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<http://www.numdam.org/item?id=BSMF_1990__118_2_193_0>
RINGS OF DIFFERENTIAL OPERATORS OVER
RATIONAL AFFINE CURVES

BY
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1. Introduction

Let $X$ and $Y$ be irreducible algebraic curves over the complex numbers, $\mathbb{C}$. Let $D(X)$ and $D(Y)$ denote their ring of differential operators, respectively. (For definition see [9]). This paper is motivated by the following open question.† Does $D(X) \cong D(Y)$ imply that $X \cong Y$? Let $\tilde{X}$ denote

(*) Supported by a grant from the National Science Foundation. Texte reçu le 29 juin 1989, révisé le 11 mai 1990.
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† G. LETZTER has now found nonisomorphic curves $X$ and $Y$ with isomorphic rings of differential operators (see [4]).

BULLETIN DE LA SOCIÉTÉ MATHEMATIQUE DE FRANCE 0037–9484/1990/193/$ 5.00
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the normalization of $X$. MAKAR–LIMANOV [5] shows that the set of ad-nilpotent elements $N(X)$ is exactly $O(X)$ whenever $O(X)$ is not a subring of a polynomial ring in one variable over $C$. He thus answers the question affirmatively for these curves. Let $\mathbb{A}^1$ denote the affine line. PERKINS [8] extends this result showing that $D(X) \cong D(Y)$ implies $X \cong Y$ whenever $\tilde{X} \neq \mathbb{A}^1$, or $\tilde{X} = \mathbb{A}^1$ but the normalization map $\pi : \tilde{X} \to X$ is not injective. Thus, in the paper, we are interested in curves $X$ such that $\tilde{X} \cong \mathbb{A}^1$ and $\pi : \tilde{X} \to X$ is injective. STAFFORD [10] shows the conjecture holds the following two examples of such curves: when $X$ is the affine line $\mathbb{A}^1$, or when $X$ is the cubic cusp $y^2 = x^3$.

For the remainder of the paper, assume that $X$ is a curve such that its normalization is isomorphic to the affine line $\mathbb{A}^1$ with an injective normalization map. We may therefore assume that the coordinate ring of $X$, denoted $O(X)$, is a subring of a polynomial ring in one variable $C[x]$ such that the integral closure of $O(X)$, written $\widetilde{O(X)}$, is equal to $C[x]$. Furthermore $D(X)$ is a subring of $C(x)[\partial]$ where $[\partial, x] = 1$. Here $\partial$ is just $\partial/\partial x$ and the element $f_n(x)\partial^n + \cdots + f_0(x)$ of $D(X)$ sends $g(x) \in O(X)$ to $f_n(x)g^{(n)}(x) + \cdots + f_0(x)g(x)$ where $g^{(n)}(x)$ denotes the $n$th derivative of $g(x)$.

PERKINS studies rings that satisfy these conditions in [8]. He shows that in many cases, $D(X)$ contains maximal commutative ad-nilpotent subalgebras not isomorphic to $O(X)$. Thus, for these curves, the set $N(X)$ of ad-nilpotent elements does not determine $O(X)$.

In this paper, we obtain an invariant for $D(X)$ and a nice description of the maximal ad-nilpotent subalgebras of $D(X)$. Set $T = C(x)[\partial]$ and set $\partial$-deg $w = n$ where $w = f_n(x)\partial^n + \cdots + f_0(x)$ is an element of $T$. Define a filtration on $T$ by $T_i = \{w \in T \mid \partial$-deg $w \leq i\}$ and hence on any subring $R$ of $T$ by $R_i = R \cap T_i$. (Note that this is the same filtration on $D(X)$ as the one defined by the order of the differential operator.) We may form the associated graded ring $\partial$-gr $R = \bigoplus R_i/R_{i-1}$. We define $\text{codim} R$ to be equal to $\dim_C \partial$-gr $C[x, \partial]/\partial$-gr $R$ for those subrings $R$ of $T$ such that $\partial$-gr $R \subset \partial$-gr $C[x, \partial]$.

Now assume that both $X$ and $Y$ are affine curves with normalization equal to the affine line and injective normalization map. By [9], both $\partial$-gr $D(X)$ and $\partial$-gr $D(Y)$ are subrings of $\partial$-gr $C[x, \partial]$ and $\text{codim} D(X)$ and $\text{codim} D(Y)$ are finite numbers.

Our main results are:

**Theorem.** — Suppose that $B$ is a maximal ad-nilpotent subalgebra of $D(X)$. Then there exists elements $x'$ and $\partial'$ in the quotient field of
C(x)[\partial] such that [\partial', x'] = 1, D(X) is a subring of C(x')[\partial'], D(X) \cap C(x') = B, and the integral closure of B is C[x']. Furthermore, \partial'-gr D(X) is a subring of \partial'-gr C[x', \partial'] and

\dim_X \partial'-gr C[x', \partial']/\partial'-gr D(X) = \text{codim} D(X).

**Corollary.** — If D(X) \cong D(Y), then \text{codim} D(X) = \text{codim} D(Y).

This result permits one to distinguish many rings of differential operators. For example, set O(X_n) = C + \sum C^k. Then it will follow from the **Corollary**, that D(X_n) \cong D(X_m) implies that n = m.

2. Graded Algebras of D(X)

In this section, \alpha and \beta are nonnegative real numbers with \alpha + \beta > 0. Define valuations V_{\alpha, \beta} on C(x)[\partial] as follows. Set

V_{\alpha, \beta}\left(w_0(x)\partial^n + w_{n-1}(x)\partial^{n-1} + \cdots + w_0(x)\right)

equal to \max\{\alpha d_m + \beta m \mid n \geq m \geq 0\} where d_m = \text{deg}(w_n(x)).

This extends the notion of valuations introduced by Dixmier in [2] for the Weyl algebra. For each valuation V_{\alpha, \beta} we may define a filtration of C(x)[\partial], and hence on any subring R of C(x)[\partial] as follows. Recall that T = C(x)[\partial]. Set T_i = \{z \in T \mid V_{\alpha, \beta}(z) \leq i\} and R_i = R \cap T_i. We may then define the associated graded algebra gr_{\alpha, \beta} R = \bigoplus R_i/R_{i-1}.

