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SCHWARTZ'S THEOREM ON MEAN PERIODIC VECTOR-VALUED FUNCTIONS

BY

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RÉSUMÉ. — Nous exposons une preuve plus simple du théorème de SCHWARTZ sur les fonctions continues à valeurs dans \mathbb{C}^N .

ABSTRACT. — A simpler proof to SCHWARTZ'S theorem for \mathbb{C}^N -valued continuous functions is provided.

1. Introduction and preliminaries

The theorem of L. SCHWARTZ on mean periodic functions of one variable states that every closed translation-invariant subspace of the space of continuous complex functions on \mathbb{R} is spanned by the polynomial-exponential functions it contains [4]. In [2, VII], J.-J. KELLEHER and B.-A. TAYLOR provide a characterization of all closed subspaces of \mathbb{C}^N -valued entire functions of exponential type which have polynomial growth on \mathbb{R} . By duality, their result generalizes Schwartz's Theorem to \mathbb{C}^N -valued continuous functions.

Our goal is to provide a simple and a direct proof to this result.

$C(\mathbb{R}, \mathbb{C}^N)$ denotes the space of continuous \mathbb{C}^N -valued functions on \mathbb{R} , with the topology of uniform convergence on compact sets. By a vector-valued polynomial exponential in $C(\mathbb{R}, \mathbb{C}^N)$, we mean a function of the form $e^{\lambda x} p(x)$, $x \in \mathbb{R}$, where $\lambda \in \mathbb{C}$ and p is a polynomial in $C(\mathbb{R}, \mathbb{C}^N)$.

THEOREM. — *Every translation-invariant closed subspace of $C(\mathbb{R}, \mathbb{C}^N)$ is spanned by the vector-valued polynomial-exponential functions it contains.*

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For the theory of mean-periodic complex functions, we refer the reader to [4], [1], [3]. We need the following notations and results.

Let $M_0(\mathbb{R})$ denote the space of complex Radon measures on \mathbb{R} having compact support. For $\mu \in M_0(\mathbb{R})$, the Laplace transform $\hat{\mu}$ of μ is the entire function defined by $\hat{\mu}(z) = \int e^{-zx} d\mu(x)$, $z \in \mathbb{C}$.

We remind that $f \in C(\mathbb{R})$ is mean periodic if $\mu * f = 0$ for some $\mu \in M_0(\mathbb{R})$, $\mu \neq 0$. For $f \in C(\mathbb{R})$, f^- is the function defined by $f^-(x) = f(x)$ if $x \leq 0$ and $f^-(x) = 0$ if $x > 0$. If f is mean-periodic, $\mu \in M_0(\mathbb{R})$, $\mu \neq 0$ and $\mu * f = 0$, then the function $\mu * f^-$ has compact support and the meromorphic function

$$F = (\mu * f^-)^\wedge / \hat{\mu},$$

which does not depend on the choice of μ , is defined to be the Laplace transform of f ([3]).

The heart of our proof is the fact that F is entire only if $f = 0$ (see [3, Theorem X]).

The dual of $C(\mathbb{R}, \mathbb{C}^N)$ is the space $M_0(\mathbb{R}, \mathbb{C}^N)$ of \mathbb{C}^N -valued Radon measures on \mathbb{R} having compact supports. One notices that $M_0(\mathbb{R})$ is an integral domain under the convolution product and $M_0(\mathbb{R}, \mathbb{C}^N)$ is a module over $M_0(\mathbb{R})$ with the coordinatewise convolution. We denote the duality by

$$\langle \mu, f \rangle = \sum_{j=1}^N (\mu_j * f_j)(0)$$

for $\mu = (\mu_j) \in M_0(\mathbb{R}, \mathbb{C}^N)$ and $f = (f_j) \in C(\mathbb{R}, \mathbb{C}^N)$. If f is a vector-valued polynomial-exponential with

$$f_j(x) = \sum_{\ell=0}^m \alpha_j^{(\ell)} x^\ell e^{\lambda x} \quad (1 \leq j \leq N),$$

we have

$$\langle \mu, f \rangle = \sum_{j=1}^N \sum_{\ell=0}^m \alpha_j^{(\ell)} \hat{\mu}_j^{(\ell)}(\lambda).$$

For any subset A of $C(\mathbb{R}, \mathbb{C}^N)$ let

$$A^\perp = \{ \mu \in M_0(\mathbb{R}, \mathbb{C}^N); \langle \mu, f \rangle = 0 \text{ for all } f \in A \}.$$

If V is a translation-invariant closed subspace of $C(\mathbb{R}, \mathbb{C}^N)$, $\text{Sp}(V)$ denotes the set of all vector-valued polynomial-exponentials that belong to V .

