (\delta+1)^{c_6} H\Big(\tau_\omega(F_{\tilde{h}})|_{u_d=S^d\omega}\Big) H(\tau_\omega\tilde{\theta}S^d\omega) \tag{9}$$

where $\tilde{\theta} = \theta_1$. Combining (1), (3), (4) and (9) we obtain

$$\max_{0\leq i,j\leq n} |\tilde{\theta}_i\omega_j - \tilde{\theta}_j\omega_i| = H\big(\tau_\omega\tilde{\theta}S^d\omega\big) \leq \|P\|_\omega \exp\big(\tfrac{3}{2}c_1 t(P)^{d+1}\big). \tag{10}$$

In particular $|\tilde{\theta}_0\omega_{\tilde{h}} - 1| \leq \frac{1}{2}$, thus $\tilde{\theta} \geq c_6$. Let $\theta_i = \tilde{\theta}_i/\tilde{\theta}_0$. $\theta \in \mathbb{V}$, $\theta_0 = 1$ and

$$|\theta - \omega| \leq \|P\|_\omega \exp\big(2c_1 t(P)^{d+1}\big).$$

THEOREM 1.1 is proved. □

Proof of THEOREM 0.4. — Let I be any proper unmixed ideal $I \in \mathbb{Z}[x_1,\ldots,x_n]$ with $I\cap\mathbb{Z} = \{0\}$, $\operatorname{rank} I = n-D$ and $\mathbb{V}_{\mathbf{P}(\mathbb{C}^n)}({}^hI)$ smooth. Using proposition 2 of [N], we can find a prime ideal P of ${}^hI$ such that $\|P\|_\omega^{1/t(P)} \leq \|{}^hI\|_\omega^{c_1/t(I)}$. $\mathbb{V}_{\mathbf{P}(\mathbb{C}^n)}(P)$ is smooth and we apply THEOREM 1.1 : if $\|{}^hI\|_\omega < \exp(-c_8 t(I)^{D+2})$, there exists $\theta \in \mathbb{V}_{\mathbf{P}(\mathbb{C}^n)}(P)$ such that

$$|\theta - \omega| \leq \|P\|_\omega^{1/2}.$$

Without any restriction, we may assume $\theta_1,\ldots,\theta_D$ algebraically independent over $\mathbb{Q}$ and θ_j algebraic over $K = \mathbb{Q}(\theta_1,\ldots,\theta_D)$. For the size of θ_j we find

$$t_K(\theta_j) \leq c_9 t(P)$$

(*see* [P3, Proposition 9]). Thus

$$\|{}^hI\|_\omega > e^{-\varphi_D\left(c_{10}t(I)\right)}$$

and THEOREM 0.4 is proved with $A_8 = \max(c_8, c_{10})$. □

Proof of THEOREM 0.5. — Let I be any proper unmixed ideal $I \in \mathbb{Z}[x_1, \dots, x_n]$ with $I \cap \mathbb{Z} = \{0\}$, rank $I = n - D$. Let us assume $\|{}^hI\|_\omega < \exp(-c_{11}t(I)^{d+1})$. Using proposition 2 of [N], we can find a prime ideal P of ${}^hI$ such that $\|P\|_\omega^{1/t(P)} \leq \|{}^hI\|_\omega^{c_{12}/t(I)}$. Let F be the Chow form of P. Now we follow the pattern of the proof of THEOREM 1.1.

Case 1. — $\max_{0 \leq h \leq n} H(\sigma_\omega F_{\tilde{h}}) \geq \|P\|_\omega^{2/3}$. Then there exists $\theta \in \mathbb{V}$ with $\theta_0 = 1$ such that $|\theta - \omega| \geq \|P\|_\omega^{1/2}$ and we find $\|{}^hI\|_\omega \geq \exp(-\varphi_D(c_{13}t(I))$.