Now the commutator [x^i\partial^j, x^k\partial^\ell] = (kj - il)x^{i+k-1}\partial^{j+\ell-1} + \text{terms with x-degree less than } i + k - 1 \text{ and } \partial\text{-degree less than } j + \ell - 1. \text{ Therefore } V_{\alpha, \beta}([x^i\partial^j, x^k\partial^\ell]) < \alpha(i + k) + \beta(\ell + j). \text{ It follows that } gr_{\alpha, \beta}(C(x)[\partial]) \text{ is a commutative algebra.}

Note that when \alpha = 0 and \beta is positive, then the filtration defined by V_{0, \beta} on D(X) is the same filtration on D(X) as the one defined by \partial\text{-deg in the introduction. We will write } \partial\text{-gr } D(X) \text{ for } gr_{0, \beta} D(X) \text{ and } \partial\text{-deg for } V_{0, \beta}. \text{ Similarly, when } \beta = 0 \text{ and } \alpha \text{ is positive the graded algebra determined by } V_{\alpha, 0} \text{ is the same as } x\text{-gr } R \text{ determined by } x\text{-deg defined in [8].}

Set gr_{\alpha, \beta} x = x \text{ and } gr_{\alpha, \beta} \partial = y. \text{ Since } D(\tilde{X}) \text{ is just the first Weyl algebra, } A_1, \text{ we have that } \partial\text{-gr } D(\tilde{X}) = C[x, y] \text{ where } \partial\text{-gr } x = x \text{ and } \partial\text{-gr } \partial = y. \text{ By [9, Proposition 3.11], it follows that } \partial\text{-gr } D(X) \text{ is a subring of } C[x, y] \text{ and by [8, Lemma 2.3], } x\text{-gr } D(X) \text{ is also a subring of } C[x, y]. \text{ In the following lemma, we extend this to other gradings.}

**Lemma 2.1.** — Let R be a subring of C(x)[\partial] such that \partial\text{-gr } R \subset C[x, y]. Then the graded algebra gr_{\alpha, \beta} R is a subring of C[x, y].
Proof. — If $\alpha = 0$ then $\text{gr}_{\alpha, \beta} R = \partial - \text{gr} R$. So we may assume that $\alpha$ is positive. Let $w$ be a typical element of $D(X)$. Write $w = g_m(x) \partial^m + \cdots + g_0(x)$ where $g_i(x) \in C(x)$ for $0 \leq i \leq m$. Set degree of $g_i(x)$ equal to $d_i$ for $0 \leq i \leq m$. Since $\partial - \text{gr} R \subset C[x, y]$, it follows that $g_m(x) \subset C[x]$ and thus $d_m \geq 0$. Set $N = V_{\alpha, \beta}(w)$. By the definition of $V_{\alpha, \beta}$, it follows that $N = \max \{d_i \alpha + i \beta \mid 0 \leq i \leq m \}$. Hence $\text{gr}_{\alpha, \beta}(w) = \sum_{0 \leq s \leq m} \gamma_s x^{d_s} y^s$ where $\gamma_s = 0$ if $V_{\alpha, \beta}(x^{d} \partial^s) < N$, and $\gamma_s x^{d_s}$ is the leading term of $g_s(x)$ if $V_{\alpha, \beta}(x^{d} \partial^s) = N$. We need to show that whenever $\gamma_s \neq 0$, we have $x^{d_s} y^s \in C[x, y]$. In particular, since $0 \leq s \leq m$, we need to show that $d_s \geq 0$ whenever $\gamma_s \neq 0$. Now $N = V_{\alpha, \beta}(w) = V_{\alpha, \beta}(g_m(x) \partial^m) = d_m \alpha + m \beta$. Hence $d_s \alpha + s \beta \geq d_m \alpha + m \beta$. Recall that $m \geq s, d_m \geq 0$, and that $\alpha$ is positive. It follows that $d_s \geq d_m \geq 0$. The lemma now follows.

Define a linear map $\phi : C(x)[\partial] \to C[x, \partial]$ as follows. Suppose that $w = g_m(x) \partial^m + \cdots + g_0(x)$ is an element of $C(x)[\partial]$. For each $i$ such that $1 \leq i \leq m$, there exists a unique polynomial $f_i(x)$ such that $\deg(g_i(x) - f_i(x)) < 0$. Set

$$\phi(w) = f_m(x) \partial^m + \cdots + f_0(x).$$

Now consider two rational functions $g_1(x)$ and $g_2(x)$ such that $\phi(g_1(x)) = f_1(x)$ and $\phi(g_2(x)) = f_2(x)$. Then clearly

$$\deg\left(\lambda_1 g_1(x) + \lambda_2 g_2(x) - (\lambda_1 f_1(x) + \lambda_2 f_2(x))\right) < 0 \quad \text{and} \quad \phi(\lambda_1 g_1(x) + \lambda_2 g_2(x)) = \lambda_1 f_1(x) + \lambda_2 f_2(x).$$

It follows that $\phi$ is a well defined linear map from $C(x)[\partial]$ to $C[x, \partial]$.

Corollary 2.2. — Let $R$ be a subring of $C(x)[\partial]$ such that $\partial - \text{gr} R \subset C[x, y]$. If $w$ is an element of $R$, then $\text{gr}_{\alpha, \beta} \phi(w) = \text{gr}_{\alpha, \beta}(w)$.

Proof. — This is clear since $\text{gr}_{\alpha, \beta}(w - \phi(w))$ does not contain any monomials $x^{d_s} y^s$ with $d_s \geq 0$.

Remark 2.3. — Note that $\phi(R)$ is a linear subspace of the first Weyl algebra $A_1 = C[x, \partial]$, but, generally speaking, is not a subalgebra. Nevertheless $\alpha, \beta$ gradings are defined on $\phi(R)$ and $\text{gr}_{\alpha, \beta} \phi(R) = \text{gr}_{\alpha, \beta} R$. Now

$$\dim C[x, y]/\partial - \text{gr} D(X) < \infty \quad ([9, \text{3.12}]) \quad \text{and} \quad \dim C[x, y]/x - \text{gr} D(X) < \infty \quad ([8, \text{Lemma 2.5}])$$

In the next proposition, we will show that these two finite numbers are equal. We will later show that this codimension is an invariant for $D(X)$. 

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Proposition 2.4. — Suppose that $R$ is a subring of $\mathbb{C}(x)[\partial]$ such that 
$\partial$-$\text{gr } R \subseteq \mathbb{C}[x,y]$ and $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial$-$\text{gr } R < \infty$. Then $\text{gr}_{\alpha,\beta} R$ is a subring of $\mathbb{C}[x,y]$ and $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\text{gr}_{\alpha,\beta} R = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial$-$\text{gr } R$.

Using Corollary 2.2 and Remark 2.3, we may replace $R$ by $\phi(R)$ and prove the following.

Proposition 2.4'. — Suppose that $R'$ is a linear subspace of the Weyl algebra $\mathbb{C}[x,\partial]$ and that $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial$-$\text{gr } R' < \infty$. Then $\text{gr}_{\alpha,\beta} R'$ is a linear subspace of $\mathbb{C}[x,y]$ and $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\text{gr}_{\alpha,\beta} R' = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\partial$-$\text{gr } R'$.

Before proving Proposition 2.4', we need some additional notation and lemmas. Set, for $i \geq 0$,

$$E_i = \mathbb{C}[x] + \mathbb{C}[x]y + \cdots + \mathbb{C}[x]y^i$$
and
$$B_i = \{ w \in R' | \partial$-$\text{gr } w \in E_i \}.$$

Note that $\bigcup_{i \geq 0} B_i = R'$. Set $E = \bigcup_{i \geq 0} E_i = \mathbb{C}[x,y]$.