By duality, V is spanned by $\text{Sp}(V)$ if and only if $\text{Sp}(V)^\perp \subset V^\perp$. Since V is translation-invariant, V^\perp is a submodule of $M_0(\mathbb{R}, \mathbb{C}^N)$ and $\mu = (\mu_j) \in V^\perp$ if and only if

$$\sum_{j=1}^N \mu_j * f_j = 0 \quad \text{for all } f = (f_j) \in V.$$

2. Main result

In this section, V denotes a given translation-invariant closed subspace of $C(\mathbb{R}, \mathbb{C}^N)$. We have to prove $\langle \mu, f \rangle = 0$ for any $\mu \in \text{Sp}(V)^\perp$ and $f \in V$. We need some more notation and three lemmas.

Let $0 \leq r \leq N$ be the rank of V^\perp as a module over $M_0(\mathbb{R})$. That means r is the greatest integer for which there exists a system $(\sigma_\ell)_{1 \leq \ell \leq r}$ where $\sigma_\ell = (\sigma_{\ell,j})_{1 \leq j \leq N} \in V^\perp$ for $1 \leq \ell \leq r$ and with a non-zero determinant of order r . We shall suppose given such a system with, say,

$$\rho = \det(\sigma_{\ell,j}; 1 \leq \ell, j \leq r) \neq 0.$$

One notices that $\hat{\rho}$ is the non identically zero entire function given by

$$\hat{\rho}(\lambda) = \det(\hat{\rho}_{\ell,j}(\lambda); 1 \leq \ell, j \leq r), \quad \lambda \in \mathbb{C}.$$

If $r = 0$, i.e. $V^\perp = \{0\}$, we take for ρ the Dirac measure at 0 and $\hat{\rho}(\lambda) = 1, \lambda \in \mathbb{C}$.

For $\mu = (\mu_j) \in M_0(\mathbb{R}, \mathbb{C}^N)$ let

$$\Delta_j(\mu) = \det \begin{vmatrix} \mu_1 & \dots & \mu_r & \mu_j \\ \sigma_{1,1} & \dots & \sigma_{1,r} & \sigma_{1,j} \\ \vdots & \ddots & \vdots & \vdots \\ \sigma_{r,1} & \dots & \sigma_{r,r} & \sigma_{r,j} \end{vmatrix} \quad (\text{for } 1 \leq j \leq N)$$

and

$$\tau_\ell(\mu) = \det \begin{vmatrix} \sigma_{1,1} & \dots & \sigma_{1,r} \\ \vdots & \ddots & \vdots \\ \sigma_{\ell-1,1} & \dots & \sigma_{\ell-1,r} \\ \mu_1 & \dots & \mu_r \\ \sigma_{\ell+1,1} & \dots & \sigma_{\ell+1,r} \\ \vdots & \ddots & \vdots \\ \sigma_{r,1} & \dots & \sigma_{r,r} \end{vmatrix} \quad (\text{for } 1 \leq \ell \leq r).$$

From the definition of r , for any $\mu \in V^\perp$

$$(1) \quad \Delta_j(\mu) = 0 \quad (\text{for } 1 \leq j \leq N).$$

By expanding the $\Delta_j(\mu)$ along the last column, (1) is equivalent to

$$(2) \quad \rho * \mu_j = \sum_{\ell=1}^r \tau_\ell(\mu) * \sigma_{\ell,j} \quad (\text{for } 1 \leq j \leq N).$$

LEMMA 1. — *Let $\lambda \in \mathbb{C}$ such that $\hat{\rho}(\lambda) \neq 0$. For $\alpha = (\alpha_j) \in \mathbb{C}^N$, the vector-exponential $e^{\lambda x} \cdot \alpha$ belongs to V if and only if*

$$(3) \quad \sum_{j=1}^N \alpha_j \hat{\sigma}_{\ell,j}(\lambda) = 0 \quad 1 \leq \ell \leq r.$$