Case 2. — $\max_{0 \leq h \leq n} H(\sigma_\omega F_{\tilde{h}}) < \|P\|_\omega^{2/3}$. dim Sing $\mathbb{V} < D$, thus there exist $\widetilde{M}$, $\tilde{h}$ such that $f_{\widetilde{M}\tilde{h}} \notin P$ and J be the ideal associated with G. We have

$$\operatorname{rank} J = n - D + 1\,; \qquad t(J) \leq c_{14}t(P)^2\,;$$
$$\|J\|_\omega \leq c_{14}t(P)^2 \max\left(\|P\|_\omega, |f_{\widetilde{M}\tilde{h}}(\omega)||\omega|^{-d\delta+1}\right).$$

Using Proposition 2 of [N], we can find a prime ideal Q of J such that

$$\|Q\|_\omega^{1/t(Q)} \leq \|P\|_\omega^{c_{15}/t(P)^2}.$$

THEOREM 0.2 ensures the existence of $\theta \in \mathbb{V}_{\mathbb{P}(\mathbb{C}^n)}(J)$ with $\theta_0 = 1$ such that

$$|\theta - \omega| \leq \|P\|^{c_{16}/t(P)^2}$$

Thus

$$\|{}^hI\|_\omega \geq \exp\Big(-t(I)^2\varphi_{D-1}\left(c_{17}t(I)^2\right)\Big).$$

Hence THEOREM 0.5 is proved with $a_9 = \max(c_{11}, c_{13}, c_{17})$. □

2. Inversion of polynomials

If we have some information on the generators of the ideal P, we can improve the results of the last section.

THEOREM 2.1. — *Let ω in $\mathbb{C}^n$ and $m \in \mathbb{N}$. There exists a positive constant c_{18} depending only on n, m and $|\omega|$ with the following properties. Let $Q_1, \ldots, Q_m$ be homogeneous polynomials in $\mathbb{Z}[x_0, \ldots, x_n]$ with $P = (Q_1, \ldots, Q_m)$ prime of rank k. If $\mathbb{V} = \mathbb{V}_{\mathbb{P}(\mathbb{C}^n)}(P)$ is a smooth variety and $\max_{1 \le i \le m} |Q_i(\omega)| < \exp(-c_{18} \max_{1 \le i \le m} t(Q_i)^{n+1})$ then there exists $\theta \in \mathbb{C}^{n+1} - \{0\}$ with $\theta_0 = 1$ and a projective variety $\mathbb{W} \supset \mathbb{V}$ such that $\theta \in \mathbb{W}$, $\dim_\theta \mathbb{W} = \dim_\theta \mathbb{V} = n - k$ and*

$$|\theta - \omega| \le \Big(\max_{1 \le i \le m} |Q_i(\omega)|\Big)^{1/2}.$$

In the proof of this theorem we use the following lemma in which we yield explicit bounds for the neighbourhoods which appear in DINI'S theorem :

LEMMA 2.1. — *Let ω in $\mathbb{C}^k$. There exists a positive constant c_{19} depending only on k and $|\omega|$ with the following properties. Let $R_1, \ldots, R_k \in \mathbb{C}[y_1, \ldots, y_k]$. Let $T = \max_{1 \le i \le k} t(R_i)$, $F = (R_1, \ldots, R_k) : \mathbb{C}^k \to \mathbb{C}^k$, $\| \cdot \|$ the euclidean norm. Let us suppose :*

$$\Big|\mathrm{Det}\big[dF(\omega)\big]\Big| \ge c_{19} e^{2kT} \|F(\omega)\|^{1/2}.$$

Then there exists $\theta \in \mathbb{C}^k$ such that $F(\theta) = 0$ and

$$|\theta - \omega| \le \|F(\omega)\|^{1/2}.$$

Proof. — Let

$$K = \overline{B}_{\mathbb{C}^k}\big(\omega, k^{-1/4}\|F(\omega)\|^{1/2}\big), \quad M = dF(\omega),$$
$$\psi : K \longrightarrow \mathbb{C}^k, \quad \psi(x) = x - M^{-1}F(x).$$

For any $x \in K$ we have :

$$\begin{aligned}\|d\psi(x)\| &\le \|M^{-1}\| \|dF(\omega) - dF(x)\| \\ &\le c_{20} e^{kT} |\operatorname{Det} M|^{-1} c_{21} e^{kT} |x - \omega| \\ &\le \frac{1}{2}.\end{aligned}$$