In Proposition 2.4', we assume that $\dim_{\mathbb{C}} E/\partial$-$\text{gr } R' < \infty$. Since $\partial$-$\text{gr } w \in E_i$ if and only if $w \in B_i$ for any $w \in R'$, it follows that $\dim_{\mathbb{C}} E_i/\partial$-$\text{gr } B_i < \infty$ for all $i \geq 0$, and that there exists an $N > 0$ such that $\dim_{\mathbb{C}} E_i/\partial$-$\text{gr } B_i = \dim_{\mathbb{C}} E/\partial$-$\text{gr } R'$ for all $i \geq N$. Hence for each $i \geq 0$, there exists an integer $M_i \geq -1$ such that for each $m > M_i$ there exists a monic polynomial $p_{i,m}(x)$ of degree $m$ in $\mathbb{C}[x]$ such that $p_{i,m}(x)y^i$ is an element of $\partial$-$\text{gr } B_i$. Furthermore, for $i \geq N$, we may assume that $M_i = -1$.

We have the following lemmas.

Lemma 2.5

Suppose that $R'$ satisfies the conditions of Proposition 2.4'. Suppose that $w = (ax^d + f_{i+1}(x))\partial^{i+1} + \cdots + f_0(x)$ is an element of $B_{i+1}$ where $\alpha \in \mathbb{C} \setminus \{0\}$ and $\deg f_{i+1}(x) < d$. Then there exists a $w' \in B_{i+1}$ such that $w' = (ax^d + g_{i+1}(x))\partial^{i+1} + g_i(x)\partial^i + \cdots + g_0(x)$ and $\deg g_k(x) \leq M_k$ for each $k$ such that $i + 1 \geq k \geq 0$.

Proof. — Let us use the following induction. Set $w_{-1} = w$. Suppose that

$$w_k = (ax^d + g_{i+1}(x))\partial^{i+1} + \cdots + g_{i-k}(x)\partial^{i-k} + f_{i-k-1}(x)\partial^{i-k-1} + \cdots + f_0(x),$$

where $\deg g_j(x) \leq M_j$, is defined. There exists $b \in B_{i-k-1}$ such that $\partial$-$\text{gr } b = (f_{i-k-1} - g_{i-k-1})y^{i-k-1}$ where $\deg g_{i-k-1} \leq M_{i-k-1}$ by the
paragraph preceding the lemma. So we can define $w_{k+1}$ as $w_k - b$, and $w'$ as $w_i$.

Let $P_i$ be the set of positive integers $m$ such that there exists a nonzero polynomial $q_{i,m}(x)$ of degree $m$ in $\mathbb{C}[x]$ with $q_{i,m}(x)y^i \in \partial\text{-gr} \mathcal{R}'$. Note that if $n$ is an integer such that $n > M_i$, then $n \in P_i$. By Lemma 2.5, it now follows that for each $m \in P_i$ there exists a monic polynomial $p_{i,m}(x)$ of degree $m$ in $\mathbb{C}[x]$ such that $b_{i,m} = p_{i,m}(x)\partial^i + g_{i-1}(x)\partial^{i-1} + \cdots + g_0(x)$ is an element of $B_i$ with $\deg g_k(x) \leq M_k$ for $i - 1 \geq k \geq 0$. Furthermore, for $i \geq N$, we may assume that $p_{i,m}(x) = x^m$. Note that the set

$$\{b_{i,m} \mid i \geq 0 \text{ and } m \in P_i\}$$

forms a basis for $R'$ over $\mathbb{C}$, and

$$\{p_{i,m}(x)y^i \mid i \geq 0 \text{ and } m \in P_i\}$$

forms a basis for $\partial\text{-gr} R'$ over $\mathbb{C}$. Thus if $w \in R'$, with $\partial\text{-gr} w = f(x)y^i$, then for $i > k \geq 0$, there exist $f_k(x) \in \mathbb{C}[x]$ with $\deg f_k(x) \leq M_k$, such that $f(x)\partial^i + f_{i-1}(x)\partial^{i-1} + \cdots + f_0(x)$ is an element of $R'$.

Set $M = \max\{M_k \mid N > k \geq 0\}$. Then we may assume that $b_{i,m} = p_{i,m}(x)\partial^i + w_{i,m}$ with $\deg w_{i,m} < \min(i, N)$ and $x\deg w_{i,m} \leq M$.

**Lemma 2.6**

Assume that $R'$ satisfies the conditions of Proposition 2.4'. For each $m \geq 0$, there exists a positive integer $S_m$ such that for all $i \geq S_m$, there is an element $c_{i,m}$ in $R'$ of the form $p_{i,m}(x)\partial^i + t_{i,m}$ with $\deg p_{i,m}(x) = m$ and $\partial\deg t_{i,m} < i$ and $x\deg t_{i,m} \leq m$. If $m > M$ we may set $S_m = 0$.

**Proof.** — If $m > M$, then we may take $c_{i,m} = b_{i,m}$. So we may assume that $m \leq M$. Consider the subset $\{b_{i,m} = p_{i,m}(x)\partial^i + w_{i,m} \mid i \geq 0\}$ of $R'$. Let $E_{M,N} = \{r \in E \mid x\deg r \leq M \text{ and } y\deg r \leq N\}$, and let $V$ be the vector space spanned by $\{w_{i,m} \mid i \geq 0\}$. Set $W = \{x\text{-gr} w \mid w \in V\} \cap E$. Note that $W$ is a subspace of $E_{M,N}$. It is clear that $E_{M,N}$ and hence $W$ is a finite dimensional subspace of $E$. So there is an $S_m > 0$ such that $W$ is spanned by a subset of

$$\{x\text{-gr} w \mid w \text{ is in the span of the set } \{w_{i,m} \mid S_m \geq i \geq 0\}\}.$$

It follows that for $i > S_m$, there exist complex numbers $\alpha_{k,m}$ for $S_m \geq k \geq 0$ such that

$$x\deg\left(w_{i,m} - \sum_{k=0}^{S_m} \alpha_{k,m} w_{k,m}\right) < 0$$

and

$$\partial\deg\left(w_{i,m} - \sum_{k=0}^{S_m} \alpha_{k,m} w_{k,m}\right) < 0.$$
We may now set $c_{i,m} = b_{i,m} - \sum_{k=0}^{S_m} \alpha_{k,m} b_{k,m}$.

The next corollary follows immediately from Lemma 2.6.

**Corollary 2.7.** — We have $\dim_{\mathbb{C}} C[x,y]/x\cdot \text{gr } R' < \infty$.

**Lemma 2.8**

Let $W$ be a linear subspace of $A_1$. Then $\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} \text{gr}_{\alpha,\beta} W$.

**Proof.** — Suppose that $W$ is a vector space and that

$$\{W_i \mid i \text{ is an integer} \}$$

is a filtration for $W$ such that the vector spaces $W_i = 0$ for $i < 0$ and $W = \bigcup_{i \geq 0} W_i$. Then clearly $W$ and $\bigoplus W_i/W_{i-1}$ are isomorphic as vector spaces. Hence $\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} \bigoplus W_i/W_{i-1}$. In particular if $W$ is a linear subspace of $A_1$, then $\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} \text{gr}_{\alpha,\beta} W$.