Proof. — Let $\alpha \in \mathbb{C}^N$. We have $e^{\lambda x} \cdot \alpha \in V$ if and only if, for every $\mu = (\mu_j) \in V^\perp$,

$$(4) \quad \langle \mu, e^{\lambda x} \cdot \alpha \rangle = \sum_{j=1}^N \alpha_j \hat{\mu}_j(\lambda) = 0.$$

This proves the “only if” part. Conversely, since $\hat{\rho}(\lambda) \neq 0$, (2) implies that for any $\mu \in V^\perp$ the equation in (4) is a linear combination of the equations (3).

LEMMA 2. — *Let $\mu \in M_0(\mathbb{R}, \mathbb{C}^N)$. If $\langle \mu, e^{\lambda x} \cdot \alpha \rangle = 0$ for all $\lambda \in \mathbb{C}$ such $\hat{\rho}(\lambda) \neq 0$ and $\alpha \in \mathbb{C}^N$ such that $e^{\lambda x} \cdot \alpha \in V$, then $\Delta_j(\mu) = 0$ for $1 \leq j \leq N$.*

Proof. — Let $\lambda \in \mathbb{C}$ with $\hat{\rho}(\lambda) \neq 0$. If μ satisfies the hypothesis, the solutions of (3) are solutions of (4), which implies that the determinants $\Delta_j(\mu) \hat{\mu}_j(\lambda)$ for $1 \leq j \leq N$ are equal to zero. Then, since $\hat{\rho}$ and the $\Delta_j(\mu) \hat{\mu}_j$ are entire functions and $\hat{\rho} \neq 0$, the $\Delta_j(\mu) \hat{\mu}_j$ are identically zero. Hence, $\Delta_j(\mu) = 0$ for $1 \leq j \leq N$.

Remark. — LEMMA 2 shows that any $\mu \in \text{Sp}(V)^\perp$ satisfies (1) and (2). If $r = 0$, $\Delta_j(\mu) = \mu_j$ for $1 \leq j \leq N$; hence $\text{Sp}(V)^\perp = \{0\}$ if $V^\perp = \{0\}$.

LEMMA 3. — *Let $\lambda \in \mathbb{C}$, $m \geq 0$ and $\mu \in \text{Sp}(V)^\perp$. There exists $\nu \in V^\perp$ such that*

$$\hat{\nu}_j^{(\ell)}(\lambda) = \hat{\mu}_j^{(\ell)}(\lambda) \quad (\text{for } 1 \leq j \leq N, 0 \leq \ell < m).$$

Proof. — Suppose the element $(\hat{\mu}_j^{(\ell)}(\lambda))_{1 \leq j < N, 0 < \ell < m}$ of \mathbb{C}^{Nm} does not belong to the subspace

$$M(\lambda, m) = \{(\hat{\nu}_j^{(\ell)}(\lambda))_{1 \leq j \leq N, 0 \leq \ell < m} ; \nu \in V^\perp\}.$$

Then there exists $(\alpha_j^{(\ell)})_{1 \leq j \leq N, 0 \leq \ell < m}$ such that

$$\sum_{j=1}^N \sum_{\ell=0}^{m-1} \alpha_j^{(\ell)} \hat{\nu}_j^{(\ell)}(\lambda) = 0 \quad \text{for } \nu \in V^\perp$$

and

$$\sum_{j=1}^N \sum_{\ell=0}^{m-1} \alpha_j^{(\ell)} \hat{\mu}_j^{(\ell)}(\lambda) \neq 0.$$

Then if

$$f_j(x) = \sum_{\ell=0}^{m-1} \alpha_j^{(\ell)} x^\ell \quad (\text{for } 1 \leq j \leq N),$$

the polynomial-exponential $f = (f_j)_{1 \leq j \leq N}$ satisfies

$$\langle \nu, f \rangle = 0 \quad (\text{for } \nu \in V^\perp),$$

therefore $f \in \text{Sp}(V)$, and

$$\langle \mu, f \rangle \neq 0,$$

and we have a contradiction, since $\mu \in \text{Sp}(V)^\perp$.