Thus, for any x in K

$$|\psi(x)-\omega| \leq \big|\psi(x)-\psi(\omega)\big| + \big|\psi(\omega)-\omega\big| \leq k^{-1/4}\big\|F(\omega)\big\|^{1/2}.$$

Hence ψ is a contraction from K to K and we can find a fixed point θ which satisfies our requirements. □

Proof of THEOREM 2.1. — Let us denote by T the maximum of the size of the polynomials Q_i. Let $M_1,\ldots,M_t$ the determinants of the $k\times k$ minors of the jacobian matrix $(\partial Q_i/\partial x_j)_{1\leq i\leq m,\,0\leq j\leq n}$. $\mathbb{V}$ is smooth, thus the ideal $(Q_1,\ldots,Q_m,M_1,\ldots,M_t)$ has rank $n+1$. We apply theorem 13 of [P1] : there exists ℓ, $1\leq \ell\leq t$ such that

$$\big|M_\ell(\omega)\big| \geq \exp(-c_{22}T^{n+1}). \tag{11}$$

Using Euler's formula we may assume $\partial/\partial x_0$ do not appear in M_ℓ. Hence we may suppose

$$M_\ell = \operatorname{Det}\Big(\frac{\partial Q_i}{\partial x_j}\Big)_{1\leq i\leq k,\,1\leq j\leq k}.$$

Let $R_i(y_1,\ldots,y_k) = Q_i(1,y_1,\ldots,y_k,\omega_{k+1},\ldots,\omega_n)$, $i=1,\ldots,k$. We apply LEMMA 2.1 : there exists $\tilde\theta\in\mathbb{C}^k$ such that $R_i(\theta)=0$ $(i=1,\ldots,k)$ and $|\tilde\theta-\tilde\omega| \leq (\max_{1\leq i\leq k}|R_i(\tilde\omega|)^{1/2}$, where $\tilde\omega=(\omega_1,\ldots,\omega_k)$. Let $\theta = (1,\tilde\theta_1,\ldots,\tilde\theta_k,\omega_{k+1},\ldots,\omega_n)$. We have :

$$|\theta-\omega| \leq \Big(\max_{1\leq i\leq k}|R_i(\tilde\omega)|\Big)^{1/2}. \tag{12}$$

Let $\mathbb{W} = \{Q_1=\cdots=Q_k=0\}$. $\theta\in\mathbb{W}$ and inequalities (11) and (12) give :

$$\operatorname{Det}\Big(\frac{\partial Q_i}{\partial x_j}\Big),\ 1\leq i\leq k_{1\leq j\leq k}\quad(\theta)\neq 0.$$

Hence

$$\dim_\theta \mathbb{V} \leq \dim_\theta \mathbb{W} \leq \dim T_{\mathbb{W},\theta} \leq n-k = \dim_\theta \mathbb{V}$$

where $T_{\mathbb{W},\theta}$ is the tangent space of $\mathbb{W}$ at θ. i) and ii) are proved. □

BIBLIOGRAPHY

[B] BROWNAWELL (W.D.). — Bounds for the degrees in the Nullstellensatz, *Ann. of Math.*, t. **126**, 1987, p. 577–591.

[M–W] MASSER (D. W.) and WÜSTHOLZ (G.). — Fields of large transcendence degree generated by values of elliptic functions, *Invent. Math.*, t. **72**, 1983, p. 407–464.

[N] NESTERENKO (JU. V.). — On algebraic independence of algebraic powers of algebraic numbers, *Math. USSR Sb.*, t. **51**, 1985, p. 9–32.

[P1] PHILIPPON (P.). — Variétés abéliennes et indépendance algébrique *I*, *Invent. Math.*, t. **70**, 1983, p. 289–318.

[P2] PHILIPPON (P.). — Critères pour l'indépendance algébrique, *Inst. Hautes Études Sci. Publ. Math.*, t. **64**, 1986, p. 5–52.

[P3] PHILIPPON (P.). — Sur les mesures d'indépendance algébrique, [Séminaire de Théorie des Nombres, Paris 1983–84], *Birkhäuser Boston Inc.*, 1985.