We are now ready to prove Proposition 2.4'.

**Proof of Proposition 2.4'.** — Note that $R'$ is a linear subspace of $\mathbb{C}[x,\partial]$. Hence, it follows from the definition of $\text{gr}_{\alpha,\beta} R'$ that $\text{gr}_{\alpha,\beta} R'$ is a linear subspace of $\text{gr}_{\alpha,\beta} \mathbb{C}[x,\partial]$. Thus we only need to prove the statement about dimensions.

Set $V_n = \{x^i y^j \mid \alpha i + \beta j \leq n \}$ for all $n \geq 0$. Note that each $V_n$ has finite dimension and that $\bigcup_{n \geq 0} V_n = \mathbb{C}[x,y]$. Set $W_n = \{w \in R' \mid \text{gr}_{\alpha,\beta} w \in V_n \}$.

Since $\text{gr}_{\alpha,\beta} R' \subseteq \mathbb{C}[x,y]$, we have that $\bigcup_{n \geq 0} W_n = R'$. Suppose that $w \in W_n$. We can write $w = p(x)\partial^k + c$ for some $p(x) \in \mathbb{C}[x]$ and $k \geq 0$ such that $\partial\cdot \deg(c) < k$ and $\alpha \deg p(x) + \beta k \leq n$. So $\partial\cdot \text{gr } w = p(x)y^k$ is also in $V_n$. Thus $\partial\cdot \text{gr } W_n \subseteq V_n$ for all $n \geq 0$.

Set $L = \alpha M + \beta N$. We will show that $\partial\cdot \text{gr } W_n = \partial\cdot \text{gr } R' \cap V_n$ for all $n \geq L$. Since $\partial\cdot \text{gr } W_n \subseteq V_n$, it is clear that $\partial\cdot \text{gr } W_n \subseteq \partial\cdot \text{gr } R' \cap V_n$. Suppose $\partial\cdot \text{gr } w = p(x)y^j$ is an element of $\partial\cdot \text{gr } R' \cap V_n$. So $\alpha \deg p(x) + \beta j \leq n$. By Lemma 2.5, we may find in $R'$ an element $w = p(x)\partial^j + g_N(x)\partial^N + \cdots + g_0(x)$ and $\deg g_k(x) \leq M_k$ for each $k$ such that $N \geq k \geq 0$. Now

$$V_{\alpha,\beta}(g_N(x)\partial^N + \cdots + g_0(x)) \leq \alpha M + \beta N = L.$$ 

Hence $V_{\alpha,\beta}(w) \leq \max\{\alpha \deg p(x) + \beta j, L\}$. If $\alpha \deg p(x) + \beta j > L$, then $V_{\alpha,\beta}(w) = \alpha \deg p(x) + \beta j \leq n$ since $p(x)y^j$ is an element of $V_n$. Hence $w \in W_n$. If $\alpha \deg p(x) + \beta j \leq L$, then $V_{\alpha,\beta}(w) \leq L \leq n$, hence again $w \in W_n$. Therefore $\partial\cdot \text{gr } W_n = \partial\cdot \text{gr } R' \cap V_n$ for all $n \geq L$.

Since $W_n$ is a linear subspace of $\mathbb{C}[x,\partial]$, by Lemma 2.8, we have that

$$\dim_{\mathbb{C}} W_n = \dim_{\mathbb{C}} \partial\cdot \text{gr } W_n$$

and

$$\dim_{\mathbb{C}} W_n = \dim_{\mathbb{C}} \text{gr}_{\alpha,\beta} W_n.$$
Furthermore, for all \( n \geq L \), we have that \( \dim_{\mathbb{C}} \partial \text{-} \text{gr} \ R' \cap V_n = \dim_{\mathbb{C}} W_n = \dim_{\mathbb{C}} \text{gr}_{\alpha, \beta} W_n \). Since \( \dim_{\mathbb{C}} V_n \) is finite, it follows that \( \dim_{\mathbb{C}} V_n / \partial \text{-} \text{gr} \ R' \cap V_n = \dim_{\mathbb{C}} V_n / \text{gr}_{\alpha, \beta} W_n \) for all \( n \geq L \). Clearly

\[
\lim_{n \to \infty} \dim_{\mathbb{C}} V_n / \partial \text{-} \text{gr} \ R' \cap V_n = \dim_{\mathbb{C}} W_n = \dim_{\mathbb{C}} \text{gr}_{\alpha, \beta} W_n.
\]

Therefore \( \dim_{\mathbb{C}} C[x, y] / \partial \text{-} \text{gr} \ R' = \dim_{\mathbb{C}} C[x, y] / \text{gr}_{\alpha, \beta} R' \).

By Corollary 2.7, we have that \( \dim_{\mathbb{C}} C[x, y] / x \text{-} \text{gr} \ R' < \infty \). So we may apply the first part of the proof with \( x \) replaced by \( \partial \) and vice versa to show that \( \dim_{\mathbb{C}} C[x, y] / x \text{-} \text{gr} \ R' = \dim_{\mathbb{C}} C[x, y] / \text{gr}_{\alpha, \beta} R' \) which completes the proof of Proposition 2.4 and therefore of Proposition 2.4.

Recall that codim \( R \) is defined to be \( \dim_{\mathbb{C}} C[x, y] / \partial \text{-} \text{gr} R \). Proposition 2.4 implies that codim \( R = \dim_{\mathbb{C}} C[x, y] / \text{gr}_{\alpha, \beta} R \) for any two nonnegative not both zero real numbers \( \alpha \) and \( \beta \). We will eventually show that codim \( R \) is an invariant of \( R \).

### 3. Ad-Nilpotent subalgebras of \( D(X) \)

Suppose that \( D(X) \cong D(Y) \). Then \( D(X) \) contains a maximal commutative ad-nilpotent subalgebra isomorphic to \( O(Y) \). So it is interesting to understand the maximal commutative ad-nilpotent subalgebras of \( D(X) \). Let \( D \) denote the quotient field of the first Weyl algebra, \( A_1 \). In this section, we show that if \( B \) is a maximal commutative ad-nilpotent subalgebra of \( D(X) \), then there exists an element \( b \in D \) such that \( B \) is a subring of \( \mathbb{C}[b] \).

**Lemma 3.1.** — Suppose that \( R \) is a subalgebra of \( D \) so that the quotient ring of \( R \) is \( D \), and that \( u \) is an element of \( D - \mathbb{C} \) that acts ad-nilpotently on \( R \). Then there exists a \( v \in D \) such that \([u, v] = 1\). Furthermore, for any \( v \in D \) such that \([u, v] = 1\), we have \( R \subseteq C_D(u)[v] \) where \( C_D(u) \) denotes the centralizer of \( u \) in \( D \).