Proof of the THEOREM. — Let $\mu = (\mu_j) \in \text{Sp}(V)^\perp, f = (f_j) \in V$ and

$$g = \sum_{j=1}^N \mu_j * f_j.$$

We have to prove that $g = 0$. By LEMMA 2, $\Delta_j(\mu) = 0$ for $1 \leq j \leq N$ and μ verifies (2); therefore

$$\rho * \sum_{j=1}^N \mu_j * f_j = \sum_{\ell=1}^r (\tau_\ell(\mu) * \sum_{j=1}^N \sigma_{\ell,j} * f_j).$$

For $1 \leq \ell \leq r$, since $\sigma_\ell \in V^\perp$, we have $\sum_{j=1}^N \sigma_{\ell,j} * f_j = 0$. So

$$\rho * g = 0.$$

Hence g is mean-periodic and the Laplace transform G of g may be defined by

$$G = (\rho * g^-)^\wedge / \hat{\rho}.$$

By ([3, Theorem X]) it is enough to prove that G is entire.

If $[a, b]$ is any interval that contains the supports of the μ_j ($1 \leq j \leq N$), $\sum \mu_j * f_j^-(x)$ is equal to $g(x)$ for $x < a$ and 0 for $x > b$. Thus the function

$$s = g^- - \sum_{j=1}^N \mu_j * f_j^-$$

has compact support. For $1 \leq \ell \leq r$, let

$$h_\ell = \sum_{j=1}^N \sigma_{\ell,j} * f_j^-.$$

By the same argument, the functions h_ℓ have compact supports and, by (2),

$$\rho * \sum_{j=1}^N \mu_j * f_j^- = \sum_{\ell=1}^r \tau_\ell(\mu) * h_\ell.$$

So

$$\rho * g^- = \sum_{\ell=1}^r \tau_\ell(\mu) * h_\ell + \rho * s;$$

$$(5) \quad G = \frac{1}{\hat{\rho}} \sum_{\ell=1}^r \tau_\ell(\mu)^\wedge \cdot \hat{h}_\ell + \hat{s}.$$

The functions \hat{s} and \hat{h}_ℓ ($1 \leq \ell \leq r$) are entire, as Laplace transforms of compactly supported functions.

For any $\nu \in V^\perp$, since $\sum \nu_j * f_j = 0$, $\sum \nu_j * f_j^-$ has compact support, and it follows by (2) that the function

$$(6) \quad \frac{1}{\hat{\rho}} \sum_{\ell=1}^r \tau_\ell(\nu)^\wedge \cdot \hat{h}_\ell \quad \text{is entire.}$$

Let $\lambda \in \mathbb{C}$ and let m be the order of $\hat{\rho}$ at λ . By LEMMA 3, we can choose $\nu \in V^\perp$ so that $\hat{\nu}_j^{(k)}(\lambda) = \hat{\mu}_j^{(k)}(\lambda)$ for $1 \leq j \leq N$, $0 \leq k < m$. Then the functions $(\hat{\nu}_j - \hat{\mu}_j) / \hat{\rho}$ for $1 \leq j \leq N$ and the functions

$$\frac{1}{\hat{\rho}} (\tau_\ell(\nu)^\wedge - \tau_\ell(\mu)^\wedge) \quad (\text{for } 1 \leq \ell \leq r)$$

are analytic at λ . It follows from (5) and (6) that G is analytic at λ .

Since λ is arbitrary, G is entire. That completes the proof of the Theorem.

BIBLIOGRAPHY

- [1] KAHANE (J.-P.). — Sur les fonctions moyennes-périodiques bornées, *Ann. Inst. Fourier*, t. **7**, 1957, p. 292–315.
 - [2] KELLHER (J.-J.) and TAYLOR (B.-A.). — Closed ideals in locally convex algebras of analytic functions, *J. Reine Angew. Math.*, t. **255**, 1972, p. 190–209.
 - [3] MEYER (Y.). — *Algebraic Numbers and Harmonic Analysis (V-9)*. Amsterdam-London, North-Holland Publishing Co, 1972.
 - [4] SCHWARTZ (L.). — Théorie générale des fonctions moyenne-périodiques, *Ann. of Math.*, t. **48**, 1947, p. 857–928.
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