**Proof.** — Define \( R_0 = C_D(u) \) and \( R_i = \{ z \in D \mid [z, u] \in R_{i-1} \} \).

Now \( R \subseteq \bigcup_{i \geq 0} R_i \) since \( u \) acts ad-nilpotently on \( R \). Let \( a \) be a nonzero element of \( R_1 - R_0 \). (Note that \( R_1 - R_0 \) is nonempty since \( u \notin \mathbb{C} \) and \( \mathbb{C} \) is the center of \( R \).) Then \( 0 \neq [u, a] = b \in R_0 \). So \([u, b^{-1}a] = b^{-1}[u, a] = 1\). Set \( v = b^{-1}a \).

Clearly \( R_0 \subseteq C_D(u) \). We will show by induction on \( i \) that

\[
R_i \subseteq C_D(u)v^i + \cdots + C_D(u) \quad \text{for all} \ i \geq 0.
\]
Assume that $R_{i-1} \subset C_D(u) v_i^{-1} + \cdots + C_D(u)$ and choose $z \in R_i$. Then $[z, u] \in R_{i-1}$, hence $[z, u] = \sum_{0 \leq m \leq i-1} f_m(u) v^m$. Then

$$[z - \sum_{0 \leq m \leq i-1} f_m(u) m!^{-1}] = 0.$$ 

Hence $z - \sum_{0 \leq m \leq i-1} f_m(u) m!^{-1} = 0 \in \mathbb{C}(u)$. Therefore

$$z \in C_D(u) v^i + \cdots + C_D(u).$$

We may define the graded algebra $v\text{-gr} C_D(u)[v]$ by setting $v\text{-gr} a = u_i w^i$ where $a = u_i v^i + \cdots + u_0$ is an element of $C_D(u)[v]$ with $u_k \in C_D(u)$ for $i \geq k \geq 0$.

We will show that $C_D(u)$ is in fact a rational function field in one variable.

The next lemma is well known. See for example [3, Corollary 3.2].

**Lemma 3.2.** — If $f \in D - \mathbb{C}$ then $C_D(f)$ is commutative.

**Lemma 3.3.** — If $u \in D$ acts ad-nilpotently on $R$, where $R$ is a subalgebra of $D$ such that the quotient ring of $R$ is $D$, then there exists $z \in D$ such that $C_D(u)$ is isomorphic to a rational function field $\mathbb{C}(z)$.

**Proof.** — Let us call an element $a \in D$ ad-nilpotent if it acts ad-nilpotently on some subalgebra $R(a)$ of $D$ such that the quotient ring of $R(a)$ is $D$. By Lemma 3.1, there exists an element $v \in D$ such that $[v, u] = 1$ and $D = C_D(u)(v)$.

We will first assume that there exists an ad-nilpotent element $a$ of $D$ with $v\text{-deg} a \neq 0$. Now for each element $c \in C_D(u)$, there exists elements $c_1 = c_1(c)$ and $c_2 = c_2(c)$ in $R(a)$ such that $c = c_1 c_2^{-1}$. It is clear that $v\text{-gr} a$ acts nilpotently by Poisson bracket action on $v\text{-gr} c_1$ and $v\text{-gr} c_2$. Let $v\text{-gr} a = a_0 w^n$, $v\text{-gr} c_1 = c_1 w^m$, and $v\text{-gr} c_2 = c_2 w^m$. (Since $c \in C_D(u)$, it is clear that $v\text{-deg} c_1 = v\text{-deg} c_2$.)

By the same arguments as in [5, Lemma 7], there exists an element $b$ in the algebraic closure of $C_D(u)$ such that $c_1 w^m = (a_0 w^n)^m/n p_1(b)$ and $c_2 w^m = (a_0 w^n)^m/n p_2(b)$ where $p_1(b)$ and $p_2(b)$ are polynomials.

Since $v\text{-deg} c = 0$, we have that $c = c_1 c_2^{-1} = c_1 c_2^{-1} = p_1(b)(p_2(b))^{-1}$. Therefore $C_D(u) \subset C(b)$. By Luroth’s theorem, $C_D(u)$ is isomorphic to a field of rational functions in one variable.

Now assume that $v\text{-deg} a = 0$ for all ad-nilpotent elements. Consider the standard generators $x$ and $\partial$ for $D$. These are ad-nilpotent elements of $D$ since they act ad-nilpotently on $\mathbb{C}[x, \partial]$. Therefore $1 = [\partial, x]$ has negative $v$-degree which is impossible.
4. Codim is an invariant of $D(X)$

In this section $R = D(X)$ for a curve $X$ satisfying the conditions of the introduction. Suppose that $u$ and $v$ are elements of $D$ with commutator $[v, u] = 1$ such that $D(X) \subset C(u)[v]$ and $v$-gr $D(X)$ is a subring of the polynomial ring in two generators, $u = v$-gr $u$ and $w = v$-gr $v$. We may define codim$_{u,v} D(X)$ as dim$_C C[u, w]/v$-gr $D(X)$. In this section, we will show that codim$_{u,v} D(X) = \text{codim} D(X)$. So codim does not depend on the embedding of $D(X)$ inside of $C(x)[\partial]$.

Note that $u$-gr $C[u, v]$ and $v$-gr $C[u, v]$ are isomorphic polynomial rings. We will identify these isomorphic rings and thus write $u$-gr $u = v$-gr $u = u$ and $u$-gr $v = v$-gr $v = w$.

**Lemma 4.1.** — Suppose that $R \subset C(u)[v] \subset D$, where $[v, u] = 1$, such that the quotient ring of $R$ is $D$, the graded algebra $v$-gr $R$ is a subset of $C[u, w]$, and codim$_{u,v} R$ is finite. Then there exist elements $u'$ and $v'$ of $D$ such that $u$-gr $v' = w$ and $u$-gr $u' = -u$, the commutator $[u', v']$ is 1, and the ring $R$ is a subring of $C(v')[u']$. Moreover, there is an isomorphism from $u'$-gr $C[u', v']$ to $u$-gr $C[u, v]$ which restricts to an isomorphism from the graded algebra $u'$-gr $R$ to $u$-gr $R$, and codim$_{u',v'} R = \text{codim}_{u,v} R$.

**Proof.** — Define subalgebras $R_i$ of $R$ for $i \geq 0$ as follows:

$$R_i = \{ z \in R \mid u\text{-deg}(z) \leq i \}.$$  

(The following argument is similar to [8, Theorem 2.7].) Now

$$u \text{-gr} [f(v)u^i, g(v)] = u \text{-gr} (-if(v)g'(v)u^{i-1}) \quad \text{for } i \geq 0.$$  

Also $u$-gr $R$ is a subset of $C[u, w]$ by Lemma 2.1. Hence, it is easy to see that $R_0$ is a maximal commutative ad-nilpotent subalgebra of $R$. Furthermore the map which sends $z$ to $u$-gr $z$ is an isomorphism of $R_0$ to $u$-gr $R_0 = u$-gr $R \cap C[w]$. By assumption, codim$_{u,v} R < \infty$, hence dim$_C C[w]/u$-gr $R_0 < \infty$. So the integral closure of $u$-gr $R_0$ is $C[w]$, and thus the integral closure of $R_0$ is $C[v']$ for some $v' \in D$ with $u$-gr $v' = w$ and $R_0 = R \cap C[v']$ for some $v' \in D$ with $u$-gr $v' = w$ and $R_0 = R \cap C[v']$. Note that $u$-gr $p(v') = p(w)$ for any polynomial $p(t) \in C[t]$.

By Lemma 3.3, $C_D(v')$ is a rational function field in one variable. Let us check that $C_D(v') = C(v')$. Let $f \in C_D(v')$. Then $u$-deg $f = 0$, because otherwise $[v', f] \neq 0$, and $u$-gr $f = r(w)$ where $r(w) \in C(w)$. Therefore $f = r(v') + f_1$ where $u$-deg $f_1 < 0$. But $f_1 \in C_D(u)$ and can not have a negative degree. Hence $f_1$ is 0. Now, according to Lemma 3.1, there exists a $u' \in D$ such that $[u', v'] = 1$ and $R \subset C(v')[w']$. 

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Suppose that \( u - \text{gr} u' = f(w)u \). Since \( u - \text{gr} v' = w \), we must have \( u - \text{gr} [u', v'] = -i f(w)u^{i-1} \) unless \( i = 0 \). If \( i = 0 \), then either \([u', v'] = 0\) or \( u - \text{deg}[u', v'] < -1 \). Since \([u', v'] = 1\), it follows that \( i \neq 0 \). Hence \(-i f(w)u^{i-1}\) must equal 1. Therefore \( i = 1 \) and \( f(w) = -1 \) and \( u - \text{gr} u' = -u \).

Suppose that \( z \) is an element of \( R \subset C(v')[u'] \). We may write \( z = f(v')(u'^j + e) \) where \( u'^-\text{deg} e < j \), and \( f(v') \) is a polynomial, and \( j \geq 0 \). Since \( u - \text{deg} v' = 0 \) and \( u - \text{deg} u' = 1 \), we must have that \( u - \text{deg} e < j \) and \( u - \text{gr} z = u - \text{gr} f(v')(u'^j) \). Since \( u - \text{gr} f(v') = f(w) \) and \( u - \text{gr} u' = -u \), it follows that \( u - \text{gr} z = f(w)(-u)^j \). Hence the isomorphism from \( u - \text{gr} C[u', v'] \) to \( u - \text{gr} C[u, v] \) which sends \( u - \text{gr} u' \) to \( u - \text{gr} u' = -u \) and \( u - \text{gr} v' \) to \( u - \text{gr} v' = w \) restricts to an isomorphism from \( u - \text{gr} R \) to \( u - \text{gr} R \). Since \( \text{codim}_{u,v} R \) is finite, by Proposition 2.4, we have that \( \text{codim}_{u,v} R = \dim C[u, w]/u - \text{gr} R \). It follow immediately that \( \text{codim}_{u,v} R = \text{codim}_{v', u} R \).

For the next three lemmas, assume that \( R \) is a subring of \( C(u)[v] \subset D \), where \( u \) and \( v \) are elements of \( D \) whose commutator is 1, and that \( v - \text{gr} R \subset C[u, w] \) with \( \text{codim}_{u,v} R < \infty \). Write \( R_0 \) for the ad-nilpotent subalgebra \( \{ z \in R \mid u - \text{gr} z = 0 \} \). We may define valuations \( V_{\alpha, \beta} \) and corresponding graded algebras on \( R \) as in Section 1 using \( u \) and \( v \) instead of \( x \) and \( \partial \). For example, \( V_{\alpha, \beta}(u^j v^j) = \alpha i + \beta j \).

**Lemma 4.2.** Suppose that \( r \) is an ad-nilpotent element of \( R \) that is not contained in \( C(u) \) and is not contained in \( R_0 \). Then there exist positive integers \( n \) and \( m \) and complex numbers \( \lambda \) and \( \gamma \) such that \( u - \text{gr} r = (\lambda u)^n \) and \( v - \text{gr} r = (\gamma w)^m \). Furthermore, \( V_{m,n}(r) = mn \).

**Proof.** Since \( r \) is not an element of \( C(u) \) and is not an element of \( R_0 \), it follows that \( u - \text{deg} r > 0 \) and \( v - \text{deg} r > 0 \). We will argue as in [2, Lemma 8.7]. We may write

\[
r = \sum_{i \geq 0, j \geq 0} \sigma_{i,j} u^i v^j + f_k(u)v^k + \cdots + f_0(u)
\]

where \( \text{deg} f_j(u) < 0 \) for \( k \geq j \geq 0 \). Clearly, \( v - \text{deg} r > k \). Let \( n \) be the smallest nonnegative integer such that \( \sigma_{j,0} = 0 \) for all \( j > n \). Let \( m \) be the smallest nonnegative integer such that \( \sigma_{0,k} = 0 \) for all \( k > m \). We claim that \( \sigma_{i,j} = 0 \) for all pairs \( i, j \) such that \( mi + nj > mn \).

Assume the claim is false. Then there exist positive real numbers \( \alpha \) and \( \beta \) and a pair of positive integers \( i \) and \( j \) with \( \sigma_{i,j} \neq 0 \), such that \( \text{gr}_{\alpha, \beta} r = \sigma_{i,j} u^i w^j \). Without loss of generality, \( \sigma_{i,j} = 1 \). First assume \( i \geq j \).
Now there exists a monic polynomial $p(t)$ such that $p(u) \in R$. Since both $\alpha$ and $\beta$ are positive, we have that $\text{gr}_{\alpha,\beta} p(u) = u^d$ where $d = \deg p(u)$. Note that $\text{gr}_{\alpha,\beta}[r, p(u)] = dj u^{i+1+d} w^{j-1}$. Suppose that
\[ \text{gr}_{\alpha,\beta} \text{ad}_r^k(p(u)) = \alpha_k u^{k(i-1)+d} w^{k(j-1)}. \]
Then
\[ \text{gr}_{\alpha,\beta} \text{ad}_r^{k+1}(p(u)) = \alpha_k (k(i-1)+d)j - i k(j-1)] u^{(k+1)(i-1)+d} w^{(k+1)(j-1)}. \]
Now $(k(i-1)+d)j - i k(j-1) = (i-j)k + dj > 0$ for all $k \geq 0$ since $i \geq j$. This contradicts the fact that $r$ is ad-nilpotent.

Now assume that $i < j$. Consider a nonconstant element $z \in R_0$. Recall that $R_0$ sits inside a polynomial algebra $C[v']$ where $v' \in D$ where $u-\text{gr} v' = w$. So $z = q(v')$ for some nonconstant polynomial $q(t)$. Since both $\alpha$ and $\beta$ are positive, it follows that $\text{gr}_{\alpha,\beta} z = w^k$ where $k = \deg q(t)$. The argument now follows as in the preceding paragraph.

We have shown that $\sigma_{i,j} = 0$ for all pairs of positive integers $i$ and $j$ such $mi + nj > nm$. In particular, $u-\text{gr} r = \sigma_{n,0} u^n$, and $v-\text{gr} r = \sigma_{0,m} w^m$, and $V_{m,n}(r) = mn.$

**Lemma 4.3.** — Suppose that $r$ is an ad-nilpotent element of $R$ that is not contained in $C(u)$ and is not contained in $R_0$. Set $n = u-\text{deg} r$ and $m = v-\text{deg} r$. Then one of the following two statements hold where $\lambda, \lambda', \gamma, \gamma'$ are elements of $C$, and $i$ is an integer such that $n \geq i \geq 0$.

1. If $n \geq m$, then $n$ is a multiple of $m$ and
   \[ \text{gr}_{m,n} r = ((\lambda u)^{n/m} + \gamma w)^m. \]

2. If $m > n$, then $m$ is a multiple of $n$ and
   \[ \text{gr}_{n,m} r = (\lambda u + (\gamma w)^{m/n})^n. \]

**Proof.** — By Lemma 4.2, both $n$ and $m$ are positive. So there exist nonzero complex numbers $\sigma_1$ and $\sigma_2$ such that $u-\text{gr} r = \sigma_1 u^n$ and $v-\text{gr} r = \sigma_2 w^m$. Now by Lemma 2.1, $\text{gr}_{m,n} R \subset C[u, w]$, and by Proposition 2.4, $\dim_C C[u, w]/\text{gr}_{m,n} R < \infty$. Hence we may apply the arguments of [2, Lemma 7.3] to the ad-nilpotent element $r$ of $R$.

In the next lemma, we will show that codim $R$ is independent of the choice of generator for $C(u)$.
LEMMA 4.4. — Suppose that $u_1$ and $v_1$ are elements of $D$ whose commutator is 1 such that $C(u) = C(u_1)$, the ring $R$ is a subring of $C(u_1)[v_1] \subset D$, and that $v_1\text{-gr} R \subset C[u_1, w_1]$ with $\text{codim}_{u_1, v_1} R < \infty$. Then $\text{codim}_{u, v} R = \text{codim}_{u_1, v_1} R$.

Proof. — Set $B = R \cap C(u) = R \cap C[u]$. Since $C(u_1) = C(u)$ and $v_1\text{-gr} R \subset C[u_1, w_1]$, we have that $B = R \cap C(u_1) = R \cap C[u_1]$. By assumption, both $\text{codim}_{u, v} R$ and $\text{codim}_{u_1, v_1} R$ are finite. Hence both $\dim_C C[u]/B$ and $\dim_C C[u_1]/B$ are finite. Therefore the integral closure of $B$ in $C(u)$ is $C[u]$ and is also $C[u_1]$. So $C[u] = C[u_1]$ and there exist integers $\alpha$ and $\beta$ such that $u = \alpha u_1 + \beta$. Since $[v_1, u_1] = 1$, we have that $[\alpha v - v_1, u] = 0$. So $v + g(u) = \alpha^{-1} v_1$ for some $g(u) \in C(u)$.

Set $v_2 = v + g(u)$. Note that $[v_2, u] = 1$ and $R \subset C(u)[v_2]$. Now $f(u)v^i = f(u)(v_2 - g(u))^i$, hence $v\text{-gr} R = v_2\text{-gr} R$ and $\text{codim}_{u, v} R = \text{codim}_{u_1, v_1} R$. Without loss of generality, we may assume that $v = v_2$ and that $v = \alpha^{-1} v_1$. The isomorphism of $C[u, w]$ to $C[u_1, w_1]$ which sends $u$ to $\alpha u_1$ and $w$ to $\alpha^{-1} w_1$ clearly induces an isomorphism from $v\text{-gr} R$ to $u\text{-gr} R$. The result now follows.

We are now ready to show that $\text{codim} D(X)$ is an invariant of $D(X)$.

THEOREM 4.5. — Suppose that $X$ is an affine curve such that the normalization of $X$ is the affine line, with the normalization map $\pi: \tilde{X} \to X$ injective. Then for any pair of elements $u$ and $v$ in $D$, such that $[v, u] = 1$, the ring $D(X)$ is a subring of $C(u)[v]$, and $v\text{-gr} D(X)$ is a subring of the polynomial ring with generators $v\text{-gr} u$ and $v\text{-gr} v$, we have that $\text{codim}_{u, v} D(X) = \text{codim} D(X)$.

Proof. — Now $D(X)$ is a subring of $C(x)[\partial]$ and $\text{codim} D(X) = \text{codim}_{x, 0} D(X)$. Assume that $u$ and $v$ are elements of $D$ such that $[v, u] = 1$, the ring $D(X)$ is a subring of $C(u)[v]$, and $v\text{-gr} D(X)$ is a subring of the polynomial ring $C[u, w]$ where $v\text{-gr} u = u$ and $v\text{-gr} v = w$. Let $r$ be a nonconstant ad-nilpotent element of $D(X)$ contained inside $C(u)$. Set $x\text{-deg} r = n$ and $\partial\text{-deg} r = m$. We will induct on $t = m + n$.

If $m = 0$, then $r$ is an element of $C(x)$ and the result now follows by LEMMA 4.4.

If $n = 0$, then $r$ is an element of $\{z \in D(X) \mid x\text{-deg} z = 0\}$, and the result follows from LEMMA 4.1 and LEMMA 4.4. Hence the theorem holds for $t = 0$.

So we may assume that both $n$ and $m$ are positive.

First assume that $n \geq m$. By LEMMA 4.3, $n$ is a multiple of $m$ and there exist elements $\lambda$, and $\gamma$ of $C$ such that $\text{gr}_{m, n} r = (\lambda x)^{n/m} + \gamma y)^m$. Hence

$$r = ((\lambda x)^{n/m} + \gamma \partial)^m + c$$
where $V_{m,n}(c) < mn$ and $x\text{-deg} c < n$ and $\partial\text{-deg} c < m$. Set $\partial_1 = \partial - (\gamma)^{-1}(\lambda x)^{n/m}$ and $x_1 = x$. Note that $((\lambda x)^{n/m} + \gamma \partial)^m = (\gamma \partial_1)^m$. Furthermore $(\partial)^i = (\partial_1 + (\gamma)^{-1}(\lambda x_1)^{n/m})^i$. It follows that $\partial_1\text{-deg} c < m$ and $\partial_1\text{-deg} r = m$. Also $x_1\text{-deg} c \leq (m - 1)n/m < n$. Since $r = (\gamma \partial_1)^m + c$, we have that $x_1\text{-deg} r < n$. By Lemma 4.4, $\text{codim}_{x_1,\partial_1} D(X) = \text{codim}_{x,\partial} D(X)$. Now $\partial_1\text{-deg} r + x_1\text{-deg} r < t$, hence the result now follows by induction for this case.

Now assume that $n < m$. By Lemma 4.1, there exist elements $x_1$ and $\partial_1$ in $D$ such that $D(X) \subset C(\partial_1)[x_1]$, $[x_1, \partial_1] = 1$, $x_1\text{-gr} \partial = x_1\text{-gr} \partial_1$, $x_1\text{-gr} R \cong x\text{-gr} R$, and $\text{codim}_{\partial_1,x_1} R = \text{codim}_{x,\partial} R$. It follows that $x_1\text{-deg} r = x\text{-deg} r = n$. If $\partial_1\text{-deg} r < m$, then the proof follows by induction.

Otherwise $\partial_1\text{-deg} r \geq m > n$ and we may apply the methods used above repeatedly to find elements $\partial_2 = \partial_1$ and $x_2 = x_1 + g(\partial_1)$ where $g(\partial_1) \in C(\partial_1)$ such that $x_2\text{-deg} r = n$ and $\partial_2\text{-deg} r < m$. The proof again follows by induction.

We are now able to obtain a nice description of the maximal ad-nilpotent subalgebras of $D(X)$.

**Corollary 4.6.** — Suppose that $X$ is an affine curve such that the normalization of $X$ is the affine line, with the normalization map $\pi : \tilde{X} \rightarrow X$ injective. Suppose that $B$ is a maximal ad-nilpotent subalgebra of $D(X)$. Then there exists an element $u$ in $D$ such that $B$ is a commutative finitely generated algebra with integral closure $C[u]$ and the centralizer of $B$ in $D(X)$ is the rational function field $C(u)$.

**Proof.** — By Lemma 3.3 and Lemma 3.4, there exists $u$ in $D$ such that $C_D(B) = C(u)$ and $B \subset C[u]$. By Lemma 3.1, there exists $v$ in $D$ such that $D(X) \subset C(u)[v]$. Recall that the set of ad-nilpotent elements of $D(X)$ is strictly larger than the maximal commutative ad-nilpotent subalgebra $O(X)$ of $D(X)$. Since $B$ is commutative, $B$ cannot contain all the ad-nilpotent elements of $D(X)$. Hence $D(X)$ contains an ad-nilpotent element $s$ not contained in $B$. By [8, Lemma 1.7], $v\text{-gr} s = \lambda w^n$ for some $\lambda \in C$ and $n > 0$. Since $s$ acts ad-nilpotently on $D(X)$, it is clear that $v\text{-gr} D(X) \subset C[u,w]$. By Theorem 4.5, $\dim C[u]/B$ is finite hence the integral closure of $B$ is $C[u]$. By Eakin’s theorem [6, Section 35], $B$ is finitely generated.

The invariant $\text{codim} D(X)$ can be used to distinguish rings of differential operators.

**Corollary 4.7.** — Suppose that $X$ and $Y$ are both affine curves with normalization equal to the affine line and with injective normalization.
maps. If \( D(X) \cong D(Y) \), then \( \text{codim} D(X) = \text{codim} D(Y) \).

Proof. — Consider both \( D(X) \) and \( D(Y) \) as subalgebras of \( C(x)[\partial] \) using the standard embedding. Let \( \phi \) be an isomorphism which maps \( D(Y) \) to \( D(X) \). Set \( u = \phi(x) \) and \( v = \phi(\partial) \). Clearly \( u \) and \( v \) satisfy the conditions of Theorem 4.5. Therefore \( \text{codim} D(Y) = \text{codim}_{u,v} D(X) = \text{codim} D(X) \).

5. Examples

In this section, we will consider two families of curves. We will calculate codimensions to show that their rings of differential operators are mutually nonisomorphic.

Recall that \( X \) is a monomial curve if \( O(X) \) is generated by monomials \( x^k \) as an algebra over \( C \). Let \( \Lambda \) be the subset \( \{ k \mid x^k \in O(x) \} \) of the integers. Define the set \( \Lambda - i \) to be \( \{ k - i \mid k \in \Lambda \} \) where \( i \) is an integer. Musson gives a complete description of \( D(X) \) in \([7]\). In particular,

\[
D(X) = \sum_{k \in \mathbb{Z}} x^k f_k(x\partial) C[x\partial]
\]

where

\[
f_k(x\partial) = \prod_{\alpha \in \Lambda - (\Lambda - k)} (x\partial - \alpha).
\]

Let \( X_n \) be the monomial curve with \( O(X_n) = C + x^n C[x] \) as coordinate ring, where \( n \) is a positive integer. Then by the previous paragraph, we have

\[
D(X_n) = \sum_{k \in \mathbb{Z}} x^k f_k(x\partial) C[x\partial]
\]

where the polynomial \( f_i \) is 1 for \( i = 0 \) and \( i \geq n \); the polynomial \( f_i \) is \( x\partial \) for \( 1 \leq i \leq n - 1 \); the polynomial \( f_i \) is

\[
(x\partial) \prod_{n-i<k\geq n} (x\partial - k) \quad \text{for} \quad -1 \geq i \geq -(n-1)
\]

and the polynomial \( f_i \) is

\[
(x\partial) \prod_{n-k<i} (x\partial - k) \prod_{-i<k<n-i} (x\partial - k) \quad \text{for} \quad i \leq -n.
\]

Note that if \( g(x\partial) \) is a monic polynomial in \( C[x\partial] \), then

\[
\partial^{-\text{gr}} g(x\partial) = x^d \partial^d \quad \text{where} \quad d = \deg g(x\partial).
\]
Hence $\partial - gr D(X_n) = \sum_{k \in \mathbb{Z}} g_k \mathbb{C}[x,y]$ where

$$
\begin{align*}
g_0 &= 1; \\
g_i &= x^{i+1}y \quad \text{for } 1 \leq i \leq n-1; \\
g_i &= x^i \quad \text{for } i \geq n; \\
g_i &= xy^{i+1} \quad \text{for } -n+1 \leq i \leq -1; \\
g_i &= y^i \quad \text{for } i \leq -n.
\end{align*}
$$

A basis for $\mathbb{C}[x,y]/\partial - gr D(X_n)$ is just $x, x^2, \ldots, x^{n-1}, y, y^2, \ldots, y^{n-1}$. Therefore $\text{codim } D(X_n) = 2(n-1)$. By Corollary 4.7, $D(X_n)$ is isomorphic to $D(X_m)$ if and only if $O(X_n) \cong O(X_m)$.

Now set $Y_{2n} = \mathbb{C} + \mathbb{C}x^2 + \cdots + \mathbb{C}x^{2n} \mathbb{C}[x]$ for $n \geq 1$. A similar calculation shows that $\text{codim } D(Y_{2n}) = n(n+1)$. Therefore $D(Y_{2n}) \cong D(Y_{2m})$ if and only if $O(Y_{2n}) \cong O(Y_{2m})$.

Consider just the curves $X_4$ and $Y_4$. Now $O(X_4) = \mathbb{C} + x^4 \mathbb{C}[x]$ and $O(Y_4) = \mathbb{C} + x^2 + x^4 \mathbb{C}[x]$. Clearly $O(X_4)$ is not isomorphic to $O(Y_4)$. But $\text{codim } D(X_4) = \text{codim } D(Y_4) = 6$. Therefore codim does not distinguish between these two rings of differential operators. We should add that it has now been shown that $D(X_4)$ and $D(Y_4)$ are actually isomorphic rings even though $O(X_4)$ and $O(Y_4)$ are not isomorphic (see [4]).

**BIBLIOGRAPHY**


